29:5742 Homework 10
Due 4/19

1. Let $f^\mu(x)$ be a four vector valued function of a four vector $x = x^\nu$ satisfying

$$\sum_{\mu,\nu} \eta_{\mu\nu}(f(x)^\mu - f^\mu(y))(f(x)^\nu - f^\nu(y)) = \sum_{\mu,\nu} \eta_{\mu\nu}(x^\mu - y^\mu)(x^\mu - y^\nu)$$

for all x and y. Show that

$$f^\mu(x) = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

where $\Lambda^\mu{}_\nu$ is a constant matrix satisfying

$$\sum_{\mu,\nu} \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

2. Let $X = \sum_{\mu=0}^3 x^\mu \sigma_\mu$ where $\sigma_0 = I$ and σ_i are the three Pauli matrices. Show that

$$x^\mu = \frac{1}{2}\mathrm{Tr}(\sigma^\mu X)$$

3. For $A = e^{\frac{\theta}{2}\sigma_z}$ compute

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma^\mu A \sigma_\nu A^\dagger)$$

4. Assume that f^μ transforms like a 4 vector under Lorentz transformations and has the form $(0, \mathbf{F})$ in the particles instantaneous rest frame $p^\mu_{rest} = (mc, 0, 0, 0)$ and assume

$$m\frac{dp^\mu}{d\tau} = m\frac{d^2x^\mu}{d\tau^2} = f^\mu$$

is a four vector equation. Find the form of f^μ is a frame where the particle has linear momentum p in the z direction.

5. Show that a general $SL(2, \mathbb{C})$ matrix A can be expressed as the product of a unitary, U, and positive matrix, P (positive eigenvalues):

$$A = UP = P'U'$$

This means that any Lorentz transformation connected to the identity can be expressed as the product of a rotation and a rotationless Lorentz transformation.

6. Let Λ be a rotationless Lorentz transformation that transforms from a frame where a particle is initially at rest to one where it is moving at half the speed of light in the z-direction. If this same transformation is applied again what is the resulting velocity of the particle in the new frame?

① $\sum_{\mu\nu} \eta_{\mu\nu} (x^\mu - y^\mu)(x^\nu - y^\nu) =$

$\sum_{\mu\nu} \eta_{\mu\nu} (f^\mu(x) - f^\mu(y))(f^\nu(x) - f^\nu(y))$

use $\frac{\partial}{\partial x^\alpha}$ set $x^\alpha = 0$

$\sum \eta_{\mu\nu} \left[\left(\frac{\partial x^\mu}{\partial x^\alpha} \right)(-y^\nu) + \frac{\partial x^\nu}{\partial x^\alpha}(-y^\mu) \right] =$

$\sum \eta_{\mu\nu} \left[\frac{\partial f^\mu}{\partial x^\alpha}(0) \, (f^\nu(0) - f^\nu(y)) + \frac{\partial f^\nu}{\partial x^\alpha}(0) \, (f^\mu(0) - f^\mu(y)) \right]$

since $\eta_{\mu\nu} = \eta_{\nu\mu}$ this becomes

$-2 \sum \eta_{\mu\nu} \, \delta^\mu_\alpha \, y^\nu = 2 \eta_{\mu\nu} \frac{\partial f^\mu}{\partial x^\alpha}(0) \, (f^\mu(0) - f^\mu(y))$

$$\boxed{\eta_{\alpha\nu} y^\nu = \sum_{\mu\nu} \frac{\partial f^\mu}{\partial x^\alpha}(0) \, (f^\nu(y) - f^\nu(0)) \, \eta_{\mu\nu}} \quad ①$$

differentiate again with respect to y^β setting $y^\beta = 0$

$\eta_{\alpha\nu} \frac{\partial y^\nu}{\partial y^\beta} = \sum \frac{\partial f^\mu}{\partial x^\alpha}(0) \frac{\partial f^\nu}{\partial y^\beta}(0) \, \eta_{\mu\nu}$

$$\boxed{\eta_{\alpha\beta} = \frac{\partial f^\mu}{\partial x^\alpha}(0) \, \eta_{\mu\nu} \frac{\partial f^\nu}{\partial x^\beta}(0)} \quad ②$$

