Lecture 26

Last time

$$d\sigma = \frac{(2\pi)^4 \hbar^2}{V} |\langle \bar{p}_1' \bar{p}_2' \| T(E+i\epsilon) \| \bar{p}_1 \bar{p}_2 \rangle|^2 \times$$
$$\delta(\bar{P}' - \bar{P}) \delta(E' - E)\, d^3 p_1'\, d^3 p_1'$$

$$\langle p_1' p_2' | T(E+i\epsilon) | p_1 p_2 \rangle =$$
$$\delta(\bar{P}' - \bar{P}) \langle \bar{p}_1' \bar{p}_2' \| T(E+i\epsilon) | p_1 p_2 \rangle$$

where

$$T(z) = V + V(z-H)^{-1} V$$

satisfies

$$\boxed{T(z) = V + V(z-H_0)^{-1} T(z)}$$

Lippman Schwinger Equation

We can also express $d\sigma$ in terms of the scattering wave function

$$|\Psi_\pm(0)\rangle = \Omega_\pm |\bar{p}_1 \bar{p}_2\rangle d^3p_1 d^3p_2 \langle p_1 p_2 | \Psi^0_\pm(0)\rangle$$

$$= |(p_1 p_2)_\pm\rangle d^3p_1 d^3p_2 \langle p_1 p_2 | \Psi^0_\pm(0)\rangle$$

where

$$|(\bar{p}_1 \bar{p}_2)_\pm\rangle = \Omega_\pm |\bar{p}_1 \bar{p}_2\rangle$$

with the understanding that this must be integrated with a wave packet. We can construct an equation for $|(\bar{p}_1 \bar{p}_2)_\pm\rangle$ using

$$|(p_1 p_2)_\pm\rangle = \lim_{t\to\pm\infty} e^{iHt/\hbar} e^{-iH_0 t/\hbar} |p_1 p_2\rangle =$$

$$= |\bar{p}_1 \bar{p}_2\rangle + \lim_{t\to\pm\infty} \int_0^t \frac{d}{dt'}\left(e^{iHt'/\hbar} e^{-iH_0 t'/\hbar}\right) |\bar{p}_1 \bar{p}_2\rangle dt'$$

$$= |p_1 p_2\rangle + \lim_{t\to\pm\infty} \left(\frac{i}{\hbar}\right) \int_0^t e^{iHt'/\hbar} V e^{-iH_0 t'/\hbar} |\bar{p}_1 \bar{p}_2\rangle dt'$$

$$= |p_1 p_2\rangle + \lim_{t\to\pm\infty} \left(\frac{i}{\hbar}\right) \int_0^t e^{i(H-E)t'/\hbar} V |p_1 p_2\rangle dt'$$

The time integral only makes sense if this is first integrated over a momentum wave packet.

If this is done we can include a factor $\lim_{\epsilon \to 0} e^{\mp \epsilon t}$ which does not change the result. With this factor we can change the order of the integration to get

$$= |\bar{p}_1 \bar{p}_2\rangle + \left(\frac{i}{\hbar}\right) \lim_{\epsilon \to} \int_0^{\pm\infty} e^{i(H-E\pm i\hbar\epsilon)/\hbar} V |\bar{p}_1 \bar{p}_2\rangle \, dt$$

$$= |\bar{p}_1 \bar{p}_2\rangle + \left(\frac{i}{\hbar}\right)(-1)\left(\frac{\hbar}{i}\right)(H - E \pm i\hbar\epsilon)^{-1} V |\bar{p}_1 \bar{p}_2\rangle$$

$$= |\bar{p}_1 \bar{p}_2\rangle + (E - H \mp i\underbrace{\hbar\epsilon}_{\epsilon'})^{-1} V |\bar{p}_1 \bar{p}_2\rangle$$

This give the solved form of $|(p_1 p_2)_\pm\rangle$

$$\boxed{|(\bar{p}_1\bar{p}_2)_\pm\rangle = |\bar{p}_1\bar{p}_2\rangle + (E \mp i\epsilon' - H)^{-1} V |\bar{p}_1\bar{p}_2\rangle}$$

As in the case of $T(z)$ we can use the second resolvent equation to write

$$|(p_1 p_2)_\pm\rangle = |\bar{p}_1\bar{p}_2\rangle + \left[(E \mp i\epsilon - H_0)^{-1} + (E \mp i\epsilon - H_0)^{-1} V (E \mp i\epsilon - H)^{-1}\right] V |p_1 p_2\rangle$$

$$= |p_1 p_2\rangle + (E \mp i\epsilon - H_0)^{-1} V \underbrace{\left[1 + (E \mp i\epsilon - H)^{-1} V\right] |\bar{p}_1\bar{p}_2\rangle}_{|(p_1 p_2)_\pm\rangle}$$

