Last time

$$S = \Omega_+^\dagger \Omega_-$$

we used

$$\langle \Psi_+^0(0) | S | \Psi_-^0(0) \rangle =$$

$$\lim_{t \to \infty} \langle \Psi_+^0(0) | e^{iH_0 t/\hbar} e^{-2iHt/\hbar} e^{iH_0 t/\hbar} | \Psi_-^0(0) \rangle$$

after quite a bit of algebra we were able to show

$$\langle \Psi_+^0(0) | S | \Psi_-^0(0) \rangle = \langle \Psi_+^0(0) | \Psi_-^0(0) \rangle +$$

$$\lim_{\epsilon \to 0} \int \langle \Psi_+^0(0) | \bar{P}_1 \bar{P}_2 \rangle d^3P_1 d^3P_2 \langle \bar{P}_1 \bar{P}_2 | \times$$

$$\langle \bar{P}_1 \bar{P}_2 | V + V(\bar{E} - H + i\epsilon)V | \bar{P}_1' \bar{P}_2' \rangle \times$$

$$\boxed{\frac{1}{2}\left(\frac{1}{\bar{E} - \bar{E}' + i\epsilon} + \frac{1}{\bar{E} - \bar{E} + i\epsilon} \right)} \times$$

$$d^3P_1' d^3P_2' \langle \bar{P}_1' \bar{P}_2' | \Psi_-^0(0) \rangle$$

the quantity in the red box is

$$\frac{1}{2}\left(\frac{1}{\frac{E+E'}{2} - E' + i\epsilon} - \frac{1}{\frac{E+E'}{2} - E + i\epsilon} \right) =$$

$$= \frac{1}{2}\left[\frac{2}{E-E'+2i\epsilon} + \frac{2}{E'-E+2i\epsilon}\right] =$$

$$= \frac{1}{E-E'+2i\epsilon} - \frac{1}{E-E'-2i\epsilon} =$$

$$= \boxed{\frac{-4i\epsilon}{(E-E')^2+4\epsilon^2}}$$

For Homework note that

$$\boxed{\lim_{\epsilon\to 0}\int \frac{-4i\epsilon}{(E-E')^2+4\epsilon^2} f(E')\,dE' = -2\pi i f(E)}$$

all that matters is that E is between the limits of integration. This is equivalent to

$$\boxed{\lim_{\epsilon\to 0} \frac{-4i\epsilon}{(E-E')^2+4\epsilon^2} = -2\pi i\,\delta(E-E')}$$

putting everything together gives

$$\boxed{\begin{aligned}\langle \bar{p}_1 \bar{p}_2 | S | \bar{p}_1' \bar{p}_2' \rangle &= \delta(\bar{p}_1 - \bar{p}_1')\,\delta(\bar{p}_2 - \bar{p}_2') \\ &\quad - 2\pi i\, \delta(E - E')\, \langle \bar{p}_1 \bar{p}_2 | V + V (E - H + i\epsilon)^{-1} V | \bar{p}_1' \bar{p}_2' \rangle\end{aligned}}$$

Where it is understood that the limit $\epsilon \to 0$ must be taken after integrating against wave packets

The physics is in the operator $T(E + i\epsilon)$ which depends on V

$$\boxed{T(z) = V + V (z - H)^{-1} V}$$

$T(z)$ is called the transition operator. This is the solved form — H must first be diagonalized to construct $(z - H)^{-1}$.

T(z) can be computed by solving an integral equation using (HW)

$$(z-H)^{-1} = (z-H_0)^{-1} + (z-H_0)^{-1} V (z-H)^{-1}$$

$$T(z) = V + V(z-H)^{-1}V =$$

$$V + V\left((z-H_0)^{-1} + (z-H_0)^{-1} V (z-H)^{-1}\right)V =$$

$$V + V(z-H_0)^{-1}\left(V + V(z-H)^{-1}V\right) =$$

$$V + V(z-H_0)^{-1}T(z), \qquad (z = E+i\epsilon)$$

$$\boxed{T(E+i\epsilon) = V - V(E+i\epsilon-H_0)^{-1}T(E+i\epsilon)}$$

This equation is called the Lippmann Schwinger equation

This is one of the main equations for scattering.