If we write these as matrix equations ① ⟹ equation ②

$\Lambda^\mu{}_\alpha = \frac{\partial f^\mu}{\partial x^\alpha}(0) \quad a^\mu = f^\mu(0)$

these equation become

$$\underline{\underline{\eta}}\, y = \underline{\underline{\Lambda}}^T \underline{\underline{\eta}}\, (f(y) - a)$$

$$\underline{\underline{\eta}} = \underline{\underline{\Lambda}}^T \eta\, \underline{\underline{\Lambda}}$$

since $\eta^2 = I$

$$I = \eta^2 = \eta \Lambda^T \eta \Lambda \qquad \Lambda^{-1} = \eta \Lambda^T \eta$$

multiply the first equation by η

$$\eta^2 y = \eta \Lambda^{-1} \eta\, (f(y) - a) = \Lambda^{-1}(f(y) - a) = y$$

$$(f(y) - a) = \Lambda y \qquad a$$

$$\boxed{f(y) = \Lambda y + a \implies f^\mu(y) = \sum \Lambda^\mu{}_\alpha y^\alpha + a^\mu}$$

for the second equation note

$$\boxed{\eta_{\alpha\beta} = \Lambda^\mu{}_\alpha\, \eta_{\mu\nu}\, \Lambda^\nu{}_\beta}$$

the only assumption that we made was $\tilde{f}(x)$ was differentiable

② $\frac{1}{2} Tr(\sigma_\mu X) = \frac{1}{2} Tr(\sigma_\mu \sum_\nu x^\nu \sigma_\nu)$

$$= \frac{1}{2} \sum_\nu Tr(\sigma_\mu \sigma_\nu) x^\nu$$

$$Tr(\sigma_0 \sigma_i) = Tr(\sigma_i \sigma_0) = Tr(\sigma_i) = 0$$
$$Tr(\sigma_0^2) = Tr(I) = 2$$
$$Tr(\sigma_i \sigma_j) = \sum i \varepsilon_{ijk} Tr(\sigma_k) = 0$$
$$Tr(\sigma_i \sigma_i) = Tr(I) = 2$$

; $\mathrm{Tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}$

$\frac{1}{2}\mathrm{Tr}(\sigma_\mu X) = \frac{1}{2}\mathrm{Tr}(\sum x^\nu \sigma_\mu \sigma_\nu)$

$= \frac{1}{2}\mathrm{Tr}(\sigma_\mu \sigma_\nu) x^\nu = \frac{2}{2}\sum_\nu \delta_{\mu\nu} x^\nu = x^\mu$

③ $\frac{1}{2}\mathrm{Tr}(\sigma_\mu e^{\frac{\rho}{2}\sigma_z} \sigma_\nu e^{\frac{\rho}{2}\sigma_z}) =$

$\frac{1}{2}\mathrm{Tr}(\sigma_\mu (I\cosh\frac{\rho}{2} + \sigma_z \sinh\frac{\rho}{2})\sigma_\nu (I\cosh\frac{\rho}{2} + \sigma_z\sinh\frac{\rho}{2}))$

consider cases

① $\mu = \nu = 0$

$\frac{1}{2}\mathrm{Tr}(I\cosh^2\frac{\rho}{2} + I\sinh^2\frac{\rho}{2} + 2\sigma_z \cosh\frac{\rho}{2}\sinh\frac{\rho}{2}) =$

$\frac{2}{2}(\cosh^2\frac{\rho}{2} + \sinh^2\frac{\rho}{2}) = 2\cosh^2\frac{\rho}{2} - 1$

recall $\cosh\frac{\rho}{2} = \sqrt{\frac{\cosh\rho + 1}{2}}$

$2\cosh^2\frac{\rho}{2} = \cosh\rho + 1$

this means $\Lambda^0{}_0 = \cosh\rho$

② $\mu = 0\ \nu = 3$ or $\nu = 0\ \mu = 3$

$\frac{1}{2}\mathrm{Tr}((\cosh\frac{\rho}{2} + \sigma_z\sinh\frac{\rho}{2})\sigma_z(\cosh\frac{\rho}{2} + \sigma_z\sinh\frac{\rho}{2}))$

$\frac{1}{2}\mathrm{Tr}(\sigma_z(\cosh^2\frac{\rho}{2} + \sinh^2\frac{\rho}{2}) + I(\cosh\frac{\rho}{2}\sinh\frac{\rho}{2} + \sinh\frac{\rho}{2}\cosh\frac{\rho}{2}))$