(1)

$$= |\bar{p}_1\bar{p}_2\rangle + (E \mp i\epsilon - H_0)^{-1} \underbrace{\left[V + V(E \mp i\epsilon - H)^{-1} V\right]}_{T(E \mp i\epsilon)} |\bar{p}_1\bar{p}_2\rangle$$

(2)

equation (1) is an integral equation for $|(p_1 p_2)_\pm\rangle$ while equation (2) expresses $|(p_1 p_2)_\pm\rangle$

in terms of $T(E\mp i\epsilon)$

$$|\Psi_\pm(t)\rangle = \int |(p_1p_2)_\pm\rangle d^3p_1 d^3p_2 \langle \bar{p}_1\bar{p}_2|\Psi^0_\pm(t)\rangle$$

where

$$|(p_1p_2)_\pm\rangle = |\bar{p}_1\bar{p}_2\rangle + (E\mp i\epsilon - H_0)^{-1} V |(p_1p_2)_\pm\rangle$$

$$= |\bar{p}_1\bar{p}_2\rangle + (E\mp i\epsilon - H_0)^{-1} T(E\mp i\epsilon) |\bar{p}_1\bar{p}_2\rangle$$

The equation for $|(p_1p_2)_\pm\rangle$ is called the Lippmann–Schwinger equation for $|(p_1p_2)_\pm\rangle$

We can express the differential cross section in terms of $|(\bar{p}_1\bar{p}_2)_\pm\rangle$. To do this use

$$\langle \bar{p}_1'\bar{p}_2'|T(E\pm i\epsilon)|(p_1p_2)\rangle =$$

$$\langle \bar{p}_1'\bar{p}_2'|(V + V(E\pm i\epsilon - H)^{-1}V)|p_1p_2\rangle =$$

$$\langle \bar{p}_1'\bar{p}_2'|V|\underbrace{(1+(E\mp i\epsilon - H)^{-1}V)|p_1p_2\rangle}_{|(p_1p_2)_\mp\rangle} =$$

$$= \langle \bar{p}_1' \bar{p}_2' | V | (\bar{p}_1 \bar{p}_2)_{\mp} \rangle \quad ①$$

we can also write this as

$$\langle \bar{p}_1' \bar{p}_2' | V (1 + (E \mp i\epsilon - H)^{-1} V) | p_1 p_2 \rangle =$$

$$\langle \bar{p}_1' p_2' | (1 + V (E \mp i\epsilon - H)^{-1}) V | p_1 p_2 \rangle =$$

$$\langle p_1 p_2 | V \underbrace{(1 + (E \pm i\epsilon - H)^{-1} V) | p_1' p_2' \rangle}_{|(p_1' p_2')_{\pm} \rangle}^{*} =$$

$$\langle p_1 p_2 | V | (p_1' p_2') \rangle_{\pm}^{*} =$$

$$\langle (\bar{p}_1' \bar{p}_2')_{\pm} | V | p_1 p_2 \rangle \quad ②$$

combining ① and ② give

$$\langle p_1' p_2' | T(E + i\epsilon) | \bar{p}_1 p_2 \rangle =$$

$$\langle \bar{p}_1' \bar{p}_2' | V | (\bar{p}_1 \bar{p}_2)_{-} \rangle =$$

$$\langle (p_1' p_2')_{+} | V | p_1 p_2 \rangle$$

in all three cases if V is translationally invariant we can factor out an overall momentum conserving δ function

This gives the following 3 equivalent expressions for the differential cross section

$$d\sigma = \frac{(2\pi)^4 \hbar^2}{V} \left| \langle \bar{p}_1' \bar{p}_2' \| T(E+i\epsilon) \| p_1 p_2 \rangle \right|^2 \times \delta(\bar{p}-\bar{p}')\,\delta(E-E')\; d^3p_1'\, d^3p_2'$$

$$= \frac{(2\pi)^4 \hbar^2}{V} \left| \langle \bar{p}_1' \bar{p}_2' \| V \| (\bar{p}_1 \bar{p}_2)_- \rangle \right|^2 \times \delta(\bar{p}'-\bar{p})\,\delta(E'-E)\; d^3p_1'\, d^3p_2'$$

$$= \frac{(2\pi)^4 \hbar^2}{V} \left| \langle (p_1 p_2)_+ \| V \| \bar{p}_1 \bar{p}_2 \rangle \right|^2 \delta(\bar{p}'-\bar{p})\,\delta(E'-E)\; d^3p_1'\, d^3p_2'$$

The equation on the last page show that the differential cross section can be evaluated by solving the Lippmann-Schwinger equation for $T(E+i\epsilon)$, $|(p,n_i)\rangle_+$, $|(p,n_i)\rangle_-$

all three give the same result.

