29:5742 Homework 9
Due 4/5

1. Assume that an electron scatters off of a potential due to a spherically symmetric electric charge density, $-e\rho(\mathbf{r})$. Find the scattering amplitude and differential cross section in the Born approximation. Show how these are related to the Fourier of this charge distribution.

2. The S operator must be unitary. On way to ensure this is to write

$$S = \frac{I - iK}{1 + iK}$$

where $K = K^\dagger$. Find the relation between matrix elements of $K(E)$ and matrix elements of the transition operator $T(E + i\epsilon)$. Find an integral equation for $\langle \mathbf{k}'|K(E)|\mathbf{k}\rangle$ in terms of the potential. How is $K(E)$ related to the phase shift.

3. Calculate the differential cross section (in terms of the transition operator matrix elements) for two-body scattering in the laboratory frame where particle 1 of mass m_1 is initially at rest, particle 2 of mass m_2 is initially moving with momentum $\mathbf{p} = \mathbf{p}_2$ and the angular distribution of particle 2 is measured. (hint - you need to find the initial relative velocity in this frame and integrate over all variables that are not measured.

4. Consider scattering of a neutron off of a proton (with radius of about $10^{-15}m$). For a 10 MeV bean of neutrons, how many partial waves would you need to accurately describe this scattering reaction?

5. Find the $l = 0$ partial wave scattering wave function $\langle r, 0, 0|k^-, 0, 0\rangle$ for the delta shell potential, $V(r) = -V_0 R\delta(r - R)$.

6. Find the $l = 0$ scattering amplitude for the potential of problem 5.

(1) $\frac{d\sigma}{d\Omega} = |F|^2$

$$F = -(2\pi)^2 \mu\hbar \langle \bar{k}' | V | \bar{k} \rangle$$

$$= -(2\pi)^2 \mu\hbar \frac{1}{(2\pi\hbar)^3} \int e^{-i(\bar{k}'-\bar{k})\cdot\bar{r}/\hbar} \int \frac{-e^2 \rho(\bar{r}')}{|r-r'|} d^3r\, d^3r'$$

Let $r'' = r - r'$ $\quad d^3r'' = d^3r$

$$= \frac{(2\pi)^2 e^2 \mu\hbar}{(2\pi\hbar)^3} \int e^{-i(\bar{k}'-\bar{k})\cdot(\bar{r}''+\bar{r}')/\hbar} \frac{\rho(r')}{|r''|} d^3r''\, d^3r'$$

$$= \frac{e}{(2\pi\hbar)^{3/2}} \int \rho(r') e^{-i(\bar{k}'-\bar{k})\cdot\bar{r}'/\hbar} d^3r' \times \int \frac{(2\pi)^2 e\mu\hbar}{(2\pi\hbar)^{3/2}} \frac{e^{-i(\bar{k}'-\bar{k})\cdot\bar{r}''/\hbar}}{r''} d^3r''$$

If we define $\bar{q} = \bar{k}' - \bar{k}$ the first integral is the Fourier transf of the potential $\tilde{\rho}(\bar{q})$

The second term is

$$(2\pi\hbar)^{3/2} \times \underbrace{\left(-(2\pi)^2 \mu\hbar\right) \langle \bar{k}' | \left(-\frac{e}{r}\right) | \bar{k} \rangle}_{\text{Coulomb scattering amplitude in the Born Approximation}}$$

$$F = (2\pi\hbar)^{3/2} \tilde{\rho}(\bar{q})\, F_{\text{Coulomb}}(\bar{q})$$

↑ point charge

② $S = \frac{1-iK}{1+iK}$ $\quad S = e^{2i\delta}$

First note

$$(1+iK)S = 1-iK$$

$$S-1 = -iK(1+S)$$

$$K = i\,\frac{S-1}{S+1} = i\,\frac{e^{2i\delta}-1}{e^{2i\delta}+1} = i\,\frac{e^{i\delta}}{e^{i\delta}}\,\frac{(e^{i\delta}-e^{-i\delta})\,1/2i}{(e^{i\delta}+e^{-i\delta})/2}$$

$$= -\frac{\sin\delta}{\cos\delta}$$

as an operator

$$\boxed{K = -\tan\delta}$$

Relation with T - In the energy basis $\quad |\vec{k}\rangle = |k\,\theta\,\phi\rangle \quad |E\hat{k}\rangle = |E\theta\phi\rangle$

$$|k\,\theta\,\phi\rangle\sqrt{k^2\frac{dk}{dE}} = |k\theta\phi\rangle k\mu = |E\theta\phi\rangle$$

