Phys 5742
Homework 4 - Due Friday 2/16

Problem 1:
Let $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$ be Cartesian components of vector operators. Find all components of a rank ($l = 2$) spherical tensor, T^2_m, constructed out of the products $U_i V_j$. (Hint: consider the structure of $Y^2_m(\theta, \phi)$))

Problem 2:
Write xy, xz and $x^2 - y^2$ as components of a rank 2 spherical tensor. The expectation value

$$Q = e\langle \alpha, j, j|(3z^2 - r^2)|\alpha, j, j\rangle$$

is called the quadrupole moment. Evaluate

$$e\langle \alpha, j, m|(x^2 - y^2)|\alpha, j, j\rangle$$

in terms of Q for all $m, -j \leq m \leq j$. (hint: use the Wigner-Eckart theorem).

Problem 3:
Show if T^m_l is a rank l spherical tensor that

$$[J_x[J_x, T^m_l]] + [J_y[J_y, T^m_l]] + [J_z[J_z, T^m_l]] = l(l+1)T^m_l.$$

Hint: Use the infinitesimal form of $U(R)T^m_l U^\dagger(R) = \sum_n T^n_l D^l_{nm}(R)$.

Problem 4: Consider the differential equation

$$\left(\frac{d^2}{dx^2} - x\right) f(x) = 0.$$

Show that the Airy function

$$Ai(x) := \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{-isx - is^3/3}$$

is a solution to this differential equation. Hint: consider

$$\int g(x) \left(\frac{d^2}{dx^2} - x\right) A_i(x) dx = 0$$

for arbitrary well behaved $g(x)$.

Problem 5: Let $V(x) = F|x|$ be a one dimensional confining potential. Find an equation for the energy eigenvalues by using the WKB approximation with suitable boundary conditions at $x = 0$.

Problem 6: Use the variational method to estimate the ground state energy of the Hamiltonian of a one-dimensional anharmonic oscillator:

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \lambda x^4$$

using a trial radial wave function of the form $N(\alpha)e^{-\alpha x^2}$ where α is a free parameter. Hint: the Gamma function is useful for computing the integrals.

① This transforms like Y^2_m

$$V^{-2}_2 = \frac{1}{4}(V_x - iV_y)^2$$

$$V^{-1}_2 = \frac{1}{2}(V_x - iV_y)V_z$$

$$V^0_2 = \frac{1}{4}(3V_z^2 - \bar{V}\cdot\bar{V})$$

$$V^1_2 = -\frac{1}{2}(V_x + iV_y)V_z$$

$$V^2_2 = \frac{1}{4}(V_x + iV_y)^2$$

The numerical factors are only up to normalization

For this problem there are 2 vectors. In this case the 2 vectors do not have to commute — for this reason it is useful to replace $V^{\pm 1}_2$ by

$$V^{\pm 1}_2 = \frac{1}{4}(\pm)(V_x \mp iV_y)V_z + \frac{1}{4}(\pm)V_z(V_x \mp iV_y)$$

If we remove the factor $\frac{1}{4}$ we get

$$(uv)_2^{-2} = (u_x - i u_y)(v_x - i v_y)$$

$$(uv)_2^{-1} = (u_x - i u_y) v_z + u_z (v_x - i v_y)$$

$$(uv)_2^{0} = (3 u_z v_z - \bar{u} \cdot \bar{v})$$

$$(uv)_2^{1} = -(u_x + i u_y) v_z - u_z (v_x + i v_y)$$

$$(uv)_2^{2} = (u_x + i u_y)(v_x + i v_y)$$

note that in these expressions all of the v's are on the right side of the u's.

