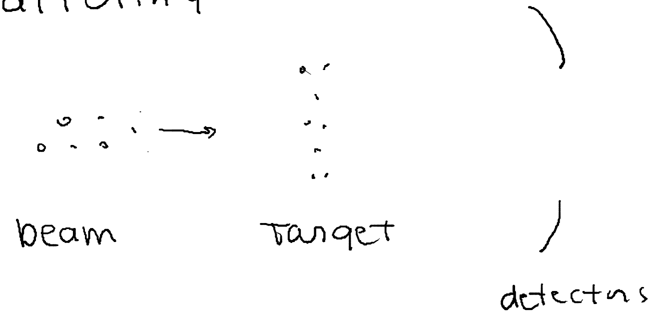
Lecture 21

Scattering



$$\rho_b = \sum_n P_{bn} |\psi_{bn}\rangle\langle\psi_{bn}|$$
$$\rho_T = \sum_n P_{Tn} |\psi_{Tn}\rangle\langle\psi_{Tn}$$

beam + target ensembles

P_{bn}, P_{Tn} classical probabilities

problem: understand properties of interaction by looking at the statistical distribution of particles detected

average beam momentum

$$\langle\vec{P}\rangle_b = \mathrm{Tr}(\bar{P}\rho_b)$$

$\langle \bar{P} \rangle_T = Tr(\bar{P} \rho_T)$

In the lab frame $\langle \vec{P}_T \rangle = 0$,

but we are not nessarily

assuming $\langle \bar{P}_T \rangle = 0$

We start by first focusing

on a single quantum

measurement

$|\Psi_-(t)\rangle$ = state of beam-

target system at time t

$|\Psi_+(t)\rangle$: detected state

at time t.

both states are solutions

of the Schrödinger equation

$$i\hbar \frac{d}{dt}|\Psi_-(t)\rangle = H|\Psi_-(t)\rangle$$

$$i\hbar \frac{d}{dt}|\Psi_+(t)\rangle = H|\Psi_+(t)\rangle$$

These have the solved form

$$|\Psi_\pm(t)\rangle = e^{-iHt/\hbar} |\Psi_\pm(0)\rangle$$

The probability that this beam target state will be measured to be in the state $|\Psi_+\rangle$ is

$$P_{bT} = |\langle\Psi_+(t)|\Psi_-(t)\rangle|^2$$

since both states are solutions of the Schrodinger equation this probability is independent of time

$$\langle\Psi_+(t)|\Psi_-(t)\rangle =$$

$$\langle\Psi_+(0)| e^{iHt/\hbar} e^{-iHt/\hbar} |\Psi_-(0)\rangle =$$

$$\langle\Psi_+(0)|\Psi_-(0)\rangle$$

The difficulty is that in general the state $|\Psi_\pm(\cdot)\rangle$ are not known. However long before the collision $|\Psi_-(t)\rangle$ looks like a freely moving beam particle and a non interacting target particle at rest, similarly when the colliding particles are beyond the range of the interaction the system looks like 2 free particles heading towards the detectors

If $-T_-$ is a time long before the collision and T_+ is a time long after the collision we can express this condition as

$$\| |\Psi_-(-T_-)\rangle - |\Psi_-^0(-T_-)\rangle \| \simeq 0$$

$$\| |\Psi_+(T_+)\rangle - |\Psi_+^0(T_+)\rangle \| \simeq 0$$

where $|\Psi_\pm^0(t)\rangle$ are solutions of the Schrodinger equation without interaction — this means that the above can be written

$$\| e^{-iH(-T_-)/\hbar} |\Psi_-(0)\rangle - e^{-iH_0(-T_-)/\hbar} |\Psi_-^0(0)\rangle \| \approx 0$$

$$\| e^{-iHT_+/\hbar} |\Psi_+(0)\rangle - e^{-iH_0T_+/\hbar} |\Psi_+(0)\rangle \| \approx 0$$

The minimum value of $T_\pm$ depends on properties of the states, but nothing changes if they are increased.

To eliminate the dependence on the structure of the state we can use

$$\lim_{t \to \pm\infty} \| e^{-iHt/\hbar} |\Psi_\pm(0)\rangle - e^{-iH_0 t/\hbar} |\Psi^0_\pm(0)\rangle \| = 0$$

since $e^{-iHt/\hbar}$ is unitary this can be replaced by

$$\lim_{t \to \pm\infty} \| |\Psi_\pm(0)\rangle - e^{iHt/\hbar} e^{-iH_0 t/\hbar} |\Psi^0_\pm(0)\rangle \| = 0$$

These are called the scattering asymptotic conditions

The scattering asymptotic condition replace the initial conditions –

The states $|\Psi_\pm^{o}(0)\rangle$ represent states of non interacting particles. We write these conditions as

$$|\Psi_\pm(0)\rangle = \lim_{t\to\pm\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\Psi_\pm^{o}(0)\rangle$$

