Scattering

Last time

$$\boxed{P = |\langle \Psi_+(t) | \Psi_-(t)\rangle|^2 = |\langle \Psi_+(0) | \Psi_-(0)\rangle|^2}$$

Initial conditions replaced by asymptotic conditions

$$\boxed{\lim_{t \to \pm\infty} \| |\Psi_\pm(t)\rangle - |\Psi^0_\pm(t)\rangle \| = 0} \qquad (1)$$

where $|\Psi^0_\pm(t)\rangle$ is a solution of the Schrodinger equation for free particles

$$|\Psi_\pm(t)\rangle = e^{-iHt/\hbar}$$
$$|\Psi^0_\pm(t)\rangle = e^{-iH_0 t/\hbar}$$

equation (1) becomes

$$\lim_{t \to \pm\infty} \| e^{-iHt/\hbar} |\Psi_\pm(0)\rangle - e^{-iH_0 t/\hbar} |\Psi^0_\pm(0)\rangle \| = 0$$

or equivalently

$$\lim_{t \to \pm\infty} \| |\Psi_\pm(0)\rangle - e^{iHt/\hbar} e^{-iH_0 t/\hbar} |\Psi^0_\pm(0)\rangle \|$$

we write this as

$$\boxed{\begin{aligned} |\Psi_\pm(0)\rangle &= \lim_{t \to \pm\infty} e^{iHt/\hbar} e^{-iHt/\hbar} |\Psi^0_\pm(0)\rangle \\ \Omega_\pm &\equiv \lim_{t \to \pm\infty} e^{iHt/\hbar} e^{-iH_0 t/\hbar} \end{aligned}}$$

It follows that

$$P = |\langle \Psi_+^\circ(0) | \Omega_+^\dagger \Omega_- | \Psi_-^\circ(0) \rangle|^2$$

$$S = \Omega_+^\dagger \Omega_-$$

is called the scattering operator

$$P = |\langle \Psi_+^\circ(0) | S | \Psi_-^\circ(0) \rangle|^2$$

* This is more useful than $|\langle \Psi_+(t) | \Psi_-(t) \rangle|^2$ because there is no time t when both $|\Psi_\pm(t)\rangle$ are simple.

summary of propertie

① $H \Omega_\pm = \Omega_\pm H_0$

② $H_0 S = S H_0$

③ $|\Psi_\pm(t)\rangle = \Omega_\pm |\Psi_\pm^\circ(t)\rangle$

④ $S S^\dagger = I$

⑤ $\langle B | \Omega_\pm = 0$ $\quad |B\rangle =$ bound state of H

existence of limit?

$$\lim_{t \to \pm\infty} \| \, |\Psi_\pm(0)\rangle - e^{iHt/\hbar} e^{-iH_0t/\hbar} |\Psi^0_\pm(0)\rangle \, \|$$

$$\lim_{t \to \infty} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\Psi^0_\pm(0)\rangle =$$

$$|\Psi_\pm(0)\rangle + \lim_{t \to \pm\infty} \int_0^t \frac{d}{dt} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\Psi^0_\pm(0)\rangle \, dt$$

$$|\Psi_\pm(0)\rangle + \lim_{t \to \pm\infty} \int_0^t \frac{i}{\hbar} e^{iHt/\hbar} V e^{-iH_0t/\hbar} |\Psi_\pm(0)\rangle \, dt$$

This will converge as $t \to \pm\infty$ if

$$\left\| \int_0^{\pm\infty} e^{iHt/\hbar} V e^{-iH_0t/\hbar} |\Psi^0_\pm(0)\rangle \, dt \right\| \le$$

$$\int_0^\infty \| e^{\pm iHt/\hbar} V e^{\mp iH_0t/\hbar} |\Psi^0_\pm(0)\rangle \| \, dt \le$$

$$\int_0^\infty \| V e^{\mp iH_0t/\hbar} |\Psi^0_\pm(0)\rangle \| \, dt < \infty$$

This last inequality is a sufficient condition for the existence of the limit. It is a condition on V, but it does not require solving the Schrodinger equation.

Typically this is finite for potentials that fall off <u>faster</u> than $1/r$

The condition

$$\boxed{\int_0^\infty \| V e^{\mp iH_0 t/\hbar} |\varphi^0_\pm(0)\rangle \| \, dt < \infty}$$

is called Cooks condition

Assuming the interaction satisfies Cooks condition

$$S = \lim_{\substack{t \to +\infty \\ s \to -\infty}} e^{iH_0 t/\hbar} e^{-iH(t-s)/\hbar} e^{-iH_0 s/\hbar}$$

letting $s = -t$

$$S = \lim_{t \to +\infty} e^{iH_0 t/\hbar} e^{-2iHt/\hbar} e^{iH_0 t/\hbar} = .$$

