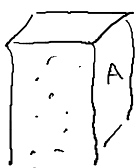
Cross sections

What is normally measured in a scattering experiment is

1. # particles / volume in the target n_T

2. # beam particles incident on target per unit time - area



vt

$$\frac{n_b \times vt \times A}{tA} = n_b \vec{v}$$

3. # detected per unit time

The number detected / time

$$\frac{dN}{dt} = n_T n_b v_{bT} \, d\sigma$$

where $d\sigma$ has units of area - it is called the differential cross section

Last time we used

$$\rho_b = \sum_{n=1}^{N} |\Phi^0_{bn}\rangle P_{nb} \langle \Phi^0_{bn}|$$

$$\rho_T = \sum_{m=1}^{m} |\Phi^0_{Tm}\rangle P_{mt} \langle \Phi^0_{Tm}|$$

these are ensembles describing the target and beam

The probability of scattering withing d^3p_1 of p_1 and d^3p_2 of p_2 is given by

$$dP = \sum_{mn} \left| \langle \bar{p}_1 \bar{p}_2 | (S - I) | \varphi_{bn} \varphi_{tn} \rangle \right|^2 \times$$

$$P_{bn} P_{Tn} \, d^3 p_1 \, d^3 p_2$$

This sums over all possible scattering of beam and target particles

In general

$$\langle p_1 p_2 | S | p_1' p_2' \rangle = \langle p_1 p_2 | p_1' p_2' \rangle$$

$$- 2\pi i \, \delta(E - E') \langle p_1 p_2 | T(E + i\epsilon) | p_1' p_2' \rangle$$

We assumed V was translationally invariant $\Rightarrow$

$$\langle \bar{p}_1 \bar{p}_2 | T(E + i\epsilon) | p_1' p_2' \rangle =$$

$$\delta(\bar{p}_1 + \bar{p}_2 - \bar{p}_1' - \bar{p}_2') \langle \bar{p}_1 \bar{p}_2 \| T(E + i\epsilon) \| p_1'' p_2'' \rangle$$

in this case

$$\langle p_1 p_2 | (S-1) | p_1' p_2' \rangle =$$

$$-2\pi i \, \delta(E-E') \, \delta(\vec{P}-\vec{P}') \times$$

$$\langle \vec{p}_1 \vec{p}_2 \| T(E+i\epsilon) \| \vec{p}_1' \vec{p}_2' \rangle$$

Last time we used this in the expression for dP on the last page and

we assumed that the wave packets in the beam and target ensembles were sharply peak near

$$\langle E \rangle = \mathrm{Tr}(\rho_b \rho_T H_0)$$
$$\langle \vec{P} \rangle = \mathrm{Tr}(\rho_b \vec{P}_b) + \mathrm{Tr}(\rho_T \vec{P}_T)$$

the mean energy and momentum of the

initial ensemble that we could make the approximations

$$\delta(E-E')\delta(E-E'') =$$

$$\delta(\langle E\rangle-E')\delta(E'-E'')$$

$$\delta(\bar{p}-\bar{p}')\,\delta(\bar{p}-\bar{p}'') =$$

$$\delta(\langle\bar{p}\rangle-\bar{p}')\,\delta(p'-p'')$$

$$\langle\bar{p}_1\bar{p}_2|T(E+i\epsilon)|\bar{p}_1'\bar{p}_2'\rangle \simeq$$

$$\langle\bar{p}_1\bar{p}_2|T(\langle E\rangle+i\epsilon)|\langle\bar{p}_b\rangle\langle\bar{p}_t\rangle\rangle$$

with these approximations the integrals over intermediate states only involve the wave packets in the beam and target ensembles