In the E basis

$$S-1 = -iK(1+S)$$

$$(\cancel{1}-2\pi i T-\cancel{1}) = -iK(1+S)$$

$$-2\pi i T = -iK(1+1-2\pi i T)$$

$$= -2iK - 2\pi K T$$

$$T = \frac{1}{\pi}K - \frac{1}{i}KT = \frac{1}{\pi}K + iKT$$

$$T = \frac{1}{\pi}K + iKT$$

as an equation in the energy basis this has the form

$$\langle \hat{k}' E \| T(E) \| \hat{k} E \rangle = \frac{1}{\pi} \langle \hat{k}' E \| K(E) \| \hat{k} E \rangle$$
$$+ i \int \langle \hat{k} E' \| K(E) \| \hat{k}'' E \rangle \, d\Omega(\hat{k}'') \langle \hat{k}'' E \| T(E) \| \hat{k} E \rangle$$

changin to the k basis gives

$$\langle \bar{k}' \| T(E) \| \bar{k} \rangle = \frac{1}{\pi} \langle \bar{k}' \| K(E) \| k \rangle$$
$$+ i \int \langle \bar{k}' \| K(E) \| \bar{k}'' \rangle \, \mu k'' \, d\Omega(\hat{k}'') \langle \bar{k}'' \| T(E) \| \bar{k} \rangle$$

$$\boxed{\langle k'' \| T(E) \| k \rangle = \frac{1}{\pi} \langle \bar{k}' \| K(E) \| \bar{k} \rangle + i \int \langle \bar{k}' \| K(E) \| \bar{k}'' \rangle \, d^3 k'' \, \delta(E'' - E) \langle \bar{k}'' \| T(E) \| \bar{k} \rangle}$$

note with partial waves this becomes an algebraic equation

To relate K to V it is useful to consider the abstract equations

$$(1) \quad T = \frac{1}{\pi} K + i K \delta T \qquad \delta = \delta(E - H_0)$$

$$(2) \quad T = V + V R T \qquad R = (E - H_0)^{-1}$$

we want to eliminate T. It is useful to express the first equation as

$$(1') = T = \frac{1}{\pi} K + \frac{1}{\pi} K (i \pi \delta) T$$

and the second equation is

$$T = V + V\,(R - i\pi\delta + i\pi\delta)\,T$$

$$(1 - V(R - i\pi\delta))\,T = V + V\,i\pi\delta\,T$$

$$T = (1 - V(R - i\pi\delta))^{-1}V + (1 - V(R - i\pi\delta))^{-1}V\,i\pi\delta\,T$$

comparing these equations

$$\frac{1}{\pi}K = (1 - V(R - i\pi\delta))^{-1}V \quad \text{or}$$

$$\frac{1}{\pi}K - \frac{1}{\pi}V(R - i\pi\delta)K = V$$

or

$$\boxed{K = \pi V + V\,(R - i\pi\delta)\,K}$$

Remark $\frac{1}{E - H_0 + i\epsilon} - i\pi\delta(E - H_0)$ is called

the principal value $P\left(\frac{1}{E - H_0 + i\epsilon}\right)$

③ $$d\sigma = \frac{(2\pi)^4 \hbar^2}{V} |\langle p_1' p_2' \| T(E+i\epsilon) \| p_1 p_2 \rangle|^2 \times d^3p_1' d^3p_2' \times$$

$$\delta(E-E')\delta(P-P')$$

In the lab frame $\bar{p}_1 = 0$ $\bar{p}_2 = m_2 V$

$$E' - E = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2} - \frac{p_2^2}{2m_2} \qquad \bar{p}_1' + \bar{p}_2' - \bar{p}_2 = 0$$