There is a corresponding equation for the wave function

$$|\Psi_\pm(0)\rangle = \Omega_\pm |\Psi^0_\pm(0)\rangle$$

We can write this as

$$|\Psi_\pm(0)\rangle = \int \Omega_\pm |\bar{P}_1 \bar{P}_2\rangle d^3p_1 d^3p_2 \langle \bar{P}_1 \bar{P}_2 |\Psi^0_\pm(0)\rangle$$

$$\equiv |(\bar{P}_1 \bar{P}_2)_\pm\rangle d^3p_1 d^3p_2 \langle \bar{P}_1 \bar{P}_2 |\Psi^0_\pm(0)\rangle$$

To find

$$|(\bar{P}_1, \bar{P}_2)_\pm\rangle = \Omega_\pm |\bar{P}_1 \bar{P}_2\rangle =$$

$$\lim_{t\to\mp\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\bar{P}_1 \bar{P}_2\rangle =$$

$$|\bar{P}_1 \bar{P}_2\rangle + \lim_{t\to\infty} \int_0^t \frac{d}{dt'}\left(e^{iHt'/\hbar} e^{-iH_0t'/\hbar}\right)|\bar{P}_1 \bar{P}_2\rangle dt'$$

$$=$$

$$|\bar{P}_1\bar{P}_2\rangle + \frac{i}{\hbar}\lim_{t\to\pm\infty}\int_0^t e^{iH t'/\hbar} V e^{-iH_0 t'/\hbar}|P_1P_2\rangle dt$$

$$= |P_1P_2\rangle + \frac{i}{\hbar}\lim_{t\to\pm\infty}\int_0^t e^{i(H-E)t/\hbar} V |P_1P_2\rangle dt$$

As in the case of $T(z)$ the time integral does not converge as $t \to \pm\infty$. without first integrating over a wave packet. We can insert $\lim_{\epsilon\to 0} e^{\mp\epsilon t}$ if we do the P_1P_2 integrals first. If we include this factor we can change the order of the integration and do the t integral first

$$= |P_1P_2\rangle + \left(\frac{i}{\hbar}\right)(-1)\left(\frac{\hbar}{i}\right)(H-E\mp i\hbar\epsilon)^{-1}V|P_1P_2\rangle$$

$$= \left(I + \lim_{\epsilon\to 0}(E-H\pm i\hbar\epsilon)^{-1}V\right)|P_1P_2\rangle$$

$$|(p_1 p_2)_\pm\rangle = \left(I + (E - H \pm i\hbar\epsilon)^{-1} V\right) |p_1 p_2\rangle$$

and

$$|\Psi_\pm(0)\rangle = \int |(p_1 p_2)_\pm\rangle \, d^3p_1 \, d^3p_2 \, \langle p_1 p_2 | \Psi^0_\pm(0)\rangle$$

we can use the second resolvent identity again to write

$$|(p_1 p_2)_\pm\rangle = |(p_1 p_2)\rangle + (E - H_0 \pm i\hbar\epsilon)^{-1}\left(1 + V(E - H \pm i\epsilon\hbar)^{-1}\right) V |p_1 p_2\rangle$$

$$= |(p_1 p_2)\rangle + (E - H_0 \pm i\hbar\epsilon)^{-1} V \underbrace{\left(1 + (E - H \pm i\epsilon\hbar)^{-1} V\right) \times |p_1 p_2\rangle}_{|(p_1 p_2)_\pm\rangle}$$

$$\boxed{\begin{aligned} &= |(p_1 p_2)\rangle + (E - H_0 \pm i\epsilon)^{-1} V |(p_1 p_2)_\pm\rangle \\ &= |(p_1 p_2)\rangle + (E - H_0 \pm i\epsilon)^{-1} T(E \pm i\hbar\epsilon) |(p_1 p_2)\rangle \end{aligned}}$$

The first of these is an equation for $|(p_1 p_2)_\pm\rangle$

$$|(p_1 p_2)_\pm\rangle = |(p_1 p_2)\rangle + (E - H_0 \pm i\epsilon\hbar)^{-1} V \, |(p_1 p_2)_\pm\rangle$$