* we also expressed

$\delta(E'-E'')\,\delta(\bar{p}'-\bar{p}'')$ using

their Fourier representation

The result (without repeating

all of the algebra) was

$$dP = (2\pi)^4 \hbar^2 \,\delta(E - \langle E\rangle)\,\delta(\bar{p} - \langle\bar{p}\rangle)\,\times$$

$$|\langle \bar{p}_1 \bar{p}_2 | T(\langle E\rangle + i\epsilon) | \langle\bar{p}_1\rangle \langle\bar{p}_2\rangle \rangle|^2$$

$$d^3p_1\, d^3p_2 \sum_{mn} P_{nb} P_{mT} \,|\phi_{nb}(\bar{x}t)|^2 \times$$

$$|\phi_{mT}(\bar{x}t)|^2 \; d\bar{x}\, dt$$

This quantity gets a

contribution whenever

a given target and

beam particle are at the

same position at the

same time

If we multiply this by the total numbers of beam and target particles we get the number of particles scattered

$$dN = \int (\;) \, d^3x \, dt$$

scattered / volume - time

$$\frac{dN}{dV\,dt} = (2\pi)^4 \hbar^2 \left| \langle p_1 p_2 || T(\langle E \rangle + i\epsilon) || \langle p_b \rangle \langle p_T \rangle \rangle \right|^2$$

$$N_b |\phi_{b_n}(\bar{x}t)|^2 \; N_T |\phi_{T_m}(x_T t)|^2 \; P_{bn} P_{Tm}$$

$$\delta(E - \langle E \rangle)\, \delta(\bar{p} - \langle \bar{p} \rangle) \; d^3p_1 \, d^3p_2$$

here $\sum P_{bn} N_b |\phi^0_{bn}(xt)|^2 = n_b(xt)$

$\sum P_{Tm} N_T |\phi^0_{mT}(xt)|^2 = n_T(xt)$

$$= n_b \, n_T \, v_{TB} \, d\sigma$$

comparing both sides gives

$$\boxed{d\sigma = \frac{(2\pi)^4 \hbar^2}{V_{bT}} |\langle p_1 p_2 | T(\langle E\rangle + i\epsilon) | \langle \bar{p}_b\rangle \langle \bar{p}_T\rangle\rangle|^2 \delta(E - \langle E\rangle)\, \delta(\bar{P} - \langle \bar{P}\rangle)\, d^3p_1\, d^3p_2}$$

normally we count all events that conserve energy and momentum - this means that we only need to measure 2 of the six variables $\bar{p}_1$ and $\bar{p}_2$, the remaining variables are used to integrate out the δ functions example

center of mass cross section

$$d^3p_1\, d^3p_2 = d^3P\, d^3k$$

$$k = p_1 - m_1 \frac{p_1+p_2}{m_1+m_2} = \frac{m_2 p_1 - m_1 p_2}{m_1+m_2}$$

k = momentum of particle 1 in the rest frame of the system

$$\int d^3p_1\, d^3p_2\, \delta(\bar{P}-\bar{P}')\, \delta(E-E') =$$

$$\int d^3P\, d^3k\, \delta(P-P')\, \delta(E-E') =$$

$$\int d^3k\, \delta(E-E') =$$

$$\int d\Omega(\hat{k})\, k^2 \frac{dk}{dE}\, dE\, \delta(E-E')$$

$$d\Omega(\hat{k})\, k^2 \frac{dk}{dE}$$

here we assume that only the center of mass scattering angles are measured. Note

$$E = \frac{P^2}{2M} + \frac{k^2}{2\mu}$$

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\frac{dE}{dk} = \frac{2k}{2\mu} = \frac{k}{\mu} \qquad \frac{dk}{dE} = \frac{\mu}{k}$$

$$k^2 \frac{dE}{dk} = k\mu$$

We also have

$$\bar{v} = \frac{\bar{k}}{m_1} - \frac{(-\bar{k})}{m_2} = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{k}{\mu}$$

$$d\sigma = \frac{(2\pi)^4 \hbar}{(k/\mu)} |\langle \bar{P}=0\ \bar{k}' \| T(E+i\epsilon) \| \bar{P}=0\ \bar{k} \rangle|^2 \, k\mu \, d\Omega(k)$$

$$\boxed{d\sigma = (2\pi)^4 \hbar^2 \mu^2 |\langle \bar{P}\bar{k}' \| T(E+i\epsilon) \| \bar{P}\bar{k} \rangle|_{\bar{P}=0}^2 \sin\theta_k \, d\theta_k \, d\phi}$$

