Lecture 25

Recall

$$d\sigma = \frac{(2\pi)^4}{V} \hbar^2 |\langle \bar{p}_1' \bar{p}_2' || T(E+i\epsilon) || \bar{p}_1 \bar{p}_2 \rangle|^2 \times$$

$$\delta(E-E')\,\delta(\bar{P}-\bar{P}')\, d^3p_1\, d^3p_2$$

where

$$T(E+i\epsilon) = V + V(E-H+i\epsilon)^{-1}V$$

and

$$P = |\langle \Psi_+(t) | \Psi_-(t) \rangle|^2 =$$

$$|\langle \Psi_+^0(t) | S | \Psi_-^0(t) \rangle|^2$$

$$\langle p_1' p_2' | S | p_1 p_2 \rangle =$$

$$\langle p_1' p_2' | p_1 p_2 \rangle - 2\pi i\, \delta(E'-E) \langle p_1' p_2' | T(E+i\epsilon) | p_1 p_2 \rangle$$

$$\langle p_1' p_2' | T(E+i\epsilon) | p_1 p_2 \rangle =$$

$$\delta(\bar{P}'-\bar{P}) \langle p_1' p_2' || T(E+i\epsilon) || p_1 p_2 \rangle$$

Because we use $S-1$ to calculate $d\sigma$ is only is non zero for scattered particles

The physics is contained in $T(E+i\epsilon)$ - called the transition operator

The nice thing about the differential cross section is that it only involves quantities that can be measured - the wave packets have disappeared.

To calculate $T(z)$ we use the second resolvent identity

$$(z-H)^{-1} = (z-H_0)^{-1} + (z-H_0)^{-1} V (z-H)^{-1}$$

in the expression for $T(z)$.

This gives

$$T(z) = V + V\{(z-H_0)^{-1} + (z-H_0)^{-1}V(z-H)^{-1}\}V$$

$$= V + V(z-H_0)^{-1}[V + V(z-H)^{-1}V]$$

$$= V + V(z-H_0)^{-1}T(z)$$

or

$$\boxed{T(z) = V + V(z-H_0)^{-1}T(z)}$$

This is an integral equation because the unknown appears on both sides of the equation.

It is called the

**Lippmann Schwinger equation**

In order to solve it it is useful to consider the case of translationally invariant potentials

$$\bar{P} = \bar{P}_1 + \bar{P}_2 \qquad M = m_1 + m_2$$

$$\bar{k} = p_1 - m_1 \frac{\bar{P}}{M} \qquad \mu = \frac{m_1 m_2}{M}$$

$$H = \frac{P^2}{2M} + \frac{k^2}{2\mu} + V$$

where

$$\langle \bar{P}' \bar{k}' | V | \bar{P} k \rangle = \delta(\bar{P} - \bar{P}') \langle \bar{k}' | V | \bar{k} \rangle$$

In this case if we take matrix elements of the equation an overall momentum conserving $\delta$ function factors out

we define

$$h = \frac{k^2}{2\mu} + V \qquad h_0 = \frac{k^2}{2\mu} \quad \varepsilon = \frac{k^2}{2\mu}$$

$$\langle P' k' | T(E + i\epsilon) | P k \rangle =$$

$$\delta(P - P') \langle \bar{k}' | \underbrace{(V + V(\varepsilon - h + i\epsilon)^{-1} V)}_{\hat{T}(\varepsilon + i\epsilon)} | k' \rangle$$

$$\boxed{\langle \bar{k}' | \hat{T}(\varepsilon + i\epsilon) | \bar{k} \rangle = \langle \bar{k}' | V | \bar{k} \rangle + \int d^3k'' \langle \bar{k}' | V | \bar{k}'' \rangle \frac{1}{\frac{k^2}{2\mu} - \frac{k''^2}{2\mu} + i\epsilon} \langle \bar{k}'' | T(\varepsilon + i\epsilon) | \bar{k} \rangle}$$

This is the actual equation that must be solved.

