29:5742 Homework 11
Due 4/26

1. Dirac Equation: Consider an equation of the form

$$i\hbar \frac{\partial}{\partial x^0}\psi(x) = (-i\hbar \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc)\psi(x)$$

where $\boldsymbol{\alpha}$ and β are constant Hermitian matrices

a Find conditions on $\boldsymbol{\alpha}$ and β for the solution $\psi(x)$ to satisfy the Klein Gordon equation.

b. Show that the matrices that solve part a must be even dimensional.

c. Show that the conditions from part a cannot be satisfied using 2×2 matrices.

2. Show, using explicit Poincaré transformations, that time translation can be expressed as a finite combination of space translations and rotationless Lorentz transformations.

3. Show for

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma} & 0 \end{pmatrix}$$

where $\tilde{\sigma} = (I, -\boldsymbol{\sigma})$ satisfies

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\eta_{\mu\nu}$$

4. Compute the anticommutator

$$\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu.$$

5. Show that

$$v(p) = S(B(p))v_+(0)$$

where

$$v_+(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

is a negative energy solution of the Dirac equation.

6. Show that

$$(\sum_\mu \gamma_\mu p^\mu - mcI)(\sum_\nu \gamma_\nu p^\nu + mcI) = 0.$$

This means that it is possible to find solutions to the soltions to the Dirac equation by applying $(\sum_\mu \gamma_\mu p^\mu + mcI)$ to any constant 4 component vector.

① $i\hbar \frac{\partial}{\partial x_0} \Psi = (-i\hbar \bar{\nabla}\cdot\bar{\alpha} + \beta mc)\Psi$

ⓐ $-\hbar^2 \frac{\partial^2}{\partial x_0^2} \Psi = \Big(-\hbar^2 \sum_{ij} \alpha^i \alpha^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + \beta^2 m^2 c^2$

$-i\hbar mc \sum (\alpha_i \beta + \beta \alpha_i)\nabla_i \Big)\Psi$

In order to get to the Klein Gordon equation we must have

$\frac{1}{2}(\alpha^i \alpha^j + \alpha^j \alpha^i) = \delta_{ij}$

$\alpha_i \beta + \beta \alpha_i = 0$

$\beta^2 = 1$

with these conditions

$\left(-\hbar^2 \frac{\partial^2}{\partial x_0^2} + \hbar^2 \nabla^2 - m^2 c^2\right)\Psi = 0$

This means α_i and β must be anticommuting matrices satisfying

$\alpha_i^2 = \beta^2 = 1$

$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ $\{\alpha_i \beta\} = 0$ $\{\beta, \beta\} = 2$

ⓑ show that $\bar{\alpha}, \beta$ must be even dimensional

since $\alpha_i = \alpha_i^\dagger$ $\beta = \beta^\dagger$ and $\alpha_i^2 = \beta^2 = 1$

they have real eigenvalues that are $+1$ or -1

since $Tr(\alpha_i) = Tr(\alpha_i \beta^2) = Tr(\beta \alpha_i \beta)$

$= Tr(-\alpha_i \beta^2) = -Tr(\alpha_i) \qquad (\alpha_i \beta = -\beta \alpha_i)$

$\therefore Tr(\alpha_i) = 0$ so it must have the same # of 1's and -1's.

$\therefore$ The dimension must be even

(c) For 2x2 there are only 3 independent traceless symmetric matrices $\vec{\sigma}$.

(2) $(\Lambda_1 a_1)(\Lambda_2 a_2) = (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1)$

choose $a_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X_3 \end{pmatrix}$

choose $\Lambda_1 = \begin{pmatrix} \cosh\rho & 0 & 0 & \sinh\rho \\ 0 & 1 & 0 & 0 \\ \sinh\rho & 0 & 0 & \cosh\rho \end{pmatrix}$

$\Lambda_1 a_2 = \begin{pmatrix} \sinh\rho \, X_3 \\ 0 \\ 0 \\ \cosh\rho \, X_3 \end{pmatrix}$

Choose $a_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ X^3 (1 - \cosh\rho) \end{pmatrix}$

Choose $\Lambda_2 = \Lambda_1^{-1}$

$(\Lambda_1 a_1)(\Lambda_2 a_2) = \left(I, \begin{pmatrix} \sinh\rho \, x^3 \\ 0 \\ 0 \\ x^3 \end{pmatrix} \right)$ ← Time shifts X^3 is unchanged

