Lecture 1

See notes on rotations in syllabus

outline

* unitary 1 parameter groups
* rotations about a fixed axis
* SU(2)
* Angular momentum
* Representations of SU(2)

Definition: A collection of unitary operators $\mathcal{U}(\lambda)$ is called a unitary 1 parameter group if $\mathcal{U}(\lambda)$ satisfies the following conditions

(1) $\mathcal{U}(0) = I$

(2) $\mathcal{U}(\lambda_2)\,\mathcal{U}(\lambda_1) = \mathcal{U}(\lambda_2+\lambda_1)$

(3) $\mathcal{U}(\lambda)^{+} = \mathcal{U}(\lambda)^{-1} = \mathcal{U}(-\lambda)$

examples:

(1) Rotations about a fixed axis

(2) Time evolution

(3) Translations in a given direction

<u>Theorem</u>: If $\mathcal{U}(\lambda)$ is a unitary 1 parameter group then

$$\mathcal{U}(\lambda) = e^{-iG\lambda}$$

where $G = G^{+}$; G is independent of λ and

$$e^{-iG\lambda} = I + \sum_{n=1}^{\infty} \frac{(-iG\lambda)^n}{n!}$$

The operator G is called the <u>infinitesimal generator</u> of $\mathcal{U}(\lambda)$

proof

$$0 = \frac{d}{d\lambda} I = \frac{d}{d\lambda}\left(\mathcal{U}(\lambda)\mathcal{U}^{+}(\lambda)\right) = \frac{d\mathcal{U}}{d\lambda}(\lambda)\mathcal{U}^{+}(\lambda) + \mathcal{U}(\lambda)\frac{d\mathcal{U}^{+}}{d\lambda}(\lambda)$$

this gives

$$\frac{d\mathcal{U}}{d\lambda}(\lambda)\mathcal{U}^{+}(\lambda) = -\mathcal{U}(\lambda)\frac{d\mathcal{U}^{+}(\lambda)}{d\lambda} = -\left(\frac{d\mathcal{U}}{d\lambda}(\lambda)\mathcal{U}^{+}(\lambda)\right)^{+}$$

define

$$G(\lambda) = i\frac{d\mathcal{U}}{d\lambda}(\lambda)\mathcal{U}^{+}(\lambda)$$

then the above equation gives

$$iG(\lambda) = -\left(iG(\lambda)\right)^{+} = iG^{+}(\lambda)$$

or $\quad G(\lambda) = G(\lambda)^{+}$

To show $G(\lambda)$ is independent of λ let $\lambda' = \lambda + c$ where c is a constant.

$$\frac{d}{d\lambda'} = \frac{d\lambda}{d\lambda'}\frac{d}{d\lambda} = \frac{d}{d\lambda'}(\lambda'-c)\frac{d}{d\lambda} = \frac{d}{d\lambda}$$

$$\frac{d}{d\lambda}u(\lambda)\cdot u^{+}(\lambda) = \frac{d}{d\lambda'}u(\lambda'-c)\cdot u^{+}(\lambda'-c) =$$

$$\frac{d}{d\lambda'}u(\lambda')\cdot \underbrace{u(-c)\,u^{+}(-c)}_{I}\,u^{+}(\lambda') =$$

$$\frac{d}{d\lambda'}u(\lambda')\,u^{+}(\lambda')$$

This implies $G(\lambda) = G(\lambda')$

using the definition

$$G = i\frac{du}{d\lambda}\cdot u^{+}(\lambda) \Rightarrow$$

$$\frac{du}{d\lambda} = -iG\,u(\lambda)$$

differentiating n times

$$\frac{d^n u}{d\lambda^n} = (-iG)^n u(\lambda)$$

we can use these to construct the Taylor series

$$e^{-iG\lambda} = I + \sum_{n=1}^{\infty} \frac{(-i\lambda)^n}{n!} G^n$$

note

(1) $$\frac{d}{d\lambda}(e^{-iG\lambda}) = \sum_{n=1}^{\infty} \frac{n\lambda^{n-1}(-iG)^n}{n!} = -iG \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(-iG)^{n-1}}{(n-1)!}$$

let $m = n-1$

$$= -iG \sum_{m=0}^{\infty} \frac{(-i\lambda)^m}{m!} G^m = -iG\, e^{-iG\lambda}$$