Another approach that is used in perturbative quantum field theory

the time translation operator in the interaction picture is

$$U_I(t',t)\,|\Psi_I(t)\rangle = |\Psi_I(t')\rangle$$

$$U_I(t',t)\,e^{iH_0t/\hbar}\,|\Psi_s(t)\rangle = e^{iH_0t'/\hbar}\,|\Psi_s(t')\rangle$$

$$e^{-iH_0t'/\hbar}\,U_I(t',t)\,e^{iH_0t/\hbar}\,|\Psi_s(t)\rangle = |\Psi_s(t')\rangle$$

but

$$|\Psi_s(t')\rangle = U_s(t'-t)\,|\Psi_s(t)\rangle$$

comparing we get

$$e^{-iH_0t'/\hbar}\,U_I(t',t)\,e^{iH_0t/\hbar} = U_s(t'-t)$$

or

$$\boxed{U_I(t',t) = e^{iH_0t'/\hbar}\,U_s(t'-t)\,e^{-iH_0t/\hbar}}$$

* note that while

$$U_I(t,t')\,U_I(t',t'') = U_I(t,t'')$$

it is not a one parameter group

$$* \quad U_I(t't) = e^{iH_0t'/\hbar}\,e^{-iH(t'-t)}\,e^{-iH_0t/\hbar}$$

using the definition of S

$$S = \lim_{\substack{t\to\infty \\ t'\to-\infty}} e^{iH_0t/\hbar}\,e^{-iH(t-t')}\,e^{-iHt'/\hbar}$$

we get

$$\boxed{S = \lim_{\substack{t \to \infty \\ t' \to -\infty}} \mathcal{U}_I(t,t')}$$

recall

$$\frac{d\mathcal{U}_I}{dt} = \frac{d}{dt}\left(e^{iH_0 t/\hbar} e^{-iH(t-t')/\hbar} e^{-iH_0 t'/\hbar}\right)$$

$$= -\frac{i}{\hbar}\left(e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}\right) e^{iH_0 t/\hbar} e^{-iH(t-t')/\hbar} e^{-iH_0 t'/\hbar}$$

$$= -\frac{i}{\hbar} V_I(t)\, \mathcal{U}_I(t,t')$$

$$\therefore \boxed{\frac{d\mathcal{U}_I(t,t')}{dt} = -\frac{i}{\hbar} V_I(t)\, \mathcal{U}_I(t,t')}$$

$$\boxed{\begin{gathered} \mathcal{U}_I(t,t) = I \quad \Rightarrow \\ \mathcal{U}_I(t,t') = I - \frac{i}{\hbar}\int_{t'}^{t} V_I(t'')\, \mathcal{U}_I(t'',t')\, dt'' \end{gathered}}$$

The solution to this equation was given by the Dyson series

$$\mathcal{U}_I(t,t') = I + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{t'}^{t} T(V_I(t_1) \quad V_I(t_n)) \times$$

$$\times \, dt_1 \cdots dt_n$$

combining this with

$$S = \lim_{\substack{t \to \infty \\ t' \to -\infty}} \mathcal{U}_I(t,t') \quad \text{gives}$$

$$\boxed{S = I + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} T(V_I(t_1) \quad V_I(t_n)) \times \, dt_1 \quad dt_n}$$

because the time interval is not finite this series does not always converge like it does for finite $t'-t$

It is instructive to check that the first non trivial term in this series gives the Born approximation.

