Exam review

① unitary one parameter groups

$$\mathcal{U}(\lambda_2)\,\mathcal{U}(\lambda_1) = \mathcal{U}(\lambda_1+\lambda_2)$$

$$\mathcal{U}(0) = I$$

$$\mathcal{U}^{\dagger}(\lambda) = \mathcal{U}^{-1}(\lambda)$$

it follows that

$$\boxed{\begin{array}{l} \mathcal{U}(\lambda) = e^{iG\lambda} \\ G = G^{\dagger} \quad G \text{ independent of } \lambda \end{array}}$$

② rotations

passive - rotate basis

active - rotate vectors

rotations about 1 axis = unitary one parameter group

$$\mathcal{U}(R) = e^{-i\bar{\theta}\cdot\bar{G}}$$

$\bar{G}$ = dimensionless $= -\frac{\bar{J}}{\hbar}$ $(\hbar = 1)$

③ vector operators

$$\boxed{\mathcal{U}(R)\, V^i\, \mathcal{U}^\dagger(R) = \sum_j V^j R_{ji}}$$

④ SU(2)

2 = 2x2 matrix

U = unitary $\quad M^{-1} = M^\dagger$

S = $\det(M) = 1$

$\underline{\underline{X}} = \bar{x}\cdot\bar{\sigma} \qquad \bar{x} = \frac{1}{2}\,\mathrm{Tr}(\bar{\sigma}\,\underline{\underline{X}})$

$\det X = -|x|^2$

$R_{ij} = \frac{1}{2}\,\mathrm{Tr}(\sigma_i M \sigma_j M^\dagger)$

$x' = M x M^\dagger$

$M = e^{-\frac{i}{2}\bar{\sigma}\cdot\bar{\theta}} = \cos(\frac{\theta}{2})I - i\,\bar{\sigma}\cdot\hat{\theta}\,\sin(\frac{\theta}{2})$

(5) commutation relations for vectors

$$[J^i \; V^j] = i \sum_k \varepsilon^{ijk} V^k$$

$\vec{J}$ is a vector

$$[J^i \; J^j] = i \sum_k \varepsilon^{ijk} J^k$$

(6) $[J_z \; J^2] = 0$ $\quad |j\,m\rangle$ basis

$$J_\pm = J_x \pm i J_y$$

$$[J_z \; J_\pm] = \pm J_\pm$$

$$J_\mp J_\pm = J^2 - J_z^2 \mp J_z$$

spectrum

$$j = 0, \tfrac{1}{2}, 1, 1, \tfrac{3}{2} \cdots$$

$$m: \quad -j \le m \le j$$

$$J_\pm |j\,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\; |j\; m \pm 1\rangle$$

(7) Wigner functions

$\langle j\mu | \mathcal{U}(R) | j'\mu' \rangle =$

$\langle j\mu | e^{-i\bar{\theta}\cdot\bar{J}} | j'\mu' \rangle =$

$D^{j}_{\mu\mu'}(R)\,\delta_{jj'}$

$\sum D^{j}_{\mu\nu}(R_2)\, D^{j}_{\nu\alpha}(R_1) = D^{j}_{\mu\alpha}(R_2 R_1)$

$\left(D^{j}_{\mu\nu}(R)\right)^{+} = D^{j\,*}_{\nu\mu}(R) = D^{j}_{\mu\nu}(R^{+})$

homogeneous polynomial in SU(2) matrix elements with real coefficients

(8) adding angular momenta

$[J^{i}_{a}, J^{j}_{b}] = 0$

$\bar{J} = \bar{J}_a + \bar{J}_b$ satisfies

$[J_i, J_j] = i \sum \varepsilon_{ijk} J_k$

$$|jm\rangle = \sum_{m_a m_b} |j_a m_a\rangle |j_b m_b\rangle \times C^{j\ j_a\ j_b}_{m\ m_a\ m_b}$$

$$|j_a m_a j_b m_b\rangle = \sum_{jm} |jm\rangle\, C^{j\ j_a\ j_b}_{m\ m_a\ m_b}$$

$$C^{j\ j_a\ j_b}_{m\ m_a\ m_b} = \left(C^{j\ j_a\ j_b}_{m\ m_a\ m_b}\right)^*$$

$= 0$ unless

$$|j_a - j_b| \le j \le |j_a + j_b|$$

$$m = m_a + m_b$$

$$|jj\rangle = |j_a j_a\rangle |j_b j_b\rangle$$

use lowering operators to get $|jm\rangle$, use $\perp$ plus Condon Shortley convention

(9) Irreducible representations

$$\boxed{D^{j_a}_{\mu_a\nu_a}(R)\, D^{j_b}_{\mu_b\nu_b}(R) =}$$

$$\sum C^{j\, j_a\, j_b}_{\mu\, \mu_a\, \mu_b}\, D^{j}_{\mu\nu}(R)\, C^{j\, j_a\, j_b}_{\nu\, \nu_a\, \nu_b}$$

$$D^{j}_{\mu\nu}(R) = \langle j\mu |\, e^{i(\bar{J}_a+\bar{J}_b)\cdot\bar{\omega}}\, | j\nu\rangle$$