② $Q = e \langle \alpha j j | (3z^2 - r^2) | \alpha j j \rangle$

using the results from problem 1

$$x_2^{-2} = x^2 - y^2 - 2xyi$$

$$x_2^{2} = x^2 - y^2 + 2xyi$$

adding gives

$$x^2 - y^2 = \tfrac{1}{2}(x_2^2 + x_2^{-2})$$

$$3z^2 - r^2 = x_2^0$$

it follows that

$$\frac{\langle \alpha j j | X_2^0 | \alpha j j \rangle}{\langle \alpha j m | X_2^2 | \alpha j j \rangle} = \frac{C^{j\,2\,j}_{j\,0\,j}}{C^{j\,2\,j}_{m\,2\,j}}$$

$$\frac{\langle \alpha j j | X_2^0 | \alpha j j \rangle}{\langle \alpha j m'' | X_2^{-2} | \alpha j j \rangle} = \frac{C^{j\,2\,j}_{j\,0\,j}}{C^{j\,2\,j}_{m''\,-2\,j}}$$

$$\therefore\ e\langle \alpha j m | (x^2 - y^2) | \alpha j j \rangle =$$

$$\frac{e}{2}\left(\langle \alpha j m | X_2^2 | \alpha j j \rangle + \langle \alpha j m | X_2^{-2} | \alpha j j \rangle \right) =$$

$$\boxed{\frac{1}{2}\,\frac{1}{C^{j\,2\,j}_{j\,0\,j}}\left(C^{j\,2\,j}_{m\,2\,j} + C^{j\,2\,j}_{m\,-2\,j} \right) Q}$$

note - the normalization factor cancel.

③ Note

$$\mathcal{U}(R)\, T_\ell^m\, \mathcal{U}^\dagger(R) = \sum_{m'} T_\ell^{m'} D^\ell_{m'm}(R)$$

$$e^{i\bar{J}\cdot\theta}\, T_\ell^m\, e^{-iJ\cdot\theta} = \sum_{m'} T_\ell^{m'} \langle \ell m' | e^{iJ\cdot\theta} | \ell m\rangle$$

differentiate with respect

to $\bar{\theta}$, set $\theta = 0$

$$i(\bar{J}\, T_\ell^m - T_\ell^m \bar{J}) = i \sum T_\ell^{m'} \langle j m' | \bar{J} | j m\rangle$$

for J_z

$$[J_z\, T_\ell^m] = m\, T_\ell^m$$

$$[J_\pm\, T_\ell^m] = \sum T_\ell^{m'} \langle j m' | j m \pm 1\rangle \sqrt{(j \mp m)(j \pm m + 1)}$$

$$[J_\pm\, T_\ell^m] = T_\ell^{m\pm 1} \sqrt{(j \mp m)(j \pm m + 1)}$$

next note

$$[J_z\, [J_z\, T_\ell^m]] = m\, [J_z, T_\ell^m] = m^2 T_\ell^m$$

$$[J_\mp\, [J_\pm\, T_\ell^m]] = [J_\mp\, T_\ell^{m\pm 1}] \sqrt{(j \mp m)(j \pm m + 1)}$$

$$T_\ell^m \sqrt{(j \pm (m \pm 1))(j \mp (m \pm 1) + 1)}\, \sqrt{(j \mp m)(j \pm m + 1)} =$$

$$(j \pm m + 1)(j \mp m)$$

$$T_\ell^m\, (j \pm m + 1)(j \mp m)$$

$$T_\ell^m\, (j(j+1) - m(m \pm 1))$$

comparing both sides

we get

$$\boxed{[J_\pm\, T_\ell^m] = T_\ell^{m\pm}\sqrt{(\ell\mp m)(\ell\pm m+1)}}$$

It follows that

$$[J_z[J_z\, T_\ell^m]] =$$

$$m\,[J_z\, T_\ell^m] =$$

$$m^2\, T_\ell^m$$

$$[J_\mp[J_\pm\, T_\ell^m]] =$$

$$[J_\mp,\, T_\ell^{m\pm 1}]\sqrt{(j\mp m)(j\pm m+1)}$$

$$T_\ell^m\sqrt{\underbrace{(j\pm(m\pm1))}_{(j\pm m+1)}(j\mp(m\pm1)+1)}\times\sqrt{(j\mp m)(j\pm m+1)}$$

