29:5742 Homework 6
Due 3/8

1. Use the definitions

$$\| |\psi\rangle \| = \langle\psi|\psi\rangle^{1/2}$$

and

$$|||O||| = \underbrace{\sup}_{\| |\psi\rangle \neq 0} \frac{\|O|\psi\rangle\|}{\| |\psi\rangle \|}$$

to show

a.

$$\|O|\psi\rangle\| \leq |||O||| \| |\psi\rangle \|$$

b.

$$|||O_1 + O_2||| \leq |||O_1||| + |||||O_2|||$$

c.

$$|||O_1 O_2||| \leq |||O_1|||\, |||O_2|||$$

2. Consider a two state system with unperturbed Hamiltonian $H_0 := E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$ with $E_2 > E_1$ and perturbing interaction for $t > 0$, $V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|$ where ω and γ are real positive constants.

   a. Assume that at time $t = 0$ the system is in the state $|1\rangle$. Find the exact probability for the system to found in each of these states at later times.

   b. Do the same calculation using first order time dependent perturbation theory.

   c. Both probabilities exhibit oscillations. Find the frequency that maximizes the amplitude of the oscillations of the probability to find the system in the second state.

   * Hint - the time dependence in the Hamiltonian can be eliminated by a rotation about the z axis.

3. A one dimensional harmonic oscillator is in its ground state for $t < 0$. For $t \geq 0$ is it subject to a time-dependent but spatially uniform force (not potential) in the x-direction

$$F(t) = F_0 e^{-i\omega t}$$

Use first order time-dependent perturbation theory to find the probability of finding the oscillator in its first excited state as a function of time.

4. Consider the Hamiltonian

$$H = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix} \qquad E_2 > E_1$$

   Assume that a and b are small. Compare the exact eigenvalues of this matrix to the eigenvalues obtained using second order degenerate perturbation theory.

5. Consider the Hamiltonian

$$H = \begin{pmatrix} E_1 & \lambda \cos(\omega t) \\ \lambda \cos(\omega t) & E_2 \end{pmatrix} \qquad \lambda = \lambda^*$$

   If λ is small and the system is initially in the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   find the probability as a function of time that is will be in the state

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

6. A one electron atom in its ground state is placed in a uniform electric field in the z direction. Obtain an approximate expression for the induced electric dipole moment of this atom by considering the expectation value of ez with respect to the perturbed ground state vector created in first order perturbation theory (assume that the ground state is non-degenerate).

* Hint - This problem involves an infinite sum - the trick to perfom the sum is to find an operator F with the property $[F, H_0] = z$. This cancels the denominator allowing one use use the completeness relation.