To solve this use an orthonormal basis

$\langle k | n \rangle \qquad \langle m | n \rangle = \delta_{mn}$

$$\langle n | T(\varepsilon + i\epsilon) | \bar{k} \rangle = \langle n | V | \bar{k} \rangle + \sum_m \underbrace{\langle n | V \frac{2\mu}{k^2 - h_0 + 2\mu i\epsilon} | m \rangle}_{K_{nm}} \langle m | T(\varepsilon + i\epsilon) | k \rangle$$

The equation can be written as

$$\boxed{\sum_m (\delta_{nm} - K_{nm}) \langle m | T(\varepsilon + i\epsilon) | \bar{k} \rangle = \langle n | V | \bar{k} \rangle}$$

This is an infinite set of coupled linear equations. In most cases of interest

$$V(E-h_0+i\epsilon) = K$$

or

$$I\, V^{1/2}(E-h_0+i\epsilon)V^{1/2}$$

is a compact operator. This means

$$K = F + \Delta$$

where $F$ is finite dimensional and $\Delta$ can be made as small as desired.

Formally this means

$$(I-F)\hat{T} = V + \Delta\hat{T}$$

$$\hat{T} = (I-F)^{-1}V + \underbrace{(I-F)^{-1}\Delta}\hat{T}$$

when this is small the equation can be solved by iteration compactness means that if we truncate the sum

$$\langle n | \hat{T}^N(z) | k \rangle = \langle n | V | k \rangle + \sum_{m=1}^{N} \langle n | V (E - h_0 + i\epsilon)^{-1} | m \rangle \langle m | \hat{T}$$

that

$$\lim_{N \to \infty} \langle n | \hat{T}^N | k \rangle = \langle n | \hat{T} | k \rangle$$

how fast this converges depends on the choice of basis.

Assume that we have solved for $\langle n | \hat{T}(E+i\epsilon) | k \rangle$.

Then

$$\boxed{\langle \bar{k}' | \hat{T}(E+i\epsilon) | \hat{k} \rangle = \langle \bar{k}' | V | k \rangle + \sum_{m}^{N} \langle \bar{k}' | V (z - h_0 + i\epsilon) | m \rangle \langle m | \hat{T}(E+i\epsilon) | N \rangle}$$

which gives an approximate solution for continuous $\vec{k}'$

When

$$\| V(\varepsilon - h_0 + i\epsilon)^{-1} \| < 1$$

this equation can be solved by iteration

$$\hat{T}(\varepsilon + i\epsilon) = V + \sum_{n=1}^{\infty} (V(\varepsilon - h_0 + i\epsilon)^{-1})^n V$$

this series is called the Born series. Typically $\| V(\varepsilon - h_0 + i\epsilon)^{-1} \| < 1$ for small $V$ or large $\varepsilon$. The simplest approximation is the lowest order approximation

$$\hat{T}(\varepsilon + i\epsilon) \underset{\text{Born}}{\simeq} V$$

This is called the Born approximation

$$\frac{d\sigma}{d\Omega_{cm}} = (2\pi)^4 \hbar^2 \mu^2 \, |\langle \bar{k}' | V | \bar{k} \rangle|^2$$

where

$$\langle \bar{k}' | V | \bar{k} \rangle = \frac{1}{(2\pi\hbar)^3} \int e^{-i(\bar{k}'-\bar{k})\cdot\bar{r}/\hbar} \, V(r) \, d^3r$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} \tilde{V}(\bar{k}'-\bar{k})$$

and $\tilde{V}(\bar{k}'-\bar{k})$ is the fourier transform of $V$

example: Yukawa potential

$$\boxed{V(r) = -\lambda \frac{e^{-\alpha r}}{r}}$$

In this case we need to first calculate the fourier transform of $V(r)$

$$\tilde{V}(\bar{k}'-\bar{k}) = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-i(\bar{k}'-\bar{k})\cdot\bar{r}/\hbar} \left(-\lambda \frac{e^{-\alpha r}}{r}\right) d^3r$$

choose coordinates so

$\bar{k}'-\bar{k}$ is along the $\hat{z}$ axis $\Rightarrow$

$$(\bar{k}'-\bar{k})\cdot\bar{r}/\hbar = |\bar{k}'-\bar{k}| r \cos\theta/\hbar$$

$$\tilde{V}(\bar{k}'-\bar{k}) = \frac{-\lambda}{(2\pi\hbar)^{3/2}} \int_0^\infty r\,dr \int_0^\pi \sin\theta\, d\theta\; 2\pi \times e^{-\alpha r - i|\bar{k}'-\bar{k}| r\cos\theta/\hbar}$$