$$(j\pm m+1)\quad(j\mp m)$$

$$T_\ell^m\,(j\mp m)(j\pm m+1)$$

$$T_\ell^m\,(j^2-m^2+j\mp m)$$

$$T_\ell^m\,(j(j+1)-m(m\pm1))$$

next note

$[J_\mp [J_\pm T^m_\ell]] =$

$[J_x \mp iJ_y [J_x \pm iJ_y, T^m_\ell]] =$

$[J_x [J_x T^m_\ell]] + [J_y [J_y T^m_\ell]]$

$\pm i [J_x [J_y T^m_\ell]] \mp i [J_y [J_x T^m_\ell]]$

next we use

$[A [B C]] + [C [A B]] + [B [C A]] = 0$

$$\pm i \Big(\underbrace{[J_x [J_y T^m_\ell]] + [J_y [T^m_\ell J_x]]} \Big)$$

$$- [T^m_\ell [J_x J_y]] =$$

$$- i [T^m_\ell J_z] =$$

$$i [J_z T^m_\ell] =$$

$$i m T^m_\ell$$

$$\pm i i m T^m_\ell = \mp T^m_\ell m$$

$\therefore$ $[J_x [J_x T^m_\ell]] + [J_y [J_y T^m_\ell]] =$

$[J_\mp [J_\pm T^m_\ell]] \pm m T^m_\ell$

putting everything together

$[J_x [J_x T^m_\ell]] + [J_y [J_y T^m_\ell]] +$

$[J_z [J_z T^m_\ell]] =$

$$= T^{\ell}_{m}\left(\ell(\ell+1) - m(m\pm 1)\right) \pm m\, T^{\ell}_{m} + m^2 T^{\ell}_{m}$$

$$= T^{\ell}_{m}\left(\ell(\ell+1) - m(m\pm 1) + m(m\pm 1)\right)$$

$$= T^{\ell}_{m}\, \ell(\ell+1)$$

which gives the desired result

4) $$\int g(x)\left(\frac{d^2}{dx^2} - x\right)\int_{-\infty}^{\infty}\frac{ds}{2\pi}\, e^{-i\left(sx+\frac{s^3}{3}\right)}\, dx =$$

$$\int g(x)\int_{-\infty}^{\infty}\frac{ds}{2\pi}\left(-s^2 - x\right) e^{-i\left(sx+\frac{s^3}{3}\right)}\, dx =$$

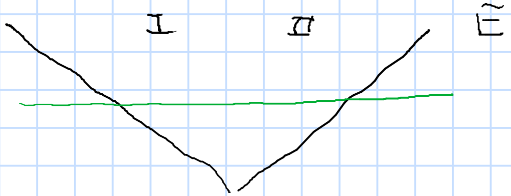
$$\int g(x)\int_{-\infty}^{\infty}\frac{ds}{2\pi}\; i\,\frac{d}{ds}\left(e^{-i\left(sx+\frac{s^3}{3}\right)}\right) dx$$

$$i\,\frac{1}{2\pi}\int g(x)\, e^{-i\left(sx+\frac{s^3}{3}\right)}\Big|_{-\infty}^{\infty} =$$

$$i\,\frac{1}{\sqrt{2\pi}}\,\tilde{g}(s)\, e^{-i\, s^3/3}\Big|_{-\infty}^{\infty}$$

where $\tilde{g}(s)$ is the Fourier transform of $g(x)$ which falls off for large $|s|$

5)



$$p_{cl}^2 = 2m(E - Fx) \qquad x > 0$$

$$p_{cl}^2 = 2m(E + Fx) \qquad x < 0$$

The WKB quantization condition is

$$\int_{-\frac{E}{F}}^{0} p_{cl}^{I}(x) + \int_{0}^{\frac{E}{E}} p_{cl}^{II}(x) = \pi(n+\tfrac{1}{2})\hbar$$

$$\int_{-\frac{E}{F}}^{0} \sqrt{2m(E+Fx)} + \int_{0}^{\frac{E}{F}} \sqrt{2m(E-Fx)} = \hbar(n+\tfrac{1}{2})\pi$$