This is the Lippmann-Schwinger equation for $|(p_1 p_2)_\pm\rangle$. The second equation relates this solution to the transition operator

$$|(p_1 p_2)_\pm\rangle = |(p_1 p_2)\rangle + (E - H_0 \pm i\hbar\epsilon)^{-1} V \, T(E - H_0 \pm i\hbar\epsilon) \, |(p_1 p_2)\rangle$$

thus in order to calculate the scattering probability it is necessary to calculate

$|(p_1 p_2)_\pm\rangle$ or $T(E \pm i\hbar\epsilon)$

by solving the Lippmann Schwinger equation.

remarks

① Let $\bar{P} = \bar{p}_1 + \bar{p}_2$

$$k = p_1 - \frac{p_1 + p_2}{M} m_1 = \frac{(M - m_1) p_1 - m_1 p_2}{M}$$

$$= \frac{m_2 \bar{p}_1 - m_1 \bar{p}_2}{M} \Longrightarrow$$

$$|p_1 p_2\rangle = |\bar{P}, \bar{k}\rangle$$

$\bar{k}$ = momentum of particle 1 in rest frame of system

For $[\bar{P}, V] = 0$ we have

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\Omega_\pm = \lim_{t \to \pm\infty} e^{iHt/\hbar} e^{-iH_0 t/\hbar} =$$

$$\lim_{t \to \pm\infty} e^{i\left(\frac{\bar{P}^2}{2M} + \frac{k^2}{2\mu} + V\right)\frac{t}{\hbar}} e^{-i\left(\frac{P^2}{2M} + \frac{k^2}{2\mu}\right)\frac{t}{\hbar}}$$

$$\lim_{t \to \pm\infty} e^{i\left(\frac{k^2}{2\mu} + V\right)\frac{t}{\hbar}} e^{-i\frac{k^2}{2\mu}\frac{t}{\hbar}}$$

This means we can replace

$$H \to h = \frac{k^2}{2\mu} + V$$

$$H_0 \to h_0 = \frac{k^2}{2\mu}$$

$$|\bar{k}_\pm\rangle = |\bar{k}\rangle + \frac{1}{\varepsilon - h \pm i\hbar\epsilon} V |k_\pm\rangle$$

$$\varepsilon = E - \frac{P^2}{2M}$$

$$|(p_1 p_2)_\pm\rangle = |\underline{P}\, k_\pm\rangle$$

The advantage of eliminating $\bar{P}$ is that the Lippmann Schwinger equation becomes an equation in 3 variables rather than 6 variables

② connection with the interaction picture

$$\mathcal{U}_s(t-t') = e^{-iH(t-t')/\hbar}$$

$$\mathcal{U}_I(t-t') = e^{iH_0 t/\hbar}\,\mathcal{U}_s(t-t')\,e^{-iH_0 t'/\hbar}$$

$$\frac{d\mathcal{U}_s}{dt} = -\frac{i}{\hbar}\,\mathcal{U}_s$$

$$\frac{d\mathcal{U}_I}{dt} = \frac{i}{\hbar}H_0\,\mathcal{U}_I(t-t') + e^{iH_0 t/\hbar}\left(-\frac{i}{\hbar}H\right)\mathcal{U}_s(t-t') \times e^{iH_0 t'/\hbar}$$

$$= -\frac{i}{\hbar}\,e^{iH_0 t/\hbar}\,V_s\,e^{-iH_0 t/\hbar}\,\mathcal{U}_I(t-t')$$

$$= -\frac{i}{\hbar}\,V_I(t)\,\mathcal{U}_I(t-t')$$

but

$$S = \lim_{\substack{t\to\infty \\ t'\to-\infty}} \mathcal{U}_I(t-t') = I$$

$$\sum \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} dt_1 \ldots dt_n\, T\left(V_I(t_1)\cdots V_I(t_n)\right)$$

In this case because $t : -\infty \to \infty$ convergence is no longer ensured.

③ Solving the Lippmann Schwinger equation

$$T = V + V(E - H_0 + i\epsilon)^{-1} T$$
$$= V + V\,R(E+i\epsilon)\,T$$

In most case for the 2 body lippmann schwinger equation the operator $VR(E+i\epsilon)$ is "compact".

