29:5742 Homework 8
Due 3/29

1. Consider a three-dimensional scattering problem for two particles of mass m scattering with a potential

$$\langle \mathbf{P}', \mathbf{k}'|V|\mathbf{P}, \mathbf{k}\rangle = -\lambda\delta(\mathbf{P}' - \mathbf{P})\frac{1}{a^2 + \mathbf{k}'^2}\frac{1}{a^2 + \mathbf{k}^2}.$$

Solve the Lippmann Schwinger equation exactly to find the scattering wave function in momentum space

$$\langle \mathbf{P}', \mathbf{k}'|\mathbf{P}, \mathbf{k}^-\rangle = \delta(\mathbf{P} - \mathbf{P}')\langle \mathbf{k}'|\mathbf{k}^-\rangle$$

2. For the potential of problem 1 find $\langle \mathbf{r}'|V|\mathbf{r}\rangle$. Write down the Lippmann Schwinger equation in coordinate space for this potential. Do not solve.

3. For the potential of problem 1 find the scattering amplitude $F(\mathbf{k}', \mathbf{k})$ and the differential cross section in the center of mass frame.

4. For the potential of problem 1 calculate the total cross section.

5. For the potential of problem 1 the Born approximation can be obtained from the exact solution by keeping only the term in the scattering amplitude that is linear in the coupling constant λ. Compute the differential cross section in the Born approximation and compare your result to the exact cross section.

6. Verify that the Born approximation in 5.) approaches the exact cross section in the high-energy limit.

General remarks - If we expand a general potential in a basis and truncate the expansion to N terms we get the approximation

$$V \approx \sum_{m,n=1}^{N} |m\rangle\langle m|V|n\rangle\langle n|$$

Potentials of this form are called separable potentials. The potential of problem 1, which has this form for $N = 1$, is the simplest example of a separable potential. A better approximate separable potential is

$$V \approx \sum_{m,n=1}^{N} V|m\rangle\langle m|V|n\rangle^{-1}\langle n|V$$

where $\langle m|V|n\rangle^{-1}$ is the inverse of the matrix $\langle m|V|n\rangle$.

Homework 8

① $\boxed{g(k) = (a^2 + k^2)^{-1}}$

$$\langle \bar{k}' | \bar{k}_- \rangle = \delta(\bar{k}' - \bar{k})$$

$$- \lambda \frac{1}{k^2/2m - k''^2/2m + i\epsilon} g(k') \underbrace{\int g(k'') \langle k'' | k_- \rangle \, d^3k''}_{h(k)}$$

To solve this multiply by $g(k')$ and integrate over k'

$$h(k) = g(k) - \lambda \underbrace{\int \frac{g(k') g(k') \, d^3k'}{k^2/2m - k'^2/2m + i\epsilon}}_{I(k)} h(k)$$

The equation has the form

$$h(k) = g(k) - \lambda I(k) h(k)$$

$$(1 + \lambda I(k)) h(k) = g(k)$$

$$h(k) = \frac{g(k)}{1 + \lambda I(k)}$$

The solution has the form

$$\boxed{\langle k' | k \rangle = \delta(\vec{k}' - \vec{k}) - \frac{\lambda}{1 + \lambda I(k)} \, \frac{g(k')\, g(k)}{k^2/2\mu - k'^2/2\mu + i\epsilon}}$$

To complete the calculation we must evaluate $I(k)$

$$I(k) = \int d^3k' \, \frac{g(k')\, g(k')}{\frac{k^2}{2\mu} - \frac{k'^2}{2\mu} + i\epsilon} =$$

$$- 2\mu \int d^3k' \left(\frac{1}{k'^2 + a^2} \right)^2 \frac{1}{k'^2 - k^2 - i\epsilon'}$$

$$- 2\mu \left(-\frac{1}{2a} \frac{d}{da} \right) \int \frac{k'^2 \, 4\pi \, dk}{(k'^2 + a^2)(k'^2 - k^2 - i\epsilon')}$$

The integral can be done using the residue theorem 

(there is no contribution from the semicircle as $k \to \infty$)

This has poles at $k = \pm ia$

and $k + i\epsilon$, $-k - i\epsilon$ — we

only pick up poles in

the upper half plane

$$(-2u)(2\pi i)(4\pi)\left(-\frac{1}{2a}\frac{d}{da}\right)\left\{\frac{-a^2}{2ia\,(-a^2-k)} + \frac{k^2}{2k\,(k^2+a^2)}\right\}$$

$$\frac{8\pi^2}{a} i \frac{d}{da}\left(\frac{1}{2}\left(\frac{k-ia}{k^2+a^2}\right)\right)$$

$$\frac{8\pi^2 i}{2a}\frac{d}{da}\left(\frac{k\cancel{-ia}}{(k\cancel{-ia})(k+ia)}\right) = \frac{4\pi^2 i}{a}\frac{(-)i}{(k+ia)^2}$$

$$= \frac{4\pi^2}{a(k+ia)^2}$$

$$\therefore \boxed{I(k) = \frac{4\pi^2}{a(k+ia)^2} = \frac{4\pi^2}{a}\frac{(k-ia)^2}{k^2+a^2}}$$

② $\langle \vec{r} | V | \vec{r}' \rangle =$

$$\int \langle \vec{r} | \vec{k} \rangle (-\lambda\, g(k))\, g(k') \langle k' | r' \rangle \, d^3k\, d^3k'$$