$|v\rangle = \sum |jm\rangle\, c_m \rightarrow$

no non trivial invariant

subspaces $\Rightarrow$ $D^{j}_{\mu\nu}(R)$

generates basis starting

with any vector

(10) integrals

$$\int dR = 1$$

$$\int dR\, D^{j}_{\mu\nu}(R) = \delta_{j0}\, \delta_{\mu 0}\, \delta_{\nu 0}$$

$$\int dR\, D^{j}_{\mu\nu}(R)\, D^{*j'}_{\mu'\nu'}(R) = \frac{\delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}}{2j+1}$$

$$Y^{l}_{m} = \sqrt{\frac{2l+1}{4\pi}}\, D^{*l}_{m0}(R)$$

(11) Wigner Eckart Theorem —

Tensor operators

cartesian tensors

$$\boxed{\mathcal{U}(R)\, T^{i_1 \dots i_n}\, \mathcal{U}^{\dagger}(R) = \sum T^{j_1 \dots j_n} R_{j_1 i_1} \cdots R_{j_n i_n}}$$

these are reducible —

rank 2

$\bar{V} \cdot \bar{W}$ $\quad \bar{V} \times \bar{W}$ $\quad \frac{1}{2}(V^i W^j + V^j W^i) - \frac{1}{3} \delta_{ij} \operatorname{Tr}(VW)$

spherical comp

$$\boxed{\begin{aligned} V_1^0 &= V_z \\ V_1^1 &= -\frac{1}{\sqrt{2}}(V_x + i V_y) \\ V_1^{-1} &= \frac{1}{\sqrt{2}}(V_x - i V_y) \\ \mathcal{U}(R)\, V_j^m\, \mathcal{U}^{\dagger}(R) &= \sum_{m'} V_j^{m'} D^j_{m'm}(R) \end{aligned}}$$

$$\langle j_a \mu_a | V_j^\alpha | j_b \nu_b \rangle =$$

$$\frac{1}{\sqrt{2j_a+1}} C^{j_a \, j \, j_b}_{\mu \, \alpha \, \nu_b} \langle j_a \| V^j \| j_b \rangle$$

$$\frac{\langle j_a \mu_a | V_j^\alpha | j_b \nu_b \rangle}{\langle j_a \mu_a' | V_j^{\alpha'} | j_b \nu_b' \rangle} = \frac{C^{j_a \, j \, j_b}_{\mu_a \, \alpha \, \nu_b}}{C^{j_a \, j \, j_b}_{\mu_a' \, \alpha' \, \nu_b'}}$$

(12) WKB

$$p_{cl} = \pm\sqrt{2m(E - V(x))}$$

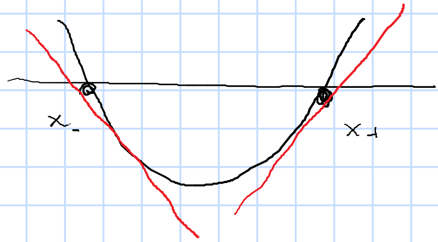
$$\psi = \frac{c}{\sqrt{|p_{cl}(x)|}} e^{\pm \frac{1}{\hbar}\int^x |p_{cl}(x)| dx} \qquad V > E$$

$$\frac{c}{\sqrt{|p_{cl}(x)|}} e^{\pm \frac{i}{\hbar}\int^x p_{cl}(x)\, dx} \qquad E > V$$

approximation fails

at classical turning

point



Linear potential

$$\frac{d^2}{d\sigma_2} Ai(\sigma) = \sigma \, Ai(\sigma)$$

$$\sigma = -\left(\frac{2mF}{\hbar^2}\right)^{1/3} (x - x_\pm)$$

$$\int_{x_-}^{x_+} p_{cl}(x)\, dx = (n+\tfrac{1}{2})\pi\hbar$$

matching conditions

$$\frac{c}{\sqrt{|p_{cl}|}} e^{\pm\frac{1}{\hbar}\int_{x_+}^{x} |p_{cl}| dx}$$

$$\frac{c}{2\sqrt{p_{cl}(x)}} \cos\left(\frac{1}{\hbar}\int_{x_-}^{x} p_{cl}(x) - \frac{\pi}{4}\right)$$

(13) Rayleigh Ritz Variational Method

$$\boxed{\begin{aligned} \langle\psi|H|\psi\rangle &\geq E_{min} \\ \langle\psi|H|\psi\rangle &\leq E_{max} \end{aligned}}$$

if there is an E_{max}

$$\langle\psi|H|\psi\rangle = E_{min}$$

$$H|\psi\rangle = E_{min}|\psi\rangle$$

$$PHP|\tilde{\Psi}_n\rangle = \tilde{E}_n|\tilde{\Psi}_n\rangle \qquad n=1\cdots N$$

$$H|\Psi_n\rangle = E_n|\Psi_n\rangle$$

$$E_n \leq \tilde{E}_n \qquad n \leq N$$

$$E_n = \tilde{E}_n \Rightarrow |\tilde{\Psi}_n\rangle = |\Psi_n\rangle$$

(14) Hellmann - Feynman

$$\frac{\partial E}{\partial \alpha} = \langle\Psi|\frac{\partial H}{\partial \alpha}|\Psi\rangle$$

Virial Theorem

$$V = \alpha r^n$$

$$\langle T\rangle = \frac{n}{2}\langle V\rangle$$

(15) non deg RS perturbation theor

$$H = H_0 + \lambda V$$

$$H_0|\Psi_n^0\rangle = E_n^0|\Psi_n\rangle$$

$$E_n = E_n^0 + \sum_{m=1} \lambda^m E_n^m$$

$$|\Psi_n\rangle = |\Psi_n^0\rangle + \sum \lambda^m |\Psi_n^m\rangle$$

$$E_n^1 = \langle \psi_n^0 | V | \psi_n^0 \rangle$$

$$E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | V | \psi_n^0 \rangle|^2}{E_n - E_m}$$

$$|\psi_n^1\rangle = \sum_{m \neq n} |\psi_m^0\rangle \frac{\langle \psi_m^0 | V | n^0 \rangle}{E_n^0 - E_m^0}$$

* see matrix example in L14