$\frac{1}{2}\cdot 2 \times 2\cosh\frac{\rho}{2}\sinh\frac{\rho}{2} = \sinh\rho$

③ $\mu = \nu = i \neq 3$

$$\frac{1}{2} \mathrm{Tr}\left(\left(\cosh\frac{\rho}{2} + \sigma_z \sinh\frac{\rho}{2}\right) \sigma_i \left(\cosh\frac{\rho}{2} + \sigma_z \sinh\frac{\rho}{2}\right) \sigma_i \right)$$

the only non zero terms a

$$\frac{1}{2} \mathrm{Tr}\left(\sigma_i^2 \cosh^2\frac{\rho}{2} - \sigma_i^2 \sinh^2\frac{\rho}{2} \right)$$

$$\frac{1}{2} \cdot 2 \left(\cosh^2\frac{\rho}{2} - \sinh^2\frac{\rho}{2}\right) = 1$$

④ $\mu = i \quad \nu = j \quad i \neq j$

$$\frac{1}{2} \mathrm{Tr}\left(\left(\cosh\frac{\rho}{2} + \sigma_z \sinh\frac{\rho}{2}\right) \sigma_i \left(\cosh\frac{\rho}{2} + \sigma_z \sinh\frac{\rho}{2}\right) \sigma_j \right)$$

the only terms that could give a non zero result -

$$\frac{1}{2} \mathrm{Tr}\left(\sinh\frac{\rho}{2} \cosh\frac{\rho}{2} \left(\sigma_z \sigma_i \sigma_j + \sigma_i \sigma_z \sigma_j\right) \right)$$

but if this vanishes if any of σ_i or $\sigma_j = \sigma_z$ and if they are not $i, j \neq z$ $\sigma_i \sigma_z \sigma_j = - \sigma_z \sigma_i \sigma_j$ so the cancel

so these terms do not contribute

$$\therefore \Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\rho & 0 & 0 & \sinh\rho \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\rho & 0 & 0 & \cosh\rho \end{pmatrix}$$

(4) $p^\mu = \gamma m v \qquad \gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

recall

$$\frac{c^2 \Delta \tau^2}{c \Delta t^2} = \frac{c^2 \Delta t^2}{c^2 \Delta t^2} - \frac{\Delta x^2}{c^2 \Delta t^2}$$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{v^2}{c^2} \qquad \frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \qquad \frac{dt}{d\tau} = \gamma$$

$$\frac{dp}{d\tau} = \frac{dt}{d\tau}\frac{dp}{dt} = \gamma \cdot \frac{d}{dt}(\gamma m v) = \gamma^2 m \frac{dv}{dt} + \gamma \frac{d\gamma}{dt} m v$$

but $\dfrac{d\gamma}{dt} = -\dfrac{1}{2}\left(\dfrac{1}{\sqrt{1 - v^2/c^2}}\right)^3 \left(-\dfrac{2\,\bar{v} \cdot \frac{dv}{dt}}{c^2}\right)$

$$= \gamma^3 \frac{\bar{v} \cdot}{c^2} \frac{d\bar{v}}{dt}$$

$\therefore \dfrac{dp}{d\tau} = \gamma^2 m \bar{a} + \gamma^4 \dfrac{\bar{v} \cdot \bar{a}}{c^2} m \bar{v} \qquad$ where $\bar{a} = \dfrac{d\bar{v}}{dt}$

when $\bar{v} = 0$

$$\frac{d\bar{p}}{d\tau} = 1 \cdot m\bar{a} + 1 \cdot \frac{0 \cdot \bar{a}}{c^2} \cdot m\bar{0} = m\bar{a}$$

$$\frac{dp^0}{d\tau} = \frac{d}{d\tau}(\gamma m c) = \frac{dt}{d\tau} \cdot \frac{d\gamma}{dt} m c = \gamma^4 \frac{\bar{v} \cdot a}{c^2} m c$$

which vanishes when $\bar{v} = 0$

$$\left.\left(\frac{dp^\mu}{d\tau}\right)\right|_{rest} = (0, m\bar{a})$$

In a general frame

$$\frac{dp^\mu}{d\tau} = \sum B(\eta)^\mu{}_\nu \left(\frac{dp^\nu}{d\tau}\right)_{rest}$$