$\therefore$ the series satisfies the differential equation and initial condition

(2) converges when applied to superpositions of eigenstates of G with maximum eigenvalue finite (this is a dense set of vectors)

Rotations about a fixed axis

In a quantum theory if we do an experiment in a rotated laboratory (isolated system) we expect to get identical quantum probabilities

$$|\psi\rangle \to |\psi'\rangle$$
$$|\phi\rangle \to |\phi'\rangle$$

$$P = |\langle\psi|\phi\rangle|^2 = |\langle\psi'|\phi'\rangle|^2$$

If this is true for every $|\psi\rangle$ $|\phi\rangle$ Wigners theorem (see notes) implies either

$$\langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle \quad \text{(unitary)}$$

$$\langle \psi' | \phi' \rangle = \langle \phi | \psi \rangle \quad \text{(antiunitary)}$$

The non trivial part is to show that these are the only possibilities

For rotations about a fixed axis

$$|\psi''\rangle \xleftarrow{T(\theta_2)} |\psi'\rangle \xleftarrow{T(\theta_1)} |\psi\rangle$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{T(\theta_1+\theta_2)}$$

which means

$$T(\theta_2)T(\theta_1) = T(\theta_1+\theta_2)\, e^{i\phi(\theta_1,\theta_2)}$$

* HW show that $T(\theta)$ with this property cannot be antiunitary

In most cases the possible phase factor can be eliminated by redefining $T(\theta)$ – the non trivial exception is transformations that shift constant velocities

$$\bar{x} \to \bar{x}' = \bar{x} - \bar{v}t$$

It then follows that for rotations about a given axis

$$|\psi\rangle \to |\psi'\rangle = \mathcal{U}(\theta)|\psi\rangle$$

where

$\mathcal{U}(\theta)$ is unitary

$\mathcal{U}(\theta_2)\mathcal{U}(\theta_1) = \mathcal{U}(\theta_1+\theta_2)$

$\mathcal{U}(0) = I$

This means for a rotation about a fixed axis $\mathcal{U}(\theta)$ is a 1 parameter unitary group, this means

$$\mathcal{U}(\theta) = e^{-i\theta G}$$

G is called the infinitesimal generator of rotations about the given axis

We can treat rotations as active or passive

passive - vector is unchanged; basis is transformed

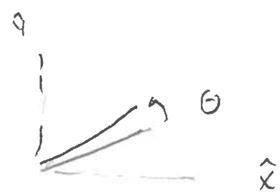
active → basis is fixed
vector changes

Following the text I will consider active transformations.

Let $\bar{V}$ be a vector operator. This means $\bar{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$ where each component of $\bar{V}$ is an operator

($\bar{x}$, $\bar{p} = -i\bar{\nabla}_x$ etc)

and the basis is fixed



$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$= \sum_{j=1}^{3} R_{ij} V_j$$

If $\bar{V}$ is an operator

$$\mathcal{U}(R)\bar{V}\mathcal{U}^{+}(R) = R^{T}\vec{V}$$

why R^{T}

$$\mathcal{U}(R_2)\mathcal{U}(R_1)\bar{V}\mathcal{U}^{+}(R_1)\mathcal{U}^{+}(R_2) =$$

$$\mathcal{U}(R_2)\, R_1^T \bar{V}\, \mathcal{U}(R_2^\dagger) = \sum R^T_{1ij}\, \mathcal{U}(R_2) \bar{V}_j\, \mathcal{U}(R_2^\dagger)$$

$$= \sum R^T_{1ij} R^T_{2jk} V_k = \sum (R_2 R_1)^T_{ij} V_j$$

(without the transpose the rotations come in the wrong order

$$e^{-iG\theta} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} e^{iG\theta} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

homework: $\frac{d}{d\theta}\Big|_{\theta=0}$

$$-i \left[G, \begin{matrix} V_x \\ V_y \\ V_z \end{matrix} \right] = \begin{bmatrix} V_y \\ -V_x \\ 0 \end{bmatrix}$$