To compute this note

$$\int \langle r | \vec{k} \rangle g(k)\, d^3k =$$

$$\frac{1}{(2\pi\hbar)^{3/2}} \int e^{-i\vec{k}\cdot\vec{r}/\hbar} \frac{d^3k}{k^2+a^2} =$$

$$\frac{1}{(2\pi\hbar)^{3/2}}\, 2\pi \int_0^{\pi} \frac{e^{-ikr\cos\theta/\hbar}\, k^2\, dk \sin\theta\, d\theta}{k^2+a^2}$$

Let $u = \cos\theta \quad du = -\sin\theta\, d\theta$

$u : 1 \to -1$

$$\frac{2\pi}{(2\pi\hbar)^{3/2}} \int_0^{\infty} \frac{k^2\, dk}{k^2+a^2} \int_{-1}^{1} e^{-ikru/\hbar}\, du =$$

$$\frac{2\pi}{(2\pi\hbar)^{3/2}} \int_0^{\infty} \frac{k^2\, dk}{k^2+a^2} \left(\frac{\hbar e^{-ikr/\hbar}}{-ikr} - \frac{\hbar e^{ikr/\hbar}}{-ikr} \right)$$

$$\frac{2\pi\hbar}{(2\pi\hbar)^{3/2}} \frac{i}{r} \int_0^{\infty} \frac{k\, dk}{k^2+a^2} \left(e^{-ikr/\hbar} - e^{ikr/\hbar} \right)$$

Let $k \to k' = -k$ in the second integral

$$= i \frac{(2\pi\hbar)}{(2\pi\hbar)^{3/2}} \frac{1}{r} \int_{-\infty}^{\infty} \frac{k\,dk}{(k^2+a^2)} e^{-ikr/\hbar}$$

this can be computed using the residue theorem - we need to close in the lower half plane $\Rightarrow$

$$i \frac{2\pi\hbar}{(2\pi\hbar)^{3/2}} \frac{1}{r} (-2\pi i) \times \left(\frac{-ia}{-2ia} e^{-ar/\hbar} \right)$$

$$\sqrt{\frac{\pi}{2\hbar}} \frac{1}{r} e^{-ar/\hbar}$$

using these in the expression for V gives

$$\boxed{\langle r | V | r' \rangle = -\frac{\lambda\pi}{2\hbar} \frac{e^{-ar/\hbar}}{r} \frac{e^{-ar'/\hbar}}{r'}}$$

$$\langle \bar{r} | k_- \rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\bar{k}\cdot\bar{r}/\hbar}$$

$$-\frac{2\pi\mu}{\hbar}\left(-\frac{\lambda\pi}{2\hbar}\right) \int \frac{e^{ik|r-r'|/\hbar}}{|r-r'|} \frac{e^{-ar'/\hbar}}{r'} \times$$

$$\int \frac{e^{-ar''/\hbar}}{r''} \langle \bar{r}'' | k_+ \rangle \, d^3r'' \, d^3r'$$

③ $F(\bar{k}'\bar{k}) = -(2\pi)^2 \mu\hbar \langle \bar{k}' | T | k \rangle$

③ $F(\bar{k}'\bar{k}) = -(2\pi)^2 \mu\hbar \langle \bar{k}' | T | \bar{k} \rangle$

$$T = V + V (E - H_0 + i\epsilon)^{-1} T$$

$$= -\lambda |g\rangle\langle g| - \lambda |g\rangle\langle g| (E - H_0 + i\epsilon)^{-1} T$$

by inspection $T = |g\rangle X$

$$|g\rangle X = -\lambda |g\rangle\langle g|$$
$$- \lambda |g\rangle\langle g| (E - H_0 + i\epsilon)^{-1} |g\rangle X$$

solving for X

$$\left(1 + \lambda \langle g| (E - H_0 + i\epsilon)^{-1} |g\rangle\right) X$$
$$= -\lambda \langle g|$$

$$X = \frac{-\lambda}{1 + \lambda \underbrace{\langle g| E - H_0 + i\epsilon)^{-1} |g\rangle}_{I(k) \text{ from problem 1}}} \langle g|$$

$$\langle k' | T | k \rangle = \frac{-\lambda\, g(k')\, g(k)}{1 + \lambda I(k)}$$

$$\boxed{F(\bar{k}'k) = \frac{2\pi^2 \mu\hbar\lambda}{1 + \lambda I(q)}\, g(k')\, g(k)}$$

④ In this case

$$\sigma_T = \int |F(\bar{k}' k)|^2 \, d\Omega = 4\pi |F(kk)|^2$$

$$\frac{16\pi^3 \mu^2 \hbar^2 \lambda^2 g^4(k)}{(1 + \lambda I(k))(1 + \lambda I(k)^*)}$$

⑤ From problem 3 $\langle k' | T | k \rangle$

becomes V by setting

$I(k) \to 0$

$$\frac{d\sigma}{d\Omega}_B = 4\pi^2 \mu^2 \hbar^2 \lambda^2 g^4(q)$$

$$\frac{d\sigma}{d\Omega_e} = \frac{d\sigma}{d\Omega_B} \frac{1}{|1 + \lambda I(k)|^2}$$

⑥ From problem 1 as $k \to \infty$

$I(k) \to 0$ which is the

Born approximation.