This is usually expressed as

$$\boxed{\frac{d\sigma}{d\Omega(k)} = ((2\pi)^2 \hbar\mu)^2 |\langle \bar{P}\bar{k}' \| T(E+i\epsilon) \| \bar{P}\bar{k} \rangle|_{\bar{P}=0}^2}$$

the quantity

$$\boxed{F(\bar{k}', \bar{k}) = -(2\pi)^2 \hbar \mu \langle \bar{k}' 0 | T(E+i\epsilon) | \bar{k} 0 \rangle}$$

is called the scattering amplitude

$$\boxed{\frac{d\sigma}{d\Omega} = |F(\bar{k}\bar{k}')|^2}$$

The total cross section is

$$\boxed{\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega}$$

Homework - Laboratory cross section

$$E = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2}$$

$$\bar{p} = \bar{P}_1 + \bar{P}_2$$

$$\int d^3p_1 d^3p_2 \, \delta(P - P') \, \delta(E - E')$$

assume we measure the angles of $\bar{P}_1' \Rightarrow$

$$\int d^3p_1' \, d^3p_2' \, \delta(\bar{P}-\bar{P}') \, \delta(E-E') =$$

$$\int d^3p_1' \, \delta(E-E') = \int p_1'^2 \frac{dp_1'}{dE'} \, d\Omega(\hat{p}_1') \, \delta(E-E')$$

$$= p_1^2 \frac{dp_1}{dE}$$

$$V = \frac{p_1}{m_1} - \frac{0}{m_2} = \frac{p_1}{m_1}$$

$$d\sigma_L = (2\pi)^4 \hbar^2 \| \langle p_1' p_2' \| T(E+i\epsilon) \| p_1 p_2 \rangle \|$$

$$\times \frac{m_1}{p_1} p_1^2 \frac{dp_1}{dE} \, d\Omega(\hat{p}_1)$$

HW compute $m_1 p_1 \frac{dp_1}{dE}$ as a function of P_1, E, $\hat{p}_1' \cdot \hat{P}$

(recall $E = \frac{(\bar{P}-\bar{P}_1)^2}{2m_2} + \frac{p_1^2}{2m_1}$)

The problem of scattering theory is to compute $d\sigma$ which is equivalent to computing

$$\langle p_1' p_2' \| T(E+i\epsilon) \| p_1 p_2 \rangle$$

where

$$\boxed{T(E+i\epsilon) = V + V(E+i\epsilon-H)^{-1}V}$$

the difficult part is

computing $(E+i\epsilon-H)^{-1}$

It satisfies

$$(E+i\epsilon-H)^{-1} =$$

$$(E+i\epsilon-H_0)^{-1} + (E+i\epsilon-H_0)V(E+i\epsilon-H)^{-1}$$

using this in the equation

at the top of the page

gives

$$T(E+i\epsilon) = V +$$

$$V\left[(E+i\epsilon-H_0)^{-1} + (E+i\epsilon-H_0)V(E+i\epsilon-H)^{-1}\right]V =$$

$$V + V(E+i\epsilon-H_0)^{-1}\left[V + V(E+i\epsilon-H)^{-1}V\right]$$

$$V + V(E+i\epsilon-H_0)^{-1}T(E+i\epsilon)$$

the equation

$$\boxed{T(z) = V + V(z-H_0)^{-1}T(z)}$$

for $z = E + i\epsilon$

is an integral equation for $T(E+i\epsilon)$.