The operators

$$\Omega_\pm \equiv \lim_{t\to\pm\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar}$$

are called Møller wave operators

It follows that the probability amplitude can be expressed as

$$\langle \Psi_+(0) | \Psi_-(0) \rangle =$$

$$\langle \Psi_+^0(0) | \Omega_+^\dagger \Omega_- | \Psi_-^0(0) \rangle =$$

$$\langle \Psi_+^0(0) | S | \Psi_-^0(0) \rangle$$

The operator

$$S = \Omega_+^\dagger \Omega_-$$

is called the scattering operator

Theorem (Intertwining)

$$H \Omega_\pm = \Omega_\pm H_0$$

Proof

$$e^{isH/\hbar} \Omega_\pm =$$

$$\lim_{t \to \pm\infty} e^{i(s+t)H/\hbar} e^{-itH_0/\hbar} =$$

$$= \lim_{t \to \infty} e^{iH(t+s)/\hbar} e^{-iH_0(t+s)/\hbar} e^{iH_0 s/\hbar}$$

Let $t' = t+s$. For fixed s $t' \to \pm\infty$ as $t \to \pm\infty$. Therefore

$$= \Omega_\pm e^{iH_0 s/\hbar}$$

differentiating with respect to s and setting $s=0$ gives

$$iH\Omega_\pm = \Omega_\pm (iH_0) \quad \text{or}$$

$$\boxed{H\Omega_\pm = \Omega_\pm H_0}$$

corollary

$$\boxed{H_0 S = S H_0}$$

$$H_0 S = H_0 \Omega_+^\dagger \Omega_- =$$
$$\Omega_+^\dagger H \Omega_- = \Omega_+^\dagger \Omega_- H_0 =$$
$$S H_0$$

comments — this is a statement of energy conservation — this makes sense because the energy does not change when the particles move freely.

The condition $H\Omega_\pm = \Omega_\pm H_0$ follow because if the eigenvalues of H and H_0 are not the same oscillations kill the state

Note if

$$H_0 |\psi^\circ\rangle = E |\psi^\circ\rangle$$

then

$$H\Omega_\pm |\psi^\circ\rangle = E\Omega_\pm |\psi^\circ\rangle$$

so $\Omega_\pm$ transform eigenstate of H_0 to eigenstates of H with the same energy. We also have

$$\Omega_\pm |\Psi_0(t)\rangle = \Omega_\pm e^{-iH_0t/\hbar} |\Psi_0(0)\rangle$$

$$= e^{-iHt/\hbar} \Omega_\pm |\Psi_0(0)\rangle$$

$$= |\Psi_\pm(t)\rangle$$

so

$$\boxed{\Omega_\pm |\Psi_0(t)\rangle = |\Psi_\pm(t)\rangle}$$

Theorem $\boxed{\Omega_+ \Omega_+^\dagger = I}$

$$\lim_{t\to\infty} \underbrace{e^{iHt/\hbar} \underbrace{e^{-iH_0t/\hbar} e^{iH_0t/\hbar}}_{I} e^{-iHt/\hbar}}_{I}$$

note if H has bound states with energy $E<0$ then

$$H|\psi_b\rangle = E_b|\psi_b\rangle$$

$$\langle\psi_b|\Omega_\pm =$$

$$\langle\psi_b|\Omega_\pm|\overset{o}{E}\rangle dE\langle\overset{o}{E}|$$

$$\int \underbrace{\langle\psi_b|E_\pm\rangle}_{0} dE\langle E^{\circ}|$$

This vanishes because H_0 only has eigenstates with positive energy while $|\psi_b\rangle$ has negative energy

$$|\psi_b\rangle \quad |E_\pm\rangle$$

are eigenstates of H with different energies

Theorem: The scattering operator is unitary

$$S^\dagger S = \Omega_-^\dagger \underbrace{\Omega_+ \Omega_+^\dagger}_{I} \Omega_-$$

$$= \Omega_-^\dagger \Omega_- =$$

$$= \lim_{t \to -\infty} \underbrace{e^{-iHt/\hbar} \underbrace{e^{iH_0 t/\hbar} e^{-iH_0 t/\hbar}}_{I} e^{iHt/\hbar}}_{I}$$

$$\boxed{S^\dagger S = I}$$

while S is unitary $\Omega_\pm$ may or may not be unitary

Structure of S

$$\langle \Psi_+^0(0) | S | \Psi_-^0(0) \rangle =$$

$$\langle \Psi_+^0(0) | \Omega_+^\dagger \Omega_- | \Psi_-^0(0) \rangle =$$

$$\lim_{\substack{t \to \infty \\ s \to -\infty}} \langle \Psi_+^0(0) | e^{iH_0 t/\hbar} e^{-iHt/\hbar} e^{iHs/\hbar} e^{-iH_0 s/\hbar} | \Psi_-^0(0) \rangle =$$

$$\lim_{t \to \infty} \langle \Psi_+^0(0) | e^{iH_0 t/\hbar} e^{-2iHt/\hbar} e^{iH_0 t/\hbar} | \Psi_-^0(0) \rangle$$