If we want to measure $\bar{p}_2'$ integrate over $\bar{p}_1'$ — this eliminates the $\delta(\bar{p}_1' + \bar{p}_2' - p_2)$

$$E' - E = \frac{p_2'^2}{2m_2} - \frac{p_2^2}{2m_2} + \frac{(p_2 - p_2')^2}{2m_1}$$

$$= \frac{p_2'^2}{2m_2} + \frac{p_2'^2}{2m_1} - \frac{p_2^2}{2m_2} + \frac{p_2^2}{2m_1} - \frac{\bar{p}_2 \cdot \bar{p}_2'}{m_1} = 0$$

$$\frac{dE'}{dp_2'} = \left(\frac{p_2'}{m_2} + \frac{p_2'}{m_1} - \frac{p_2 \cos\theta}{m_1}\right) = \left(\frac{p_2'}{\mu} - \frac{p_2 \cos\theta}{m_1}\right)$$

$$\frac{dp_2'}{dE'} = \frac{1}{\frac{p_2'}{\mu} - \frac{p_2 \cos\theta}{m_1}} = \frac{m_1 \mu}{m_1 p_2' - \mu p_2 \cos\theta} = \frac{\frac{m_1^{\cancel{2}} m_2}{m_1 + m_2}}{\cancel{m_1} p_2' - \frac{\cancel{m_1} m_2}{m_1 + m_2} p_2 \cos\theta}$$

$$= \frac{m_2 m_1}{(m_1 + m_2) p_2' - m_2 p_2 \cos\theta}$$

$$\frac{d\sigma}{d\Omega(p_2')} = \frac{(2\pi)^4 \hbar^2}{(p_2/m_2)} |\langle p_1' p_2' \| T(E+i\epsilon) \| \bar{0}, \bar{p}_2 \rangle|^2 \frac{m_2 m_1}{(m_1 + m_1) p_2' - p_2 m_2 \cos\theta}$$

$$\boxed{\frac{d\sigma}{d\Omega(p_2')} = \frac{(2\pi)^4 \hbar^2 m_2^2 m_1 |\langle \bar{p}_1' \bar{p}_2' \| T(E+i\epsilon) \| 0\, \bar{p}_2 \rangle|^2}{(m_1 + m_2) p_2 p_2' - p_2^2 m_2 \cos\theta}}$$

④ Classically the maximum orbital momentum is about

$R = 10^{-15}\,m = 1\,fm$

$p = 10\ MeV/c$

$$\frac{L}{\hbar} = \frac{10\ MeV/c\ \ 1\,fm}{197\ MeV\,fm/c} = .05$$

This suggest $l=0$ or $l=0,1$ should be sufficient to treat scattering at this momentum.

⑤ $$\langle r0|k_-0\rangle = \frac{4\pi i^0}{(2\pi\hbar)^{3/2}} j_0\left(\frac{kr}{\hbar}\right) - \frac{(4\pi)^2}{(2\pi\hbar)^3}\pi k\mu \int j_0\left(\frac{kr_<}{k}\right) h_0^+\left(\frac{kr_>}{\hbar}\right)$$

$$\times(-V_0 R)\delta(r'-R)\,r'^2\langle r0|k_-0\rangle$$

$$= \frac{4\pi}{(2\pi\hbar)^{3/2}} j_0\left(\frac{kr}{\hbar}\right) + \frac{(4\pi)^2\pi k\mu\, V_0 R^3}{(2\pi\hbar)^3} \times$$

$$j_0\left(\frac{kr_<}{\hbar}\right) h_0^+\left(\frac{kr_>}{\hbar}\right)\langle R0|k_-0$$