Let $u = \cos\theta \quad du = -\sin\theta\, d\theta$

$u: 1 \to -1$

$$= -\frac{\lambda 2\pi}{(2\pi\hbar)^{3/2}} \int_0^\infty r\,dr \int_{-1}^{1} du\; e^{-\alpha r - i|\bar{k}'-\bar{k}| r u/\hbar}$$

$$= -\frac{2\pi\lambda}{(2\pi\hbar)^{3/2}} \int_0^\infty \cancel{r}\, dr\; e^{-\alpha r} \left[ \frac{1}{-i|\vec{k}'-\vec{k}|\cancel{r}/\hbar} \times \left( e^{-i|\vec{k}-\vec{k}'|r/\hbar} - e^{i|\vec{k}'-\vec{k}|r/\hbar} \right) \right]$$

$$= -\frac{2\pi\lambda}{(2\pi\hbar)^{3/2}} \frac{i\hbar}{|\vec{k}'-\vec{k}|} \left( \frac{-1}{-\alpha - i|\vec{k}'-\vec{k}|/\hbar} - \frac{-1}{-\alpha + i|\vec{k}'-\vec{k}|/\hbar} \right)$$

$$= \frac{-\lambda}{(2\pi\hbar)^{1/2}} \frac{i}{|\vec{k}'-\vec{k}|} \frac{-2i|\vec{k}'-\vec{k}|/\hbar}{\alpha^2 + |\vec{k}'-\vec{k}|^2/\hbar^2}$$

$$= -\frac{4\pi\lambda}{(2\pi\hbar)^{3/2}} \frac{1}{\alpha^2 + |\vec{k}'-\vec{k}|^2/\hbar^2}$$

since energy is conserved

$k^2 = k'^2$

$$|\vec{k}'-\vec{k}|^2 = k^2 + k'^2 - 2kk'\cos\theta$$
$$= 2k^2(1-\cos\theta) =$$
$$= 2k^2\, 2\sin^2\left(\frac{\theta}{2}\right)$$
$$= 4k^2\sin^2(\theta/2)$$

$$= -\frac{4\pi\lambda}{(2\pi\hbar)^{3/2}} \frac{1}{\alpha^2 + 4k^2\sin^2\theta/2\,/\hbar^2}$$

This give the following expression for the differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(2\pi)^4 \hbar^2 \mu^2}{(2\pi\hbar)^3} |\langle k' | V | k \rangle|^2 =$$

$$\frac{(2\pi)^4 \hbar^2 \mu^2}{(2\pi\hbar)^3} \frac{16\pi^2\lambda^2}{(2\pi\hbar)^3} \left(\frac{1}{\alpha^2 + 4k^2 \sin^2(\frac{\theta}{2})/\hbar^2}\right)^2$$

$$\boxed{\frac{\mu^2}{\hbar^4} \frac{16\lambda^2}{(\alpha^2 + 4k^2 \sin^2(\frac{\theta}{2})/\hbar^2)^2}}$$

If we take the limit $\alpha \to 0$ we get the first Born approximation for the Coulomb potential

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\mu^2}{\hbar^4} \frac{16\lambda^2\hbar^4}{16 k^4 \sin^4(\frac{\theta}{2})} = \frac{\mu^2\lambda^2}{k^4 \sin^4(\frac{\theta}{2})}$$

you can check the units $\to$ area.

equation for wave function

$$|\Psi_\pm(0)\rangle = \Omega_\pm |\Psi^0_\pm\rangle =$$

$$\int \Omega_\pm |p_1 p_2\rangle\, d^3p_1\, d^3p_2 \langle p_1 p_2 | \Psi^0_\pm(0)\rangle$$

we define

$$|(p_1 p_2)_\pm\rangle \equiv \Omega_\pm |\bar{p}_1 \bar{p}_2\rangle$$

$$\boxed{\begin{aligned} &\int |(p_1 p_2)_\pm\rangle\, dp_1\, dp_2 \langle p_1 p_2 | \Psi^0_\pm(0)\rangle = \\ &\int \Omega_\pm |\bar{p}_1 \bar{p}_2\rangle\, d^3p_1\, d^3p_2 \langle p_1 p_2 | \Psi^0_\pm(0)\rangle = \\ &\Omega_\pm |\Psi^0_\pm(0)\rangle = |\Psi_\pm\rangle \end{aligned}}$$