which gives

$$[G, V_x] = i V_y$$
$$[G, V_y] = -i V_x$$
$$[G, V_z] = 0$$

(for a positive rotation (ccw) about the z axis

This exercise can be repeated for rotations about the x any y axes

$$[G_x\ V_x] = 0$$

$$[G_x\ V_y] = i V_z$$

$$[G_x\ V_z] = -i V_y$$

$$[G_y\ V_x] = -i V_z$$

$$[G_y\ V_y] = 0$$

$$[G_y\ V_z] = i V_x$$

These can be summarized if we let $V_1 = V_x$ $V_2 = V_y$ $V_3 = V_z$ and

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{213} = -\varepsilon_{321} = -\varepsilon_{132} = 1$

0 otherwise

$$[G_i\ V_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} V_k$$

These relations are the same for any operator that transforms like a vector under rotations.

If we have a Hamiltonian that is invariant with respect to rotations

$$\mathcal{U}(\theta) H \mathcal{U}^{\dagger}(\theta) = H$$

Exercise show

$$[G, H] = 0$$

which means that G is a conserved quantity; the conserved quantity for rotationally invariant systems is the angular momentum

since $\mathcal{U}(\theta) = e^{-iG\theta}$; $G\theta$ must be dimensionless for this to make sense, since θ is dimensionless $\Rightarrow$

$$G = J_z/\hbar$$

where $\hbar$ has units of angular momentum.

In what follows I use units where $\hbar = c = 1$.

$$G_i = J_i/\hbar$$

SU(2)

It is sometimes useful to represent vectors by 2x2 traceless Hermitian matrices rather than by their components

$$X = \bar{x} \cdot \bar{\sigma} = \begin{pmatrix} x^3 & x^1 - ix_2 \\ x^1 + ix_2 & -x^3 \end{pmatrix}$$

$$\bar{x} = \frac{1}{2} \mathrm{Tr}(\bar{\sigma} X)$$

properties of X

$X = X^\dagger$ (this is equivalent to saying $\bar{x} = \bar{x}^*$)

$\mathrm{Tr}(X) = 0$

$\det(X) = -\bar{x} \cdot \bar{x} = -\|\bar{x}\|^2$

These properties are preserved under transformations of the form

$$X \to X' = WXW^\dagger = WXW^\dagger$$

where $WW^\dagger = 1$ and $\det W = 1$

$$\det X' = \det W \det X \det W^\dagger$$

if $\det W = e^{i\varphi}$ $\det(W^\dagger) = e^{-i\varphi}$ and the phases cancel

also note $X \to -X$ corresponds to a space reflection, which cannot be represented by $UXU^\dagger$

Finally

$$X' = WXW^\dagger = (-W)X(-W)^\dagger$$

where both

$$\det(W) = \det(-W) = 1$$

from this we conclude

$$X' = WXW^\dagger$$

corresponds to a rotation, and there are at least 2 U's for a given 3×3 rotation

$$\bar{x}' = \frac{1}{2}\mathrm{Tr}(\bar{\sigma}X') =$$

$$= \frac{1}{2}\mathrm{Tr}(\bar{\sigma}W\bar{\sigma}\cdot\bar{x}\,W^\dagger)$$

$$= \frac{1}{2}\mathrm{Tr}(\bar{\sigma}W\sigma_i W^\dagger)x^i$$

$$= R\bar{x}$$

Homework:

show that a general SU(2) matrix can be expressed as

$$W = e^{-\frac{i}{2}\theta \hat{n}\cdot\bar{\sigma}}$$
$$= \cos\left(\frac{\theta}{2}\right) I - i\hat{n}\cdot\bar{\sigma} \sin\left(\frac{\theta}{2}\right)$$

Homework

show that a general SU(2) matrix can be expressed as

$$W = e_0 \underline{\underline{I}} + i\vec{e}\cdot\bar{\sigma}$$

where $e_0^2 + \vec{e}\cdot\vec{e} = 1$

Homework

show that $W = e^{-\frac{i}{2}\theta \hat{z}\cdot\bar{\sigma}} \Rightarrow$

$$R_{ij} = \frac{1}{2} Tr(\sigma_i W \sigma_j W^{\dagger})$$

corresponds to a rotation about the z axis through an angle θ

Angular Momentum

A general angular momentum operator satisfies

$$[J_i, J_j] = i \sum_k \epsilon_{ijk} J_k$$

Homework: show that the commutation relations imply

$$[\bar{J}^2, J_i] = 0$$

where $\bar{J}^2 = J_1^2 + J_2^2 + J_3^2$.