$$= \sum B(\eta)^\mu{}_\nu \cdot (0, m\bar{a})$$

$$\frac{dp^0}{d\tau} = \sum B(\eta)^0{}_i \, m a_i = f^0$$

$$\frac{d\bar{p}}{d\tau} = \sum_j B^i(\eta)^i{}_j \, m a_j = f^i$$

for motion in the z direction

$$\frac{dp^0}{d\tau} = \sinh\eta \; m a_z$$

$$\frac{d\bar{p}}{d\tau} = \hat{z} \cosh\eta \; m a_z + \hat{x} \, m a_x + \hat{y} \, m a_y$$

in terms of η $\quad \cosh\eta = \frac{p^0}{mc} = \frac{\gamma m c}{mc} = \gamma$

$$\sinh\eta = \frac{p}{mc} = \frac{m\gamma v}{mc} = \frac{\gamma v_z}{c}$$

$$\frac{dp^0}{d\tau} = \frac{\gamma v_z}{c} \cdot m a_z = \gamma \frac{\bar{v}}{c} \cdot (m\bar{a}) = \gamma \frac{\bar{v}}{c} \cdot \bar{F}$$

$$\frac{d\bar{p}}{d\tau} = \gamma \frac{\bar{v} \cdot \bar{F}}{v^2} \bar{v} + F_\perp$$

$$= \gamma \frac{\bar{v} \cdot F}{v^2} \bar{v} + \left(F_\perp + \frac{\bar{v} \cdot F}{v^2} \bar{v} - \frac{\bar{v} \cdot F}{v^2} \bar{v}\right)$$

$$= F + (\gamma - 1) \frac{\bar{v} \cdot F}{v^2} \bar{v}$$

(5) In general for $\det A = 1$

$A^\dagger A$ and $AA^\dagger$ are hermitian with det 1 and have positive eigenvalues $\lambda \cdot \frac{1}{\lambda} = 1$ $\lambda > 0$

$$P = (AA^\dagger)^{1/2} \quad P' = (A^\dagger A)^{1/2}$$

then

$$A = (AA^\dagger)^{\frac{1}{2}} (AA^\dagger)^{-1/2} A = A (A^\dagger A)^{-1/2} (A^\dagger A)$$

(1) both $(AA^\dagger)^{1/2}$ and $(A^\dagger A)^{1/2}$ are positive Hermitian

(2) $\mathcal{U} = (AA^\dagger)^{-1/2} A$

$$\mathcal{U}\mathcal{U}^\dagger = (AA^\dagger)^{-1/2} (AA^\dagger) (AA^\dagger)^{-1/2} = 1$$

$$\mathcal{U}' = A (A^\dagger A)^{-1/2}$$

$$\mathcal{U}'^\dagger \mathcal{U}' = (A^\dagger A)^{-1/2} A^\dagger A (A^\dagger A)^{-1/2} = 1$$

which shows

$$A = P\mathcal{U} = \mathcal{U}' P'$$

(6) use the z direct

$$\begin{pmatrix} \gamma & 0 & 0 & \gamma \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c \\ 0 \\ 0 \\ \gamma v \end{pmatrix}$$

If we do this a second time

$$\begin{pmatrix} \gamma & 0 & 0 & \gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma c \\ 0 \\ 0 \\ \gamma v \end{pmatrix} = \begin{pmatrix} \gamma^2 c + \gamma^2 \frac{v^2}{c^2} \\ 0 \\ 0 \\ \gamma^2 v + \gamma^2 v \end{pmatrix}$$

the new γ is

$$\gamma' = \gamma^2 \left(1 + \frac{v^2}{c^2}\right) = \frac{1 + v^2/c^2}{1 - v^2/c^2} = \frac{c^2+v^2}{c^2-v^2}$$

$$\gamma' = \frac{1}{\sqrt{1 - v'^2/c^2}} \qquad \gamma'^2 = \frac{1}{1 - v'^2/c^2} = \left(\frac{c^2+v^2}{c^2-v^2}\right)^2$$

$$1 - \frac{v'^2}{c^2} = \left(\frac{c^2-v^2}{c^2+v^2}\right)^2$$

$$\frac{v'^2}{c^2} = 1 - \left(\frac{c^2-v^2}{c^2+v^2}\right)^2 = \frac{(c^2+v^2)^2 - (c^2-v^2)^2}{(c^2+v^2)^2}$$

$$= \frac{2c^2v^2 + 2c^2v^2}{(c^2+v^2)^2}$$

$$\boxed{v' = c\frac{2cv}{c^2+v^2}}$$