we write this as

$$= I + \lim_{t \to \infty} \int_0^t \frac{d}{dt'} \left(e^{iH_0 t'/\hbar} e^{-2iHt'/\hbar} e^{iH_0 t'/\hbar} \right) dt' =$$

$$= I + \lim_{t \to \infty} \int_0^t \left(-\frac{i}{\hbar}\right) \Big[e^{iH_0 t'/\hbar} V e^{-2iHt'/\hbar} e^{iH_0 t'/\hbar} +$$

$$e^{iH_0 t'/\hbar} e^{-2iHt'/\hbar} V e^{iH_0 t'/\hbar} \Big]$$

This expression is only meaningful when applied to $\langle \varphi^0_+(0)|$, $|\varphi^0_-(0)\rangle$.

For 2 body scattering

$$H_0 = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} = E(P_1, P_2)$$

$$I = \int d^3p_1\, d^3p_2\ |\bar{P}_1 P_2\rangle\langle \bar{P}_1 P_2|$$

with wave packets

$$\langle \Psi_+^0(c)| S |\Psi_-^0(c)\rangle = \langle \Psi_+(c)| \Psi_-(c)\rangle$$

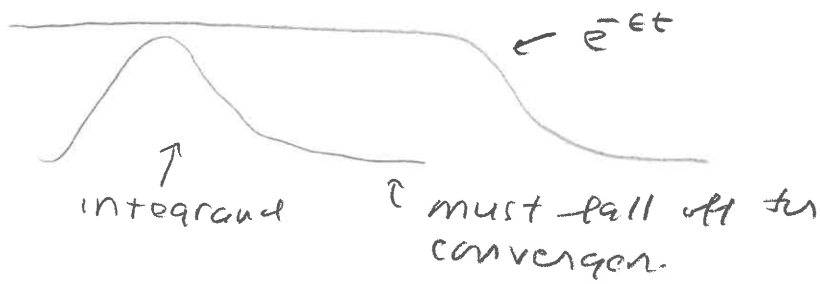
$$\lim_{t\to\infty} \left(-\frac{i}{\hbar}\right) \int_0^t \langle \Psi_+^0(c)|\bar{P}_1\bar{P}_2\rangle\, d^3p_1\, d^3p_2$$

$$\langle \bar{P}_1 \bar{P}_2| \left[V e^{-\frac{i}{\hbar}(2H - E(P_1P_2) - E(P_1'P_2'))t} + e^{-\frac{i}{\hbar}(2H - E(P_1P_2) - E(P_1'P_2'))t} V \right] |P_1'P_2'\rangle\, d^3p_1'\, d^3p_2'$$

$$\langle P_1' P_2' | \Psi_-^0(c)\rangle$$

If we did the t integral before the $\bar{P}_1\ \bar{P}_2\ \bar{P}_1'\ \bar{P}_2'$ integrals we would have exponents with $it \to i\infty$ which make no sense. On the other hand we know for short range potentials this exists but we must first do the $P_1\ P_2\ P_1'\ P_2'$ integrals

If this integral exists we can insert $\lim_{\epsilon \to} e^{-\epsilon t}$ and it will not change the value of a convergent integral



Inserting the factor $e^{-\epsilon t}$ allows one to change the order of integration

$$\int dx \left[\int dy\, f(yx) \right] =$$

$$\lim_{\epsilon \to} \int dx \left[\int dy\, f(xy)\, e^{-\epsilon x} \right]$$

$$\lim_{\epsilon \to} \int dy \int f(xy)\, e^{-\epsilon x}\, dx$$

define

$$E = E(\bar{P}_1 \bar{P}_2)$$

$$E' = E(\bar{P}_1' \bar{P}_2')$$

$$\mathbb{E} = \frac{1}{2}(E + E')$$

$$\langle \psi_+^0(c) | S | \psi_-^0(c) \rangle = \langle \psi_+^0(c) | \psi_-^-(c) \rangle$$

$$\lim_{\epsilon \to 0} \left(-\frac{i}{\hbar}\right) \int d^3p_1\, d^3p_2\, d^3p_1'\, d^3p_2' \int_0^\infty dt \quad \langle \psi_+^0(c) | \bar{p}_1 \bar{p}_2 \rangle \times$$

$$\langle p_1 p_2 | V e^{-\frac{i}{\hbar}(2H - 2E - i\hbar\epsilon)t} + e^{-\frac{i}{\hbar}(2H - 2E - i\hbar\epsilon)t} V | p_1' p_2' \rangle$$

$$\langle p_1' p_2' | \psi_-^0(c) \rangle$$

It is now possible to perform the t integral – suppressing the

$$\int d^3p_1\, d^3p_2\, d^3p_1'\, d^3p_2' \quad \langle \psi_+^0(c) | p_1 p_2 \rangle \cdots \langle p_1' p_2' | \psi_-^0(c) \rangle$$