This means that it is possible to find simultaneous eigenstates of $\bar{J}^2$ and J_z; $|n\mu\rangle$

$$\bar{J}^2 |n\mu\rangle = n\,|n\mu\rangle$$

$$J_z |n\mu\rangle = \mu\,|n\mu\rangle$$

(normally the RHS would be multiplied by $\hbar^2$ or $\hbar$)

Define $J_\pm = J_x \pm i J_y$

Homework: show that the angular momentum commutation relation imply $[J_z J_\pm] = \pm J_\pm$.

This means

$$J_z J_\pm |n\mu\rangle = (J_\pm J_z \pm J_\pm)|n\mu\rangle =$$

$$= (\mu \pm 1) J_\pm |n\mu\rangle$$

This means that $J_\pm |n\mu\rangle$ is an eigenstate of $\bar{J}^2$ and J_z with eigenvalue $\mu \pm 1$.

Exercise: Show

$$J_\mp J_\pm = \bar{J}^2 - J_z^2 \mp J_z$$

given this we can compute the norm of $J_\pm |n\mu\rangle$

$$\| J_\pm |n\mu\rangle \|^2 = \langle n\mu | J_\mp J_\pm | n\mu\rangle =$$

$$\langle n\mu | J^2 - J_z^2 \mp J_z | n\mu\rangle =$$

$$(n - \mu^2 \mp \mu) = n - \mu(\mu \pm 1)$$

note that the norm of a vector must be positive. For fixed n if we keep increasing or decreasing μ the expression above eventually becomes negative

this means that there must be a maximum and minimum value of μ statisfying

$$J_+ |\eta\, \mu_{max}\rangle = J_- |\eta\, \mu_{min}\rangle = 0$$

this require

$$\eta - \mu_{max}(\mu_{max}+1) =$$
$$\eta - \mu_{min}(\mu_{min}-1) = 0$$

These are quadratic equations for μ_{max} in terms of μ_{min}

$$\mu_{min}^2 - \mu_{min} - \mu_{max}(\mu_{max}+1)$$

$$\mu_{min} = \frac{1}{2}\left(1 \pm \sqrt{1+4\mu_{max}^2+4\mu_{max}}\right)$$

$$= \frac{1}{2}\left(1 \pm (1+2\mu_{max})\right)$$

$$= \begin{cases} \frac{1}{2}(2)(1+\mu_{max}) = \mu_{max}+1 \\ \frac{1}{2}(2)(-\mu_{min}) = -\mu_{max} \end{cases}$$

for the first root $\mu_{min} > \mu_{max}$; so it is not the correct solution. This gives

$$\mu_{min} = -\mu_{max}$$

$$\eta = \mu_{max}(\mu_{max}+1)$$
$$= \mu_{min}(\mu_{min}-1)$$

It is customary to call

$$j \equiv \mu_{max}$$

$$\eta = j(j+1)$$

we relabel the states by

$$|j\mu\rangle := |\eta\mu\rangle$$

$$J_z|j\mu\rangle = \mu|j\mu\rangle$$

$$\vec{J}^2|j\mu\rangle = j(j+1)|j\mu\rangle$$

recall

$$\langle\eta\mu|J_\mp J_\pm|\eta\mu\rangle =$$

$$\eta - \mu(\mu\pm1) =$$

$$j(j+1) - \mu(\mu\pm1) =$$

$$(j\mp\mu)(j\pm\mu+1)$$

This means

$$\boxed{\begin{aligned} J_\pm|j\mu\rangle &= \sqrt{(j\mp\mu)(j\pm\mu+1)}\,|j\,\mu\pm1\rangle \\ &= \sqrt{j(j+1)-\mu(\mu\pm1)}\,|j\,\mu\pm1\rangle \end{aligned}}$$