Next write the limit as the integral of a derivativ

$$= \langle \Psi_+^0(0) | \Psi_-^0(0) \rangle +$$

$$\int_0^\infty \langle \Psi_+^0(0) | \frac{d}{dt} \left(e^{iH_0 t/\hbar} e^{-2iHt/\hbar} e^{iH_0 t/\hbar} \right) | \Psi_-(0) \rangle =$$

$$= \langle \Psi_+^0(0) | \Psi_-^0(0) \rangle +$$

$$\int_0^t \langle \Psi_+^0(0) | \left\{ e^{iH_0 t/\hbar} \left(-i(H-H_0)\tfrac{1}{\hbar}\right) e^{-2iHt/\hbar} e^{iH_0 t/\hbar} + e^{iH_0 t/\hbar} e^{-2iHt/\hbar} \left(-i(H-H_0)\tfrac{1}{\hbar}\right) e^{iH_0 t/\hbar} \right\} | \Psi_-^0(0) \rangle \, dt$$

$$= \langle \psi_+^0(0) | \psi_-^0(0) \rangle$$

$$(-1) - \frac{i}{\hbar} \int_0^t \langle \psi_+^0(0) | \Big\{ e^{iH_0 t/\hbar} V e^{-2iHt/\hbar} e^{iH_0 t/\hbar} + e^{iH_0 t/\hbar} e^{-2iHt/\hbar} V e^{iH_0 t/\hbar} \Big\} | \psi_-^0(0) \rangle \, dt$$

In order to do the time integral expand $|\psi_\pm^0(0)\rangle$ in terms of a complete set of eigenstates of H_0

$$I = \int dE \, |E\rangle\langle E|$$

(there is an implied sum/integral over other variables)

$$= \langle \psi_+^0(0) | \psi_-^0(0) \rangle +$$

$$\frac{i}{\hbar} \int_0^t dt \; \langle \psi_+^0(0) | E_1 \rangle dE_1 \Big\{ \langle E_1 | V e^{-2i(H - \frac{E_1 + E_2}{2})t/\hbar} | E_2 \rangle + \langle E_1 | e^{-2i(H - \frac{E_1 + E_2}{2})t/\hbar} V | E_2 \rangle \Big\}$$

$$\langle E_2 | \psi_0^-(0) \rangle \; dE_1 \, dE_2$$

For this to makes sense the E_1 and E_2 integral must be evaluated before doing the t integral

* If the integrals are done in the correct order including a factor $e^{-\epsilon t}$ does not change the integrals provided ϵ is small enough.

* With the ϵ the order does not matter. This means that it is possible to do the t integrals first — then taking $\epsilon \to 0$ after doing the $E_1 E_2$ integrals

$$= \langle \psi_+^0(0) | \psi_-^0(0) \rangle + \frac{i}{\hbar} \int \langle \psi_0^+(0) | E_1 \rangle \, dE_1$$

$$\langle E_1 | \left\{ V \frac{1}{-\frac{2i}{\hbar}(H - \frac{E_1+E_2}{2} - \epsilon)} + \frac{1}{-\frac{2i}{\hbar}(H - \frac{E_1+E_2}{2} - \epsilon)} V \right\}$$

$$dE_2 \langle E_2 | \psi_-^0 \rangle$$

Let $\bar{E} = (E_1 + E_2)/2$

$$= \langle \psi_+^0(0) | \psi_-^0(0) \rangle + \int \langle \psi_0^+(0) | E_1 \rangle \, dE_1$$

$$\langle E_1 | \left\{ V \frac{1}{-2(H - \bar{E} - \frac{i\hbar}{2}\epsilon)} + \frac{1}{-2(H - \bar{E} - \frac{i\hbar}{2}} \right\}$$

$$| E_2 \rangle \, dE_2 \langle E_2 | \psi_-^0(0) \rangle$$

Next we use the sec

resolvent identity

$$\frac{1}{z - H} = \frac{1}{z - H_0} + \frac{1}{z - H_0} V \frac{1}{z - 1}$$

$$= \frac{1}{z - H_0} + \frac{1}{z - H} V \frac{1}{z - H_0}$$

where Z is any complex numbers where $(Z-H)^{-1}$ and $(Z-H_0)^{-1}$ exist

The proof follows by multiplying

$$(Z-H_0)[\quad](Z-H)$$

$$(Z-H)[\quad](Z-H_0)$$

where $[\quad]$ is the resolvent identity

$$= \langle \Psi_+^0 | \Psi_-^0 \rangle + \int \langle \Psi_+^0 | E_1 \rangle dE_1$$