Note

$$(E+i\epsilon-H_0)^{-1} = (\underbrace{\frac{P^2}{2M}+\frac{k^2}{2\mu}}_{\text{numbers}}+i\epsilon-\underbrace{\frac{\bar{P}^2}{2M}-\frac{k^2}{2\mu}}_{\text{operators}})^{-1}$$

since $\bar{P}$ is conserved it cancels

$$\langle \bar{k}' \| T(\tfrac{k'^2}{2\mu}+i\epsilon) \| k' \rangle = \langle \bar{k}' | V | \bar{k}' \rangle + \int \langle \bar{k}' \| V | k'' \rangle \frac{d^3k''}{k'^2/2\mu - k''^2/2\mu + i\epsilon} \langle k'' \| T(\tfrac{k'^2}{2\mu}+i\epsilon) \| \bar{k}' \rangle$$

this is the actual integral equation

It can be solved using a basis

$$\langle m \| T(z) \| \bar{k} \rangle = \langle m | V | \bar{k} \rangle +$$

$$\sum_n \langle m | V (k^2/2\mu + i\epsilon - h_0)^{-1} | n \rangle$$

$$\langle n | T(z) \| k \rangle$$

where $h_0 = H_0 - \frac{P^2}{2M}$. Once this is solved

$$\langle \bar{k}' \| T(z) \| k \rangle = \langle \bar{k}' | V | k \rangle +$$

$$\sum_n \langle \bar{k}' | V (z - h_0)^{-1} | n \rangle \langle n \| T(z) \| \bar{k} \rangle$$

$$z = k^2/2\mu + i\epsilon$$

In many cases $V(z-h_0)^{-1}$ is a compact operator which means

$$V(z-h_0)^{-1} = F + \Delta$$

where F is a finite dimensional matrix and Δ is small

$$F = \sum_{mn=1}^{N} |m\rangle f_{mn} \langle n|$$

$$T = V + (F + \Delta) T \rightarrow$$

$$(1 - F) T = V + \Delta T$$

$$T = (1-F)^{-1} V + \underbrace{(1-F^{-1}) \Delta \cdot T}_{\text{small - can be solved by iteration.}}$$

when $\| V (z - H_0)^{-1} \| < 1$ the Lippmann Schwinger equation can be solved by iteration

$$\boxed{T(z) = V + \sum_{n=1}^{\infty} \left(V (z - H_0)^{-1} \right)^n V}$$

This series is called the Born series. The first approximation

$$T(z) \simeq V$$

is called the born approximation

$$(d\sigma)_{born} = (2\pi)^4 \hbar^2 |\langle k' | V | k\rangle|^2$$

where

$$\langle \bar{k}' | V | \bar{k}\rangle = \int \langle \bar{k}' | \bar{r}\rangle \, d^3r \, V(r) \langle \bar{r} | \bar{k}\rangle$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-i\bar{k}' \cdot \bar{r}/\hbar} \, V(r) \frac{1}{(2\pi\hbar)^{3/2}} e^{i\bar{k} \cdot \bar{r}/\hbar}$$

$$= \frac{1}{(2\pi\hbar)^3} \int V(\bar{r}) e^{i(\bar{k} - \bar{k}') \cdot \bar{r}/\hbar}$$

$$= \tilde{V}(\bar{k} - \bar{k}')$$

which is the Fourier transform of the potential

The transition operator is als closely related to the scattered wave function

$$|\Psi_\pm\rangle =$$

$$\lim_{t\to\pm\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\Psi_\pm^0\rangle =$$

$$|\Psi_\pm^0\rangle + \lim_{t\to\pm\infty} \int_0^t \frac{d}{dt} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\Psi_\pm^0\rangle$$

$$|\Psi_\pm^0\rangle + \lim_{t\to\pm\infty} \int_0^t \frac{i}{\hbar} e^{iHt/\hbar} V e^{-iH_0t/\hbar} |\Psi_\pm(0)\rangle =$$

$$\int \left[|p_1 p_2\rangle + \lim_{t\to\pm\infty} \int_0^t \frac{i}{\hbar} e^{iHt/\hbar} V e^{-iEt/\hbar} |p_1 p_2\rangle \right] d^3p_1\, d^3p_2\, \langle p_1 p_2 | \Psi_\pm^0(0)\rangle$$

as before - this only makes sense if the $\bar{p}_1 \bar{p}_2$ integrals are done before the t integral - but if we do the $p_1 p_2$ integral first we can insert $e^{\mp \epsilon t}$.

