Phys 5742
Homework 5 - Due Friday 3/1

1. Use the variational method to estimate the ground state energy of the Hamiltonian of a one-dimensional anharmonic oscillator:

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \lambda x^4$$

using a trial radial wave function of the form $N(\alpha)e^{-\alpha x^2}$ where α is a free parameter. Hint: the Gamma function is useful for computing the integrals.

2. Consider the variational functional

$$F[|\psi\rangle] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

Show that the states $|\psi\rangle$ that make this functional stationaly are eigenstates of H. Assume that

$$|\psi\rangle = |\psi_0\rangle + \delta|\psi\rangle$$

and use lagrange multipliers to fix the normalization.

3. Consider a Hamiltonian that is known to be bounded above and below. This happens, for example, in systems involving a finite number of spins. Find a variational principle that gives a lower bounds on the largest eigenvalue.

4. Show that

$$\int_{-\infty}^{\infty} Ai(x-y)Ai(x-y')dx = \delta(y-y').$$

5. A particle of mass m experiences a constant gravitational force $F = -mg$ in one dimension. Find the eigenstates of the corresponding quantum Hamiltonian.

6. Let $X = ctI + \mathbf{x}\cdot\boldsymbol{\sigma}$ be a 2×2 matrix where $\boldsymbol{\sigma}$ represents the three Pauli matrices considered as a vector and c is the speed of light in a vacuum. Let A be a 2×2 matrix with $det(A) = 1$. Consider the transformation

$$X' = AXA^\dagger$$

   a. Show that $(c\tau)^2 := (ct)^2 - \mathbf{x}\cdot\mathbf{x} = (ct')^2 - \mathbf{x}'\cdot\mathbf{x}'$.
   b. Show that sucessive transformation of this type preserve $(c\tau)^2$. This group is called $SL(2,C)$ - it is closely related to the group of Lorentz transformations, although as in $SU(2)$ both A and $-A$ give the same transformation.
   c. Show that for $\sigma_\mu = (I, \sigma_1, \sigma_2, \sigma_3)$ that

$$(ct, x_1, x_2, x_3) = \frac{1}{2}\mathrm{Tr}(\sigma_\mu X)$$

Solutions HW #5

① integrals

$$\int_0^\infty e^{-x} x^{s-1}\, dx = \Gamma(s)$$

$$\Gamma(s+1) = s\,\Gamma(s)$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\, dx =$$

$u = \alpha x^2$

$du = 2\alpha x\, dx$

$= 2\alpha \left(\frac{u}{\alpha}\right)^{1/2} dx$

$dx = \frac{1}{2}\frac{1}{(\alpha u)^{1/2}}\, du$

$$2\int_0^\infty x^{2n} e^{-\alpha x^2}\, dx$$

$$2\int_0^\infty \left(\frac{u}{\alpha}\right)^n e^{-u} \frac{1}{2}\frac{du}{(\alpha u)^{1/2}}$$

$$\alpha^{-n-\frac{1}{2}} \int_0^\infty u^{n+\frac{1}{2}-1} e^{-u} = \frac{1}{(\alpha)^{n+1/2}} \Gamma\left(n+\frac{1}{2}\right)$$

$$\boxed{\therefore \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} = \frac{1}{\alpha^{n+1/2}} \Gamma\left(n+\frac{1}{2}\right)}$$

$n = 0$ $\quad \int e^{-\alpha x^2} = \frac{\Gamma(1/2)}{\alpha^{1/2}} = \sqrt{\frac{\pi}{\alpha}}$

$n = 1$ $\quad \int x^2 e^{-\alpha x^2} = \frac{\Gamma(3/2)}{\alpha^{3/2}} = \frac{\frac{1}{2}\sqrt{\pi}}{\alpha\sqrt{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$

$n = 2$ $\quad \int x^4 e^{-\alpha x^2} = \frac{\Gamma(5/2)}{\alpha^{5/2}} = \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{\alpha^2\sqrt{\alpha}} = \frac{3}{4\alpha^2}\sqrt{\frac{\pi}{\alpha}}$

step 1 compute the normalization coefficient

$$1 = N^2 \int e^{-2\alpha x^2} = \sqrt{\frac{\pi}{2\alpha}}$$

$$\boxed{N^2 = \sqrt{\frac{2\alpha}{\pi}} \qquad N = \left(\frac{2\alpha}{\pi}\right)^{1/4}}$$

step 2 compute

$$\langle \psi | \tfrac{1}{2} p^2 | \psi \rangle =$$

$$N^2 \int e^{-\alpha x^2} \left(-\frac{1}{2}\frac{d^2}{dx^2}\right) e^{-\alpha x^2} dx =$$

$$N^2 \int e^{-\alpha x^2} \left(-\frac{1}{2}\frac{d}{dx}\right)\left(-2\alpha x e^{-\alpha x^2}\right) dx =$$