$$\frac{1}{2} \langle E_1 | \left\{ V \frac{1}{E-H+i\epsilon'} + \frac{1}{E-H-i\epsilon'} V \right\} | E_2 \rangle dE_2$$

$$\times \langle E_2 | \Psi_-^0(0) \rangle$$

$$\epsilon' = \frac{\hbar}{2}\epsilon$$

$$= \langle \Psi_+^0(0) | \Psi_-^0(0) \rangle + \int dE_1\, dE_2 \, \langle \Psi_+^0(0) | E_1 \rangle \times$$

$$\langle E_1 | \frac{i}{2} \left\{ V \left(\frac{1}{\bar{E} - H_0 + i\epsilon'} + \frac{1}{\bar{E} - H + i\epsilon'} V \frac{1}{\bar{E} - H_0 + i\epsilon'} \right) + \right.$$

$$\left. \left(\frac{1}{\bar{E} - H_0 + i\epsilon'} + \frac{1}{\bar{E} - H_0 + i\epsilon'} V \frac{1}{\bar{E} - H + i\epsilon'} \right) V \right\} | E_2 \rangle$$

$$\langle E_2 | \Psi_-^0(0) \rangle$$

In the form the H_0's can be replaced by E_1 or E_2

on the left

$$\frac{1}{\bar{E} - E_1 + i\epsilon'} = \frac{1}{\frac{1}{2}(E_1 + E_2) - E_1 + i\epsilon} = \frac{2}{E_2 - E_1 + 2i\epsilon}$$

on the right

$$\frac{1}{\bar{E} - E_2 + i\epsilon} = \frac{1}{\frac{1}{2}(E_1 + E_2) - E_2 + i\epsilon} = \frac{2}{E_1 - E_2 + 2i\epsilon}$$

Using these in the above

$$\langle \Psi_+^0(0) | \Psi_-^0(0) \rangle + \int dE_1\, dE_2 \, \langle \Psi_+^0(0) | E_1 \rangle$$

$$\langle E_1 | V + V(\bar{E} - H + 2i\epsilon')V | E_2 \rangle \times$$

$$\frac{2}{2} \left(\frac{1}{E_2 - E_1 + 2i\epsilon'} + \underbrace{\frac{1}{E_1 - E_2 + 2i\epsilon'}}_{-\frac{1}{E_2 - E_1 - 2i\epsilon}} \right)$$

note

$$\frac{1}{E_2-E_1+2i\epsilon'} - \frac{1}{E_2-E_1-2i\epsilon'} =$$

$$\frac{-4i\epsilon'}{(E_2-E_1)^2+4\epsilon'^2}$$

In the limit $\epsilon' \to 0$

$$-2i\pi\,\delta(E_2-E_1)$$

To show this consider

$$\int \frac{-4i\epsilon'}{(E_2-E_1)^2+4\epsilon^2} f(E_1)\,dE_1' =$$

$$-\frac{i}{\epsilon}\int \frac{1}{\left(\frac{E_2-E_1}{2\epsilon}\right)^2+1} f(E_1)\,dE_1$$

Let $u = \frac{E_1-E_2}{2\epsilon}$ $\quad du = \frac{dE_1}{2\epsilon}$

$$-2i\int \frac{du}{u^2+1} f(E_2+2\epsilon u)\,du$$

as $\epsilon \to 0$ this becomes

$$-2if(E_2)\int \frac{du}{u^2+1} = -2\pi i\int \delta(E_2-E_1)\, f(E_1)\,dE_1$$

This gives an energy conserving δ function $\Rightarrow E_1 = E_2 = \bar{E}$

so the expression is

$$\langle \Psi_+^\circ(0) | S | \Psi_-^\circ(0) \rangle = \langle \Psi_+^\circ(0) | \Psi_-^\circ(0) \rangle$$

$$\int \langle \Psi_+^\circ(0) | E \rangle \, dE \times \langle E |$$

$$\left[-2\pi i \, \delta(E - E') \left[V + V \frac{1}{E - H + i\epsilon} V \right] \right.$$

$$| E' \rangle \, dE' \langle E' | \Psi_-^\circ(0) \rangle$$

The operator

$$\boxed{T(E + i\epsilon) = V + V (E - H + i\epsilon)^{-1} V}$$

is called the transition operator

$$\boxed{S = I - 2\pi i \, \delta(E - E') \, T(E + i\epsilon)}$$