with this factor included

we can change the order

of the t and $p_1 p_2$ integrals

$$\boxed{|\Psi_\pm(0)\rangle = \int |(p_1 p_2)^\pm\rangle \, d^3p_1 \, d^3p_2 \, \langle p_1 p_2 | \Psi^0_\pm(0)\rangle}$$

where

$$|(p_1 p_2)^\pm\rangle = |\bar{p}_1 \bar{p}_2\rangle +$$

$$\lim_{\epsilon \to 0} \frac{i}{\hbar} \int_0^{\mp\infty} e^{i(H - E \pm i\hbar\epsilon)t/\hbar} \, V \, |p_1 p_2\rangle \, dt =$$

$$|\bar{p}_1 \bar{p}_2\rangle + \frac{i}{\hbar}(-1)\frac{\hbar}{i}\frac{1}{(H - E \pm i\epsilon\hbar)} V |p_1 p_2\rangle =$$

$$|p_1 p_2\rangle + \frac{1}{E \mp i\epsilon\hbar - H} V \, |p_1 p_2\rangle$$

$$\therefore \quad \boxed{|(p_1 p_2)^\pm\rangle = |\bar{p}_1 \bar{p}_2\rangle + (E \mp i\epsilon - H)^{-1} V |\bar{p}_1 p_2\rangle}$$

using the second resolvent identities gives

$$|(p_1 p_2)^{\pm}\rangle = |\bar{p}_1 \bar{p}_2\rangle +$$

$$\left((E \mp i\epsilon - H_0)^{-1} + (E \mp i\epsilon - H_0)^{-1} V (E \mp i\epsilon - H)^{-1} \right) V |p_1 p_2\rangle$$

$$\simeq |\bar{p}_1 \bar{p}_2\rangle + (E \mp i\epsilon - H_0)^{-1} V \left(|\bar{p}_1 \bar{p}_2\rangle + (E \mp i\epsilon - H)^{-1} V |p_1 p_2\rangle \right.$$

$$= |p_1 p_2\rangle + (E \mp i\epsilon - H_0)^{-1} V |(p_1 p_2)^{\pm}\rangle$$

This is an integral equation for $|(\bar{p}_1 \bar{p}_2)^{\pm}\rangle$ - it is also called the Lippmann Schwinger equation

$$\boxed{|(p_1 p_2)^{\pm}\rangle = |p_1 p_2\rangle + (E \mp i\epsilon - H_0)^{-1} V |(p_1 p_2)^{\pm}\rangle}$$

The equation at the top of the page can also be expressed as

$$|(p_1 p_2)^{\pm}\rangle = |\bar{p}_1 \bar{p}_2\rangle + (E \mp i\epsilon - H_0)^{-1} [V + V (E \mp i\epsilon - H)^{-1} V \ldots$$

or

$$\boxed{|(p_1 p_2)^{\pm}\rangle = |(\bar{p}_1 \bar{p}_2)\rangle + (E \mp i\epsilon - H_0)^{-1} T(E \mp i\epsilon) |p_1 p_2\rangle}$$

which shows that knowing $T(E \pm i\epsilon)$ can be used to construct $|(p_1 p_2)^{\pm}\rangle$

Note

$$\langle p_1 p_2 | T(E \pm i\epsilon) | p_1 p_2 \rangle =$$

$$\langle p_1 p_2 | V (1 + (E \pm i\epsilon - H_0)^{-1} T(z)) | p_1 p_2 \rangle =$$

$$\langle p_1 p_2 | V | (p_1 p_2)^{\mp} \rangle$$

note the $\pm$ on $i\epsilon$ have the reverse sign of $|(p_1 p_2)^{\pm}\rangle$

$$\boxed{\begin{aligned} &\langle \bar{p}_1' \bar{p}_2' | T(E \pm i\epsilon) | \bar{p}_1 \bar{p}_2 \rangle = \\ &\langle \bar{p}_1' \bar{p}_2' | V | (\bar{p}_1 \bar{p}_2)^{\mp} \rangle = \\ &\langle (p_1' p_2')^{\pm} | V | p_1 p_2 \rangle \end{aligned}}$$

the last equation follows by taking the adjoint of the first equation and changing the sign of ϵ