What this means is that

$$VR(E+i\epsilon) = F_N + \Delta$$

where F_N is finite dimensional

$$F_N = \sum_{mn=1}^{k} |m\rangle f_{mn} \langle n|$$

$$|||\Delta||| < 1$$

Then we have

$$T(z) = V + (F_N + \Delta) T(z)$$

$$(1 - F_N) T(z) = V + \Delta T(z)$$

$$T(z) = \underbrace{(1-F_N)^{-1}}_{\text{inversion of an } N\times N \text{ matrix}} V + \underbrace{(1-F_N)^{-1}\Delta}_{\text{small}} T(z)$$

$$= D + K T(z)$$

$$(1 - K) T(z) = D$$

$$\boxed{T(z) = (I + K + K^2 + K^3 \cdots) D}$$

which converges for $|||K||| < 1$

What this means in practice

$$V = \sum_{mn} |m\rangle\langle m|V|n\rangle\langle n|$$

$$T(z) = \sum_{mn} |m\rangle\langle m|T(z)|n\rangle\langle n|$$

$$\langle m|T(z) = \langle m|V + \sum_{n} \langle m|V|n\rangle\langle n|R|k\rangle\langle k|T$$

$$\sum' \left(\delta_{mn} - \langle m | V R | n \rangle\right) \langle n | T = \langle n | V$$

$$| n \rangle T = \sum_m (I - VR)^{-1}_{nm} \langle m | V$$

given $| n \rangle T$

$$\boxed{T(z) = V + \sum_m VR | m \rangle \langle m | T}$$

In principle the sum is infinite but if VR is compact we can ignore all but a finite number of basis functions

$\times\checkmark$

$$P = |\langle \Psi_+(t) | \Psi_-(t) \rangle|^2 =$$

$$|\langle \Psi^0_+(t) | S | \Psi^0_-(t) \rangle|^2$$

$$\langle p_1 p_2 | S | p_1' p_2' \rangle =$$

$$\langle p_1 p_2 | p_1' p_2' \rangle - 2\pi i \, \delta(E - E') \times$$

$$\langle p_1 p_2 | T(E + i\epsilon) | p_1' p_2' \rangle$$

$$\boxed{T(E + i\epsilon) = V + V(E + i\epsilon - H_0)^{-1} T(E + i\epsilon)}$$

This means solving for $T(E+i\epsilon)$ gives the probability

The same $T(E+i\epsilon)$ can be used for different initial and final wave packets

differential cross sections

* in a real experiment we dont know all of the details about the initial and final wave packets

* The beam can be describe by a density matrix

$$\rho_B = \sum_n |\underset{\sim}{\psi}^{o}_{n_n}(t)\rangle P_{Bn} \langle \underset{\sim}{\psi}^{o}_{n_n}(t)|$$

where P_{bn} is the classical probability that the beam has free beam particle in state $|\psi^{0}_{n-}(t)\rangle$

The mean beam momentum is

$$\langle \bar{P} \rangle_b = Tr(\bar{P} \rho_b)$$

the mean spin projection of the beam particle is

$$\langle S_z \rangle = Tr(S_z \rho_b)$$

We can do the same with the ensemble of target particles

$$\rho_T = \sum |\psi^{0}_{Tn}(t)\rangle P_{Tn} \langle \psi^{0}_{Tn}(t)|$$

$$P = \sum \langle \psi^{0}_{i} | S \rho_B \rho_T S^{+} | \psi^{0}_{i} \rangle \rightarrow$$

$$\sum \langle \psi_i | (S-1) \rho_B \rho_T (S-1)^{+} | \psi_i \rangle$$

Here we replace S by $S-1$ which corresponds to the particles that scatter — there should be essentially no unscattered particles in the detectors — except along the beam line

If we want the differential probability that particle 1 is detected to have momentum within d^3p_1 of $\bar{p}_1$ and particle 2 is detected to have momentum within d^3p_2 of $\bar{p}_2$ then it becomes

$$dP = \langle p_1 p_2 | (S-1) \rho_B \rho_T (S-1)^{+} | p_1 p_2 \rangle \, d^3p_1 \, d^3p_2$$