$$\langle \bar{p}_1' \bar{p}_2' | S_{BORN} | p_1 p_2 \rangle =$$

$$\langle p_1' p_2' | p_1 p_2 \rangle$$

$$- 2\pi i \, \delta(E' - E) \langle p_1'' p_2'' | V | p_1 p_2 \rangle$$

using the Dyson expansion

$$\langle p_1 p_2'' | S_{Born} | p_1 p_2 \rangle =$$

$$\langle \bar{p}_1' \bar{p}_2' | \bar{p}_1 \bar{p}_2 \rangle - \frac{i}{\hbar} \int_{-\infty}^{\infty} \langle p_1' p_2' | V_I(t') | p_1 p_2 \rangle dt' =$$

$$\langle \bar{p}_1' \bar{p}_2' | \bar{p}_1 \bar{p}_2 \rangle - \frac{i}{\hbar} \int_{-\infty}^{\infty} e^{i(E' - E)t/\hbar} \langle p_1 p_2'' | V_S | p_1 p_2 \rangle$$

$$\langle \bar{p}_1' \bar{p}_2' | p_1 p_2 \rangle - \frac{i}{\hbar} \langle p_1' p_2' | V | p_1 p_2 \rangle 2\pi\hbar \times$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(E' - E)t/\hbar} d\left(\frac{t}{\hbar}\right)$$

$$\langle \bar{p}_1' \bar{p}_2' | \dot{p}_1 \dot{p}_2 \rangle - 2\pi i \, \delta(E' - E) \langle \bar{p}_1 \bar{p}_2' | V | \bar{p}_1 \bar{p}_2 \rangle$$

which is exactly the same expression obtained using the Lippmann Schwinger equation

In quantum field theory the Dyson expansion is use to develop perturbation theory - Feynman diagrams are rules to compute terms in the Dyson expansion.

Scattering with wave functions

For translationally invariant potentials it is useful to change variables

$$\bar{r}_1 \bar{r}_2 \longrightarrow R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \qquad \bar{r} = \bar{r}_1 - \bar{r}_2$$

$$d^3R\, d^3r = d^3r_1\, d^3r_2$$

this can be inverted

$$r_1 = R + \frac{m_2}{M}\bar{r} \qquad M = m_1 + m_2$$

$$r_2 = R - \frac{m_1}{M}\bar{r}$$

$$\langle r' R' | V | \bar{r} \bar{R} \rangle = \delta(\bar{R} - R') \, \delta(\bar{r}' - \bar{r}) \, V(r)$$

Recall in momentum space

$$P = p_1 + p_2 \qquad \bar{k} = \frac{m_2 p_1 - m_1 p_2}{M} \qquad E = \frac{k^2}{2\mu} + \frac{\bar{P}^2}{2M}$$

$$|(p_1 p_2)_\pm\rangle = |(P k)_\pm\rangle$$

$$\langle \bar{P}' \bar{k}' | (P k)_\pm \rangle = \langle \bar{P}' \bar{k}' | \bar{P} \bar{k} \rangle$$

$$+ \int \underbrace{\langle \bar{p}' \bar{k}' | (E \mp i\epsilon - H_0)^{-1} V | (P k)_\pm \rangle}$$

$$\langle \bar{P}' \bar{k}' | \left(\cancel{\frac{P^2}{2M}} + \frac{k^2}{2\mu} \mp i\epsilon - \cancel{\frac{P'^2}{2M}} - \frac{k'^2}{2\mu} \right)^{-1}$$

when E is the initial energy — the quantity commutes with $\bar{P}$

$$\langle \bar{P}' \bar{k}' | (\bar{P} \bar{k})_\mp \rangle = \delta(\bar{P}' - \bar{P}) \langle \bar{k}' | (\bar{k})_\mp \rangle$$

The equation for $\langle \bar{k}' | (k_\mp) \rangle$ has the form

$$\langle \bar{k}' | \bar{k}_{\mp} \rangle = \delta(\bar{k}' - \bar{k})$$

$$+ \int \frac{1}{\frac{k^2}{2\mu} \mp i\epsilon + \frac{k'}{2\mu}} \langle k' | V | k'' \rangle d^3k'' \langle \bar{k}'' | k_{\mp} \rangle$$

next we write this equation

in coordinate space

$$\langle \bar{r} | \bar{k}_{\pm} \rangle = \langle \bar{r} | \bar{k} \rangle$$

$$+ \langle \bar{r} | \bar{k}' \rangle \frac{d^3k'}{\frac{k^2}{2\mu} \mp i\epsilon - \frac{k'^2}{2\mu}} \langle k' | r'' \rangle V(r'') dr''$$

$$\langle r'' | \bar{k}_{\pm} \rangle$$

we define

$$g\left(\frac{k^2}{2\mu} \mp i\epsilon, r, r'\right) =$$

$$\int \langle \bar{r} | \bar{k}' \rangle \frac{d^3k'}{\frac{k^2}{2\mu} \mp i\epsilon - \frac{k'^2}{2\mu}} \langle \bar{k}' | \bar{r}'' \rangle =$$

$$\frac{1}{(2\pi\hbar)^3} \int e^{i\bar{k}' \cdot (\bar{r} - \bar{r}')/\hbar} \frac{d^3k'}{\frac{k^2}{2\mu} \mp i\epsilon - \frac{k''}{2\mu}}$$