$$[\cdots] = \left(-\frac{i}{\hbar}\right)(-1)\left(-\frac{\hbar}{i}\right) \langle p_1 p_2 | [V (2H - 2E - i\hbar\epsilon)^{-1} + (2H - 2E - i\hbar\epsilon)^{-1} V | p_1' p_2' \rangle$$

$$= \frac{1}{2} \langle p_1 p_2 | V \frac{1}{E - H + i\hbar\frac{\epsilon}{2}} + \frac{1}{E - H + \frac{i\hbar\epsilon}{2}} V | p_1' p_2' \rangle$$

we define $\epsilon' = \frac{\hbar\epsilon}{2}$ we use the second resolvent identity for $H = H_0 + V$

$$\frac{1}{E - H + i\epsilon} = \frac{1}{E - H_0 + i\epsilon} + \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H + i\epsilon}$$

$$= \frac{1}{E - H_0 + i\epsilon} + \frac{1}{E - H + i\epsilon} V \frac{1}{E - H_0 + i\epsilon}$$

using this in the above give

$$[\cdots] = \frac{1}{2} \langle p_1 p_2 | \left[V \left(\frac{1}{E - H_0 + i\epsilon'} + \frac{1}{E - H + i\epsilon'} V \frac{1}{E - H_0 + i\epsilon} \right) + \right.$$

$$\left. \left(\frac{1}{E - H_0 + i\epsilon} + \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H + i\epsilon} \right) V \right] | p_1' p_2' \rangle$$

$$= \frac{1}{2} \langle p_1 p_2 | V + V \frac{1}{E - H + i\epsilon'} V | p_1' p_2' \rangle \times$$

$$\left(\underbrace{\frac{1}{E - E' + i\epsilon'}}_{\frac{E-E'}{2}} + \underbrace{\frac{1}{E - E + i\epsilon'}}_{\frac{E'-E}{2}} \right)$$

$$[\cdots = \frac{1}{2} \langle p_1 p_2 | \left(V + V (E - H + i\epsilon')^{-1} V \right) | p_1' p_2' \rangle$$

$$- \frac{i 8\epsilon'}{(E - E') + 4\epsilon'^2}$$

as $\epsilon' \to 0$ this becomes $-2\pi i \, \delta(E - E')$

To see this note

$$\lim_{\epsilon \to 0} \int \frac{-i 8\epsilon'}{(E - E')^2 + 4\epsilon'^2} f(E') \, dE'$$

$$\lim_{\epsilon \to} \int \frac{1}{4\epsilon'^2} \frac{-8i\epsilon'}{\left(\frac{E-E'}{2\epsilon'}\right)^2 + 1} f(E') \, dE'$$

let $u = \frac{E' - E}{2\epsilon'}$ $\quad du = \frac{dE'}{2\epsilon'}$ $\quad E' = E + 2\epsilon u$

$$\lim_{\epsilon\to 0} \int \frac{-i\,8\epsilon'}{(E-E')^2+4\epsilon''} f(E')\,dE' =$$

$$\lim_{\epsilon\to 0} \int \frac{-4i\,du}{(u^2+1)} f(E+2\epsilon u)$$

as $\epsilon \to 0$ gets large so we can extend the integral to $\int_{-\infty}^{\infty}$

$$\int_{-\infty}^{\infty} \frac{du}{u^2+1} = \pi$$

as long as $|f| < \infty$ we can use the dominated convergence theorem to take the integral inside the integral

$$\to \quad -4\pi i\, f(E) = -4\pi i \int \delta(E-E') f(E')$$

this gives

$$[\cdots] = \frac{1}{2}(-4\pi i)\,\delta(E-E')\langle p_1 p_2 | V + V(\bar{E}-H+i\epsilon')^{-1} V | p_1' p_2' \rangle$$

with the δ function $\bar{E} = E = E'$

$$\langle \Psi_+(\epsilon) | \Psi_-(\epsilon) \rangle = \int \langle \Psi_+^0(\epsilon) | p_1 p_2 \rangle\, dp_1 dp_2$$

$$\langle p_1 p_2 | \underline{(I - 2\pi i\,\delta(E-E')\,(V + V(E-H+i\epsilon')^{-1} V)}\, | p_1' p_2' \rangle$$

$$dp_1' dp_2' \langle p_1' p_2' | \Psi_-^0(\epsilon) \rangle$$

This is usually expressed as

$$\boxed{\begin{aligned} S &= I - 2\pi i\, \delta(E-E')\, T(E+i\epsilon') \\ T(z) &= V + V(z-H)^{-1}V \end{aligned}}$$

$T(z)$ is called the transition operator and the energy conserving δ function only makes sense when evaluated between eigenstates of H_0.