$$|(p_1 p_2)_\pm\rangle = \lim_{t\to\pm\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\overline{p_1 p_2}\rangle$$

$$= |\bar{p}_1 \bar{p}_2\rangle + \lim_{t\to\infty} \int_0^t \frac{d}{dt}\left(e^{iHt/\hbar} e^{-iH_0t/\hbar}\right) |p_1 p_2\rangle\, dt$$

$$= |\bar{p}_1 \bar{p}_2\rangle + \frac{i}{\hbar} \lim_{t\to\infty} \int_0^t e^{iHt/\hbar}\, V\, e^{-iH_0t/\hbar} |p_1 p_2\rangle\, dt$$

as in the case of $T(z)$ we need to integrate over the wave packet. This can be fixed by including the factor $\lim_{\epsilon \to 0} e^{\mp \epsilon t}$ then we get the integral

$$\lim_{\epsilon \to 0} \int_0^\infty e^{i(H-E\pm i\epsilon\hbar)t/\hbar} |p_1 p_2\rangle \, dt =$$

$$\lim_{\epsilon \to 0} -\frac{\hbar}{i} (H-E\pm i\epsilon\hbar)^{-1} V |p_1 p_2\rangle$$

using this with the expression on the last page gives

$$|(p_1 p_2)_\pm\rangle = |p_1 p_2\rangle - (H-E\pm i\epsilon\hbar)^{-1} V |p_1 p_2\rangle$$

We can use the second resolvent equation to get an equation for $|(p_1 p_2)_\pm\rangle$ or an expression for $|(\bar{p}_1\bar{p}_2)_\pm\rangle$ in terms of $T(E\mp i\epsilon)$

$$|(p_1p_2)_\pm\rangle = |p_1p_2\rangle + \left[(E\mp i\epsilon\hbar - H_0)^{-1} + (E\mp i\epsilon\hbar - H_0)^{-1}V(E\mp i\epsilon\hbar - H)^{-1}V\right] \times |p_1p_2\rangle$$

(1) $$= |\bar{p}_1\bar{p}_2\rangle + (E\mp i\epsilon\hbar - H_0)^{-1}V\left[|p_1p_2\rangle + (E\mp i\epsilon\hbar - H)^{-1}V \times |p_1p_2\rangle\right]$$

$$= |\bar{p}_1\bar{p}_2\rangle + (E\mp i\epsilon\hbar - H_0)^{-1}V\,|(p_1p_2)_\pm\rangle$$

This gives the Lippmann Schwinger equation for $|(\bar{p}_1\bar{p}_2)_\pm\rangle$

$$\boxed{|(p_1p_2)_\pm\rangle = |\bar{p}_1\bar{p}_2\rangle + (E\mp i\epsilon\hbar - H_0)^{-1}V\,|(p_1p_2)_\pm\rangle}$$

(2) $$= |\bar{p}_1\bar{p}_2\rangle + (E\mp i\epsilon\hbar - H_0)^{-1}\left\{V + V(E\mp i\epsilon\hbar - H)^{-1}V\right\}|p_1p_2\rangle$$

$$\boxed{\begin{aligned}&= |\bar{p}_1\bar{p}_2\rangle + (E\mp i\epsilon\hbar - H_0)^{-1}T(E\mp i\epsilon)\,|p_1p_2\rangle\\ &= |(p_1p_2)_\pm\rangle\end{aligned}}$$

To use this solution to calculate the cross section note

$$\langle p_1' p_2' | T(E \pm i\epsilon) | p_1 p_2 \rangle =$$

$$\langle \bar{p}_1' \bar{p}_2' | V | (p_1 p_2)_- \rangle =$$

$$\langle p_1 p_2 | T(E \mp i\epsilon) | p_1' p_2' \rangle^* =$$

$$\langle p_1 p_2 | V | (p_1' p_2')_+ \rangle^* =$$

$$\langle (p_1 p_2)_+ | V | p_1 p_2 \rangle$$

This means we can use

$$\langle p_1' p_2' \| T(E + i\epsilon) \| \bar{p}_1 \bar{p}_2 \rangle =$$

$$\langle \bar{p}_1' p_2' \| V \| (p_1 p_2)_- \rangle =$$

$$\langle (p_1' p_2')_+ \| V \| \bar{p}_1 \bar{p}_2 \rangle$$

where we factored the total momentum conserving out of all 3

$$d\sigma = \frac{(2\pi)^4 \hbar}{V} |\langle \bar{p}_1' \bar{p}_2' \| T(E+i\epsilon) \| \bar{p}_1 \bar{p}_2 \rangle|^2 \, \delta(E-E')\,\delta(\bar{P}-\bar{P}')\, d^3p_1'\, d^3p_2'$$