=

$$= \sum_{mn} \left| \langle \bar{P}_1 \bar{P}_2 | (S - I) | \phi^0_{n\underline{b}} \phi^0_{m\underline{T}} \rangle \right|^2 P_{nb} P_{mT}$$

expand the beam and target wave functi. in the ensembl.

$$dP = \sum_n \left| \int \langle \bar{P}_1 \bar{P}_2 | (S - 1) | \bar{P}_1' \bar{P}_2' \rangle \, \phi_{nb}(P_1') \, \phi_{m+}(P_2') \, d^3P_1' d^3P_2' \right|^2$$

$$P_{nb} P_{m+} \times d^3P_1 d^3P_2$$

In what follows I assume that V commutes with $\bar{P}$

then

$$\langle P_1 P_2 | T | P_1 P_2 \rangle = \underbrace{\delta(\bar{P} - \bar{P}')}_{} \langle P_1 P_2 || T || P_1 P_2 \rangle$$

δ function in total momentum

Then

$$\langle P_1 P_2 | S - 1 | P_1' P_2' \rangle =$$

$$- 2\pi i \, \delta(E - E') \, \delta(\bar{P} - \bar{P}') \langle \bar{P}_1 P_2 || T || P_1' P_2' \rangle$$

with these the differential probability becomes

$$dP = \sum_{mn} P_{mb} P_{nT} \int d^3p_1' \, d^3p_2' \, d^3p_1'' \, d^3p_2'' \, |-2\pi i|^2$$

$$\delta(E-E')\,\delta(\bar{p}-\bar{p}')\,\delta(E-E'')\,\delta(\bar{p}-\bar{p}'')$$

$$\langle p_1 p_2 | T | p_1' p_2' \rangle \, \phi_{bm}(p_1') \, \phi_{Tn}(p_2')$$

$$\langle p_1'' p_2'' | T^{\dagger} | p_1 p_2 \rangle \, \phi^*_{bm}(p_1'') \, \phi^*_{Tn}(p_2'')$$

$$d^3p_1 \, d^3p_2$$

not integrated

next replace the δ function by

$$\delta(E-E')\,\delta(E-E'')\,\delta(\bar{p}-\bar{p}')\,\delta(\bar{p}-\bar{p}'') =$$

$$\delta(E-E')\,\delta(E'-E'')\,\delta(\bar{p}-\bar{p}')\,\delta(\bar{p}'-\bar{p}'') =$$

$$\delta(E-E')\,\delta(\bar{p}-\bar{p}') \times \frac{1}{(2\pi)^4} \int d\tau \, d^3x \, \frac{1}{\hbar^4} \, e^{-i(E'-E'')t/\hbar} \, e^{i(\bar{p}'-\bar{p}'')\cdot\bar{x}/\hbar}$$

so far everything is exact — we would like to remove the dependence of the structure of the wave packet

To d. the let $\langle \bar{p}_1 \rangle \langle \bar{p}_2 \rangle$ the ensemble averages of the beam and target particles

* replace

$$\langle p_1 p_2 \| T(E+i\epsilon) \| p_1' p_2' \rangle$$

by

$$\langle p_1 p_2 \| T(E+i\epsilon) \| \langle p_1 \rangle \langle p_2 \rangle \rangle$$

and

$$\delta(E-E_i)\,\delta(\bar{p}-\bar{p}_i) = \delta(E-\langle E \rangle)\,\delta(\bar{p}-\langle p \rangle)$$