$$N^2 \int e^{-\alpha x^2} \left(-\frac{1}{2}\right)\left(-2\alpha + 4\alpha^2 x^2\right) e^{-\alpha x^2} =$$

$$N^2 \left(\alpha \int e^{-2\alpha x^2} dx - 2\alpha^2 \int x^2 e^{-2\alpha x^2}\right) =$$

$$\boxed{N^2 \left(\alpha \frac{1}{N^2} - 2\alpha^2 \frac{1}{4\alpha} \frac{1}{N^2}\right) = \frac{1}{2}\alpha}$$

step 3 compute

$$\langle \psi | \tfrac{1}{2} x^2 | \psi \rangle =$$

$$\boxed{N^2 \frac{1}{2} \int e^{-2\alpha x^2} x^2 = \frac{1}{2} N^2 \frac{1}{4\alpha} \frac{1}{N^2} = \frac{1}{8\alpha}}$$

step 4 comput

$$\langle \psi | \lambda x^4 | \psi \rangle = \lambda \int N^2 e^{-2\alpha x^2} x^4 dx =$$

$$\boxed{N^2 \frac{3}{16\alpha^2} \frac{\lambda}{N^2} = \frac{3\lambda}{16\alpha^2}}$$

adding these three terms gives

$$\langle \psi | H | \psi \rangle = \frac{1}{2}\alpha + \frac{1}{8\alpha} + \frac{3\lambda}{16\alpha^2}$$

step 5 find the critical values of α

$$\frac{d}{d\alpha}\langle \psi | H | \psi \rangle = \frac{1}{2} - \frac{1}{8\alpha^2} - \frac{6\lambda}{16\alpha^3} = 0$$

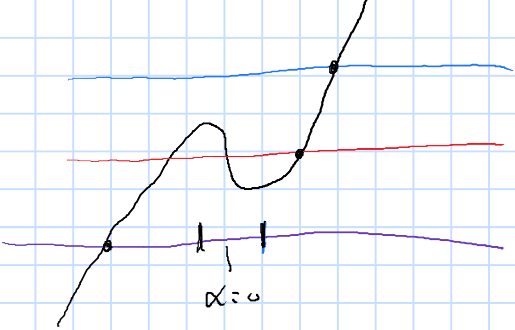
multiply by $2\alpha^3$

$$\alpha^3 - \frac{1}{4}\alpha - \frac{3}{4}\lambda = 0$$

this is a cubic equation - it has 3 roots - this is negative as $\alpha \to -\infty$ and positive as $\alpha \to +\infty$ so there is at least 1 real root

check the turning points

$$3\alpha^2 - \frac{1}{4} = 0 \qquad \alpha = \pm\frac{1}{2\sqrt{3}}$$



The value of the roots depend on λ but

Let α^* be the positive root $\Rightarrow$

$$\langle\psi|H|\psi\rangle = \frac{1}{2}\alpha^* - \frac{1}{2\alpha^*} - \frac{3}{16}\lambda\frac{1}{(\alpha^*)^2}$$

(2) $F[\phi_0 + \lambda\delta\phi]$

$$\langle\phi_0+\lambda\delta\phi|H|\phi_0+\lambda\delta\phi\rangle - \eta\langle\phi_0+\lambda\delta\phi|\phi_0+\lambda\delta\phi\rangle$$

$$\frac{dF}{d\lambda}[\phi_0+\lambda\delta\phi] = 0 =$$

$$\langle\phi_0|H|\delta\phi\rangle + \langle\delta\phi|H|\phi_0\rangle$$
$$-\eta\langle\phi_0|\delta\phi\rangle - \eta\langle\delta\phi|\phi_0\rangle$$

replace $\delta\phi$ by $i\delta\phi$

$$0 = i\langle\phi_0|H|\delta\phi\rangle - i\langle\delta\phi|H|\phi_0\rangle$$
$$-\eta i\langle\phi_0|\delta\phi\rangle + i\langle\delta\phi|\phi_0\rangle$$

cancel the factor of i in

the second equation and

subtract from the first

equation

$$2\langle\delta\Phi|H|\Phi_0\rangle - \eta\, 2\langle\delta\Phi|\Phi_0\rangle = 0$$

since this must hold

for all $|\delta\Phi\rangle$ we get

$$2H|\Phi_0\rangle = 2\eta|\Phi_0\rangle \quad \text{or}$$

$$H|\Psi_0\rangle = \eta\,|\Phi_0\rangle$$

multiply by $\langle\Phi_0|$

$$\frac{\langle\Phi_0|H|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle} = \eta$$

$\Rightarrow$

(1) $|\Phi_0\rangle$ is an eigenstate of H_0 with eigenvalue η.