To do this integral choose a coordinate system so $\bar{r}-\bar{r}'$ is in the z direction. Then the integral becomes

$$\frac{2\mu}{(2\pi\hbar)^3}\int \frac{e^{ik'|r-r'|\cos\theta/\hbar}}{k^2 \mp 2\mu i\epsilon - k'^2} k'^2 dk' \sin\theta \, d\theta \, d\phi$$

let $\epsilon' = 2\mu\epsilon$ and do the ϕ integral. Let $u = \cos\theta$ $du = -\sin\theta \, d\theta$

$u: 1 \to (-1)$

$$\frac{(2\mu)(2\pi)}{(2\pi\hbar)^3}\int_0^\infty k'^2 dk' \int_{-1}^{1} du \frac{e^{ik'|r-r'|u/\hbar}}{k^2 \mp i\epsilon' - k'^2} =$$

$$\frac{(2\mu)(2\pi)}{(2\pi\hbar)^3}\int_0^\infty k'^2 dk' \frac{\hbar}{ik'|r-r'|} \times \left(\frac{e^{ik'|r-r'|/\hbar} - e^{-ik'|r-r'|/\hbar}}{k^2 \mp i\epsilon' - k'^2} \right) =$$

$$-\frac{(2\mu)(2\pi\hbar)i}{(2\pi\hbar)^3|r-r'|}\int_0^\infty \frac{k' dk'}{k^2 \mp i\epsilon' - k'^2}\left(e^{ik'|r-r'|/\hbar} - e^{-ik'|r-r'|/\hbar}\right)$$

In the second exponential

let $k'' = -k'$ $dk'' = -dk'$

k'': $0, -\infty$ This gives

$$= -i\frac{(2\mu)(2\pi\hbar)}{(2\pi\hbar)^3\,|r-r'|}\left[\int_0^\infty + \int_{-\infty}^0\right]\frac{k'\,dk'}{k^2 \mp i\epsilon - k'^2}\, e^{ik'|r-r'|/\hbar}$$

$$= \frac{-2i\mu}{(2\pi\hbar)^2\,|r-r'|}\int_{-\infty}^{\infty}\frac{-k'\,dk'\, e^{ik'|r-r'|/\hbar}}{\left(k' - \sqrt{k^2 \mp i\epsilon}\right)\left(k' + \sqrt{k^2 \mp i\epsilon}\right)}$$

This can be computed using the residue theorem.

In order to get no contribution from the semicircle at ∞ we must use the path  –

pick up pole in contour

for the lower sign

$$k' = k + i\epsilon''$$

for the upper sign

$$k' = -k + i\epsilon'$$

$$-i \frac{2\mu}{(2\pi\hbar)^2} \frac{-(\pm k)}{\mp 2k} (2\pi i) \frac{e^{\mp ik|r-r'|/\hbar}}{|r-r'|} =$$

$$- \frac{\mu}{2\pi\hbar} \frac{e^{\mp ik|r-r'|/\hbar}}{|r-r'|}$$

* note that the sign of ϵ affects the sign in the expment. The integral equation becom

$$\boxed{\langle \bar{r} | k\pm \rangle = \langle \bar{r} | k \rangle + - \frac{\mu}{2\pi\hbar} \int \frac{e^{\mp ik|r-r'|/\hbar}}{|r-r'|} V(r') \, d^3r' \langle r' | k_\pm \rangle}$$

This is the coordinate space version of the Lippmann Schwinger equation for the wave function

since $|k\rangle_\pm\rangle$ is an eigenstate of $h = \frac{k^2}{2\mu} + V$ with energy $\frac{k^2}{2\mu}$

$$\langle \bar{r} \,|\, \bar{k}_\pm, t\rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\frac{k\cdot r}{\hbar} - i\frac{k^2}{2\mu}\frac{t}{\hbar}}$$

$$- \frac{\mu}{2\pi\hbar} \int \frac{e^{\mp ik|r-r'|/\hbar - i\frac{k^2}{2\mu}\frac{t}{\hbar}}}{|r-r'|} V(r')\, d^3r' \langle r|k_\pm, 0\rangle$$

has the form of an incoming or outgoing spherical wave