* These replacements allows us to move these terms outside of the integral

* It assumes $\langle p_1 p_2 \| T \| p_1' p_2' \rangle$ varies slowly on the width of the wave packet

$$dP = 4\pi^2 \sum P_{nb} P_{nT} \, d^3p_1 \, d^3p_2$$

$$\delta(E - \langle E \rangle)\,\delta(\bar{p} - \langle \bar{p} \rangle)$$

$$\| \langle p_1 p_2 | T(E + i\epsilon) | \langle p_1 \rangle \langle p_2 \rangle \rangle \|^2 \frac{1}{(2\pi\hbar)^4} \int d^3x \, dt$$

$$\int \phi_n(p_1') e^{i(E_1' t - \bar{p}_1' \cdot \bar{x})/\hbar} d^3p_1'$$

$$\int \phi_n^*(p_1'') e^{-i(E_1'' t - i p_1'' \cdot \bar{x})/\hbar} d^3p_1''$$

$$\int \phi_b(p_2') e^{i(E_2' t - p_2' \cdot \bar{x})/\hbar} d^3p_2'$$

$$\int \phi_b^*(p_2'') e^{-i(E_2'' t - p_2'' \cdot y)/\hbar} d^3p_2''$$

$$= (2\pi)^4 \hbar^2 \, P_{nb} P_{nT} \, d^3p_1 \, d^3p_2$$

$$\delta(E - \langle E \rangle)\,\delta(\bar{p} - \langle p \rangle)\, |\langle p_1 p_2 | T(E + i\epsilon) | \langle p_1 \rangle \langle p_2 \rangle|^2$$

$$\int d^3x \, dt \, |\hat{\phi}_{nb}(\bar{x}t)|^2 |\phi_{mT}(xt)|^2$$

note that

$$\sum P_{nb} P_{mT} |\hat{\phi}_{nb}(\bar{x}t)|^2 |\phi_{mT}(\bar{x}t)|^2$$

represents the probability that a beam and target particle will be at the same place at the same time

$$dP = d^3p_1\, d^3p_2\, \| \langle p_1 p_2 | T(E+i\epsilon) | \langle p_1\rangle \langle p_2\rangle \rangle \|^2$$

$$(2\pi)^4 \hbar^2\, \delta(E - \langle E\rangle)\, \delta(\bar{p} - \langle \bar{p}\rangle) \int d^3x\, dt$$

$$\sum P_{nb} P_{mT} |\hat{\Phi}_n(xt)|^2 |\hat{\Phi}_m(xt)|^2$$

If we multiply the total number of beam and target particle

$N_b P_{nb}$ = # beam particles in state n

$N_T P_{mT}$ = # target particle in state m

$\frac{dN}{d^3x\, dt}$ = # particles scattered within d^3p_1 of p_1 and d^3p_2 of p_2 per unit time / unit volum

$$= d^3p_1\, d^3p_2\, \| \langle p_1 p_2 | T(E+i\epsilon) | \langle p_1\rangle \langle p_2\rangle \rangle \|^2$$

$$(2\pi)^4 \hbar^2\, \delta(\bar{E} - \langle \bar{E}\rangle)\, \delta(\bar{p} - \langle \bar{p}\rangle)$$

$$\underbrace{\left| \frac{\#\ \text{beam at}\ xt}{\text{volum}} \right|}_{N_b P_{bn} |\hat{\Phi}_n(xt)|^2} \underbrace{\left| \frac{\#\ \text{target at}\ xt}{\text{volum}} \right|}_{N_T P_{Tm} |\hat{\Phi}_m(xt)|}$$

note

$$\frac{dN}{dt\,dV} \propto \rho_{\text{target}} \times \rho_{\text{beam}} \cdot v$$

the constant of proportionality is called the differential cross section

#/volume

$$\frac{dN}{dt\,dV} = \rho_T \, \rho_B \, V_{BT} \, d\sigma$$

$$= (2\pi)^4 \hbar^2 \, \| \langle p_1 p_2 \| T \| \langle p_1 \rangle \langle p_2 \rangle \rangle \|^2$$

$$\delta(E - \langle E \rangle) \, \delta(\bar{p} - \langle p \rangle) \, d^3p_1 \, d^3p_2$$

$$\rho_B(x\,t) \, \rho_T(x\,t)$$

This gives

$$\boxed{\begin{aligned} d\sigma = \frac{(2\pi)^4 \hbar}{V_{BT}} \, & \| \langle p_1 p_2 \| T \| \langle p_1 \rangle \langle p_2 \rangle \rangle \|^2 \\ & \delta(E - \langle E \rangle) \, \delta(\bar{p} - \langle \bar{p} \rangle) \, d^3p_1 \, d^3p_2 \end{aligned}}$$