$$= \frac{(2\pi)^4 \hbar}{V} |\langle \bar{p}_1' \bar{p}_2' \| V \| (p_1 p_2)_- \rangle|^2 \, \delta(E-E')\,\delta(\bar{P}-\bar{P}')\, d^3p_1'\, d^3p_2'$$

$$= \frac{(2\pi)^4 \hbar}{V} |\langle (p_1' p_2')_+ \| V \| \bar{p}_1 \bar{p}_2 \rangle|^2 \, \delta(E-E')\,\delta(\bar{P}-\bar{P}')\, d^3p_1'\, d^3p_2'$$

Note: the scattering operator can also be computed using the Dyson expansion

The time evolution operator in the interaction picture

U

$$\mathcal{U}_s(t-t') = e^{-iH(t-t')/\hbar}$$

$$\mathcal{U}_I(t,t') = e^{iH_0 t}\, e^{-iH(t-t')}\, e^{-iH_0 t'}$$

In this form it is

clear that

$$S = \lim_{\substack{t \to \infty \\ t' \to -\infty}} e^{iH_0 t/\hbar} e^{-iH(t-t')} e^{-iH_0 t'/\hbar}$$

$$\frac{d}{dt} \mathcal{U}_I(t,t') = -\frac{i}{\hbar} V_I(t) \mathcal{U}_I(t,t')$$

$$\mathcal{U}_I(t,t') = I - \frac{i}{\hbar} \int_{t'}^{t} V_I(t'') \mathcal{U}_I(t'',t')\, dt''$$

The solution is the Dyson

series

$$\mathcal{U}_I(t,t') = I + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \int_{t'}^{t} \times$$

$$T(V_I(t_1) \cdots V_I(t_n))\, dt_1 \, dt_n$$

taking the limits

$$S = \mathcal{U}(\infty, -\infty) =$$

$$I + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right) \int_{-\infty}^{\infty} T(V_I(t_1) \cdots V_I(t_n))\, dt_1 \cdots dt_n$$

unlike the case of time dependent perturbation when the time interval was finite, this does not generally converge when $t', t \to (-\infty, \infty)$

It is instructive to consider the first Born approximation

$$-\frac{i}{\hbar}\langle \bar{p}\bar{k}|V_I(t)|\bar{p}'\bar{k}'\rangle =$$

$$-\frac{i}{\hbar}\int_{-\infty}^{\infty} e^{i\left(\frac{p^2}{2m}+\frac{k^2}{2m}\right)\frac{t'}{\hbar}}\langle k|V|k'\rangle e^{-i\left(\frac{p'^2}{2m}+\frac{k'^2}{2m}\right)\frac{t'}{\hbar}}\,dt' \times$$

$$\delta(p-p') =$$

$$-\frac{i}{\hbar}\delta(p-p')\langle k|V|k'\rangle 2\pi\hbar\,\delta\left(\frac{k'^2}{2m}-\frac{k''^2}{2m}\right) =$$

$$-2\pi i\,\delta(p-\bar{p}')\,\delta(E-E')\langle \bar{k}|V|\bar{k}'\rangle$$

which is exactly what was obtained using the first Born approximation in the Lippmann Schwinger equation

* The Lippmann Schwinger equation can be used to treat cases where the Dyson expansion fails

* The Dyson expansion is used for perturbative scattering in quantum field theory. The contributions can be expressed in terms of "Feynman diagrams"

* Our treatment of the Coulomb problem in the Born approximation is not justified because $S2I$ as defined does not exist for the Coulomb problem — but the result is the exact answer.

* it is not hard to see that the total Coulomb cross section is infinite due to the $\sin^4(\frac{\theta}{2})$ in the denominator which vanishes in the forward direction.