(2) for $\langle\Phi_0|\Phi\rangle = 1$ η is the expectation value of $\langle\Psi|H_0|\Psi\rangle$

③ $\langle \psi | H | \psi \rangle =$

$$\sum_n \langle \psi | \phi_n \rangle E_n \langle \phi_n | \psi \rangle$$

$$\sum_n |\langle \psi | \phi_n \rangle|^2 E_n =$$

$$\sum_n |\langle \psi | \phi_n \rangle|^2 (E_n - E^\uparrow + E^\uparrow) =$$

where $E^\uparrow$ is the largest

eigenvalue

$$= E^\uparrow - \sum_n \underbrace{|\langle \psi | \phi_n \rangle|}_{\geq 0} \underbrace{(E^\uparrow - E_n)}_{\geq 0}$$

$$\langle \psi | H | \psi \rangle + \sum_n |\langle \psi_n | \phi_n \rangle|^2 (E^\uparrow - E_n) = E^\uparrow$$

$$\langle \psi | H | \psi \rangle \leq E^\uparrow$$

④ recall

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(sx + \frac{s^3}{3})} dx$$

$$\int_{-\infty}^{\infty} Ai(x-y) Ai(x-y') dy =$$

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi)^2} e^{-i(s(x-y) + \frac{s^3}{3})} e^{-i(t(x-y') + \frac{t^3}{3})} ds\, dt\, dx$$

The y integral

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-i(s+t)x}\, dx = \delta(s+t)$$

with this the expression on the last page becomes

$$= \frac{1}{2\pi} \int_{-\infty} ds\, e^{-is(y-y')} = \delta(y-y')$$

⑤ $F = -mg \qquad V = mgz$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dz^2} + mgz\psi = E\psi$$

Let $\quad \sigma = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3} \left(z - \frac{E}{mg}\right)$

$$\frac{d\sigma}{dz} = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}$$

$$\frac{d^2}{dz^2} = \left(\frac{d\sigma}{dz}\right)^2 \frac{d^2}{d\sigma^2} = \left(\frac{2m^2 g}{\hbar^2}\right)^{2/3} \frac{d^2}{d\sigma^2}$$

The Schrodinger equation becomes

$$\left(\frac{2m^2 g}{\hbar^2}\right)^{2/3} \frac{d^2}{d\sigma^2} = -\frac{2m}{\hbar^2}(E - mgz)\psi$$

$$= \frac{2m^2 g}{\hbar^2}\left(z - \frac{E}{mg}\right)\psi$$

$$= \left(\frac{2m^2 g}{\hbar^2}\right)^{2/3} \sigma \psi$$

$$\frac{d^2}{d\sigma^2}\psi(\sigma) = \sigma\psi(\sigma)$$

So

$$\langle z|\psi\rangle = N\,Ai(\sigma)$$

$$= N\,Ai\left(\left(\frac{2m^2g}{\hbar^2}\right)^{1/3}\left(z-\frac{E}{mg}\right)\right)$$

where N is a normalization constant

(6) $$X = \begin{pmatrix} ct+z & x-iy \\ x+iy & ct-z \end{pmatrix}$$

a) $$\det(X) = (ct+z)(ct-z)-(x-iy)(x+iy)$$

$$= (ct)^2 - x^2 - y^2 - z^2$$

$$\det(X') = \det AXA^{+} =$$

$$= \det A \det X \det A^{+}$$

$$= 1 \cdot \det X \cdot 1$$

$$\therefore (ct')^2 - \bar{x}'^2 = (ct)^2 - \bar{x}^2$$

b) $X' = A X A^+$ $\qquad \det A = \det B = 1$

$X'' = B X' B^+$

$X'' = B A X A^+ B^+$

$= (BA) X (BA)^+$

$\det X'' = \underbrace{\det B}_{1} \underbrace{\det A}_{1} \det X \underbrace{\det A^+}_{1} \underbrace{\det B^+}_{1}$

$(ct)''^2 - \bar{x}'' \cdot \bar{x}'' = (ct)^2 - \bar{x} \cdot \bar{x}$

c) $\det(\sigma_\mu X$

$\det\left(\sigma_\mu \sum_\nu X^\nu \sigma_\nu\right)$

but

$\sigma_\mu \sigma_\mu = I$

$\sigma_0 \sigma_\mu = \sigma_\mu \sigma_0 = \sigma_\mu \qquad \mu = 1, 2, 3$

$\sigma_i \sigma_j = -\sigma_j \sigma_i = i \varepsilon_{ijk} \sigma_k$

since $\mathrm{Tr}(\sigma_\mu) = 0$ for $\mu = 1, 2, 3$

and $\mathrm{Tr}(I) = 2$

$\det(\sigma_\mu X^\nu \sigma_\nu) = 2 \sum \delta_{\mu\nu} X^\nu$

$= 2X^\mu$

or

$$X^{\mu} = \frac{1}{2} \mathrm{Tr}\left(\sigma_{\mu} X\right)$$