3 $$\gamma_\mu \gamma_\nu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_\nu \\ \tilde{\sigma}_\nu & 0 \end{pmatrix} = \begin{pmatrix} \sigma_\mu \tilde{\sigma}_\nu & 0 \\ 0 & \tilde{\sigma}_\mu \sigma_\nu \end{pmatrix}$$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = \begin{pmatrix} \sigma_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu & 0 \\ 0 & \tilde{\sigma}_\mu \sigma_\nu + \tilde{\sigma}_\nu \sigma_\mu \end{pmatrix}$$

cases $\mu = \nu = 0$

$$\gamma_0 \gamma_0 + \gamma_0 \gamma_0 = \begin{pmatrix} I + I & 0 \\ 0 & I + I \end{pmatrix} = 2I$$

$$\gamma_i \gamma_i + \gamma_i \gamma_i = \begin{pmatrix} -2\sigma_i^2 & 0 \\ 0 & -2\sigma_i^2 \end{pmatrix} = -2I$$

$$\gamma_0 \gamma_i + \gamma_i \gamma_0 = \begin{pmatrix} -I\sigma_i + \sigma_i I & 0 \\ 0 & I\sigma_i - \sigma_i I \end{pmatrix} = 0$$

$$\gamma_i \gamma_j + \gamma_j \gamma_i = \begin{pmatrix} -\sigma_i \sigma_j - \sigma_j \sigma_i & 0 \\ 0 & -\sigma_i \sigma_j - \sigma_j \sigma_i \end{pmatrix} = 0 \qquad i \neq j$$

$$\therefore \{\gamma_\mu \gamma_\nu\} = -2\eta_{\mu\nu}$$

④ for the commutator from the above

$$\gamma_0 \gamma_i - \gamma_i \gamma_0 = 2\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

$$\gamma_i \gamma_j - \gamma_j \gamma_i = 2\begin{pmatrix} -\sigma_i \sigma_j & 0 \\ 0 & -\sigma_i \sigma_j \end{pmatrix} = -2\,\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

(5) consider

$$S(B(p))\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}B(p) & 0\\ 0 & B^{-1}(p)\end{pmatrix}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} := v(p)$$

$$\left(\sum \gamma_\mu p^\mu - mc\right) S(B(p))\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} =$$

$$S(B(p))\, S^{-1}(B(p))\left(\sum \gamma_\mu p^\mu - mc\right) S(B(p))\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} =$$

$$S(B(p))\Big(\sum \gamma_\mu \underbrace{S^{-1}(B(p))\gamma_\mu S(B(p))}_{\sum_\nu \gamma_\nu \Lambda^\nu{}_\mu (B^{-1}(p))\, p^\mu} p^\mu - mc\Big)\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

$$S(B(p))\begin{pmatrix} -mc & p_0(\text{rest}) \\ p_0(\text{rest}) & -mc\end{pmatrix}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

$$S(B(p))\begin{pmatrix} -mc - p_0(\text{rest}) \\ p_0(\text{res}) + mc\end{pmatrix}$$

This will vanish if $p_0(\text{rest}) = -mc$

$$p^\mu = B(p)^\mu{}_\nu \begin{pmatrix}-mc\\0\\0\\0\end{pmatrix} = \begin{pmatrix}-\sqrt{\vec{p}+mc^2}\\ -p_x\\ -p_y\\ -p_z\end{pmatrix}$$

which gives a negative energy solutn

(6)

$$\sum_{\mu}(\gamma_\mu p^\mu - mcI)\sum_{\nu}(\gamma_\nu p^\nu - mcI) =$$

$$\left(\sum_{\mu\nu} p^\mu p^\nu \gamma_\mu \gamma_\nu - m^2c^2\right) =$$

$$\left(\frac{1}{2}\sum_{\mu\nu} p^\mu p^\nu \{\gamma_\mu \gamma_\nu\} - m^2c^2\right) =$$

$$\left(-2\eta_{\mu\nu} p^\mu p^\nu - m^2c^2\right) =$$

$$\left((p^0)^2 - \bar{p}^2 - m^2c^2\right) = 0$$