<u>equations</u>

$$(z-H_0)^{-1} = \int \frac{|p_1 p_2\rangle\, d^3p_1\, d^3p_2 \langle p_1 p_2|}{z - E(p_1 p_2)}$$

is known; to find $T(z)$

note

$$(z-H)^{-1} = (z-H_0)^{-1} + (z-H_0)^{-1}V(z-H)^{-1}$$

$$T(z) = V + V[(z-H_0)^{-1} + (z-H_0)^{-1}V(z-H)^{-1}]V$$

$$= V + V(z-H_0)^{-1}(V + V(z-H)^{-1}V)$$

$$= V + V(z-H_0)^{-1}T(z)$$

The equation

$$\boxed{T(z) = V + V(z-H_0)^{-1} T(z)}$$

is an integral equation for $T(z)$ in terms of known quantities. It is called the Lippmann Schwinger equation.

There is a relation equation for $|\psi_\pm(t)\rangle$

$$|\psi_\pm(0)\rangle = \lim_{t \to \pm\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\psi^0_\pm(0)\rangle$$

$$= |\psi^0_\pm(0)\rangle + \lim_{t \to \pm\infty} \int_0^t \frac{d}{dt}\left(e^{iHt/\hbar} e^{-iH_0t/\hbar}\right) |\psi^0_\pm(0)\rangle$$

$$= |\psi^0_\pm(0)\rangle + \lim_{t \to \pm\infty} \frac{i}{\hbar} \int_0^t e^{iHt'/\hbar} V e^{-iH_0t'/\hbar} |\psi^0_\pm(0)\rangle \, dt'$$

expand $|\psi^0_\pm(0)\rangle$ in eigenstates of H_0

$$= \int d^3p_1 d^3p_2 \left(|\bar{p}_1 \bar{p}_2\rangle + \lim_{t \to \pm\infty} \frac{i}{\hbar} \int_0^t e^{i(H-E)t/\hbar} V |p_1 p_2\rangle \right) \langle p_1 p_2 | \psi^0_\pm(0)\rangle$$

As in the case of $T(z)$

(1) convergence requires computing the $p_1 p_2$ integrals before the t integral

(2) if this is done in the correct order including $e^{\mp \epsilon t}$ for

sufficiently small ϵ does not change the result

(3) With the $e^{\mp \epsilon t}$ we can change the order of integration

$$= \int dp_1 dp_2 \left[|p_1 p_2\rangle + \lim_{\epsilon\to 0} \frac{i}{\hbar}\frac{\hbar}{i}(-1)(H - E \mp i\epsilon)^{-1} V |p_1p_2\rangle \right] \langle p_1 p_2 | \Psi^0_\pm(t)\rangle$$

$$|\Psi_\pm(t)\rangle = \int dp_1 dp_2 \times \underbrace{\left(1 + \frac{1}{E\pm i\epsilon - H} V\right)|p_1p_2\rangle}_{|(p_1p_2)^\pm\rangle} \langle p_1 p_2|\Psi^0_\pm(t)\rangle$$

$$\boxed{|\Psi_\pm(t)\rangle = \int dp_1 dp_2\, |(p_1p_2)^\pm\rangle\langle p_1p_2|\Psi^0_\pm(t)\rangle}$$

$$|(p_1p_2)^\pm\rangle = |(p_1p_2)\rangle + (E\pm i\epsilon - H)V|(p_1p_2)\rangle$$

$$= |p_1p_2\rangle + \left(\frac{1}{E\pm i\epsilon - H_0} + \frac{1}{E\pm i\epsilon - H_0} V \frac{1}{E\pm i\epsilon H}\right) V |p_1p_2\rangle$$

$$= |p_1p_2\rangle + \frac{1}{E\pm i\epsilon - H_0}\underbrace{\left(V + V\frac{1}{E\mp i\epsilon - H}V\right)}_{T(E\pm i\epsilon)}|p_1p_2\rangle$$

$$= |p_1p_2\rangle + \frac{1}{E\pm i\epsilon - H_0} V \Big(\underbrace{|p_1p_2\rangle + \frac{1}{E\pm i\epsilon - H} V |p_1p_2\rangle}_{|(p_1p_2)^\pm\rangle}\Big)$$

① $|(P_1P_2)^{\pm}\rangle = |(P_1P_2)\rangle + (E \pm i\epsilon - H_0)^{-1} V |(P_1P_2)^{\pm}\rangle$

This is called the Lippmann Schwinger equation for the wave function

② $|(P_1P_2)^{\pm}\rangle = |(P_1P_2)\rangle + (E \pm i\epsilon - H_0)^{-1} T(E \pm i\epsilon) |P_1 P_2\rangle$

This shows that $|(P_1P_2)^{\pm}\rangle$ can be constructed directly from $T(z)$