set $r=R$ to find $\langle R0|k_-0\rangle$

$$\langle R0|k_-0\rangle = \frac{4\pi}{(2\pi\hbar)^{3/2}} j_0\left(\frac{kR}{\hbar}\right) + \frac{(4\pi)^2\pi\mu k}{(2\pi\hbar)^3} V_0 R^3 j_0\left(\frac{kR}{\hbar}\right) h_0^+\left(\frac{kR}{\hbar}\right)$$

$$\langle R0|k_-\rangle$$

$$\langle R0|k0\rangle = \frac{\frac{4\pi}{(2\pi\hbar)^{3/2}} j_0\left(\frac{kR}{\hbar}\right)}{1 - \frac{(4\pi)^2\mu\pi k V_0 R^3}{(2\pi\hbar)^3} j_0\left(\frac{kR}{\hbar}\right) h_0^+\left(\frac{kR}{\hbar}\right)}$$

note that $j_0(x) = \frac{\sin x}{x}$ $h_0^+(x) = \frac{e^{ix}}{x}$

using these note

$$\langle R\,0 | k\,0_- \rangle = \frac{\frac{4\pi}{(2\pi\hbar)^{3/2}} \left(\sin\left(\frac{kR}{\hbar}\right) \frac{\hbar}{kR} \right)}{1 - \frac{(4\pi)^2 \mu \pi k V_0 R^3}{(2\pi\hbar)^3} \frac{\hbar^2}{k^2 R^2} \sin\left(\frac{kR}{\hbar}\right) e^{i\frac{kR}{\hbar}}}$$

$$= \frac{\frac{4\pi}{(2\pi\hbar)^{3/2}} \frac{\hbar}{kR} \sin\left(\frac{kR}{\hbar}\right)}{1 - \frac{2\mu V_0 R}{\pi k} \sin\left(\frac{kR}{\hbar}\right) e^{ikR/\hbar}}$$

using this in the solved eq

$$\langle r\,0 | k\,0_- \rangle = \frac{4\pi}{(2\pi\hbar)^{3/2}} \left[\frac{\hbar}{kr} \sin\left(\frac{kr}{\hbar}\right) + \frac{2 V_0 R^2 \mu}{k^2 r^2} \cdot \frac{\sin\left(\frac{kR}{\hbar}\right) \sin\left(\frac{kr_<}{\hbar}\right) e^{i kr_>/\hbar}}{1 - \frac{2\mu V_0 R}{\pi R} \sin\left(\frac{kR}{\hbar}\right) e^{ikR/\hbar}} \right]$$

(c) To find the $l=0$ scattering amplitude

$$F = -(2\pi)^2 \mu \hbar \langle k | t_0 | k \rangle$$

$$= -(2\pi)^2 \mu \hbar \langle k\,0 | V | k\,0_- \rangle$$

$$= -(2\pi)^2 \mu \hbar \int \langle k\,0 | r \rangle V(r) r^2 dr \langle r\,0 | k\,0_- \rangle$$

$$= -(2\pi)^2 \mu \hbar (-V_0 R^3) \frac{4\pi i^0}{(2\pi\hbar)^{3/2}} j_0(kR/\hbar) \langle R\,0 | k\,0_- \rangle$$

From problem 5

$$F = \frac{\dfrac{(2\pi)^2 4\pi V_0 R^3 \mu \hbar}{(2\pi\hbar)^{3/2}} \dfrac{4\pi}{(2\pi\hbar)^{3/2}} \, j_0\left(\frac{kR}{\hbar}\right) j_0\left(\frac{kR}{\hbar}\right)}{1 - \dfrac{2\mu V_0 R}{\pi R} \sin\left(\dfrac{kR}{\hbar}\right) e^{i\frac{kr}{\hbar}}} \qquad \frac{\hbar^2}{k^2 R^2}$$

$$F = \frac{8\pi V_0 R \mu}{k^2} \frac{\sin^2\left(\frac{kR}{\hbar}\right)}{1 - \dfrac{2\mu V_0 R}{\pi R} \sin\left(\dfrac{kR}{\hbar}\right) e^{ikr/\hbar}}$$