let $u_+ = 2m(E+Fx) \qquad u_- = 2m(E-Fx)$

$du_+ = 2mF\,dx \qquad du_- = -2mF\,dx$

$$\int_{0}^{\sqrt{2mE}} u^{1/2}\frac{du}{2mF} - \int_{\sqrt{2mE}}^{0} u^{1/2}\frac{du}{2mF} = \hbar\pi(n+\tfrac{1}{2})$$

$$\frac{2}{3}\left(\sqrt{2mE}\right)^{3/2}\frac{1}{2mF} + \frac{2}{3}\left(\sqrt{2mE}\right)^{3/2}\frac{1}{2mF} = \hbar\pi(n+\tfrac{1}{2})$$

$$(2mE)^{3/4} = \frac{3}{2}mF\pi\hbar(n+\tfrac{1}{2})$$

$$\boxed{E_n = \frac{1}{2m}\left(\frac{3}{2}mF\pi\hbar(n+\tfrac{1}{2})\right)^{4/3}}$$

This solution ignores the discontinuity in the potential at the origin.. To treat this note p^2 and $|x|$ are both even functions which means that if $\langle x|\psi\rangle$ is a solution so is $\langle -x|\psi\rangle$ are linear combinations of the two

$$\langle x|\psi_e\rangle = \langle x|\psi\rangle + \langle -x|\psi\rangle$$

$$\langle x|\psi_o\rangle = \langle x|\psi\rangle - \langle -x|\psi\rangle$$

The odd one vanishes at 0 while the even one has vanishing derivatives at 0 - here the Airy functions give the exact solution

$$\psi(x) = N\,Ai\left(-\left(\frac{2mF}{\hbar^2}\right)^{1/3}\left(x+\frac{E}{F}\right)\right)$$

The boundary condition at 0 means that

$$-\left(\frac{2mF}{\hbar^2}\right)^{1/3}\left(\frac{E}{m}\right) = \sigma_n$$

where σ_n is the n^{th} 0 of the Airy function – recall $x = 0$ is in the physically allowed region where $Ai(\sigma)$ oscillates

(b) First compute the normalization integral

$$1 = N^2 \int_{-\infty}^{\infty} e^{-2\alpha x^2} dx$$

$$u = \sqrt{2\alpha}\, x \qquad dx = \frac{1}{\sqrt{2\alpha}} du$$

$$1 = N^2 \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-u^2} du$$

Let $v = u^2 \quad dv = 2u\,du = 2v^{1/2} du$

$$1 = N^2 \frac{1}{\sqrt{2\alpha}}\, 2 \int_0^{\infty} \frac{dv}{2} v^{-\frac{1}{2}} e^{-v}$$

$$= N^2 \frac{1}{\sqrt{2\alpha}} \int_0^{\infty} dv\, v^{\frac{1}{2}-1} e^{-v}$$

$$= N^2 \frac{1}{\sqrt{2\alpha}} \Gamma\left(\tfrac{1}{2}\right) = N^2 \sqrt{\frac{\pi}{2\alpha}}$$

(here I chose to compute the integral using the Gamma function)

$$N = \sqrt[4]{\frac{2\alpha}{\pi}} \qquad p = -i\frac{d}{dx}$$

$$\langle \psi | H | \psi \rangle =$$

$$N^2 \int e^{-\alpha x^2} \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \lambda x^4\right) e^{-\alpha x^2} =$$

note $\frac{d}{dx} e^{-\alpha x^2} = -2\alpha x e^{-\alpha x^2}$

$$\frac{d^2}{dx^2} e^{-\alpha x^2} = -2\alpha e^{-\alpha x^2} + 4\alpha^2 x^2 e^{-\alpha x^2}$$

$\langle \psi | H | \psi \rangle =$

$$N^2 \int_{-\infty}^{\infty} e^{-2\alpha x^2} \left(-\frac{1}{2}(-2\alpha + 4\alpha^2 x^2) + \frac{1}{2} x^2 + \lambda x^4 \right) dx$$

Let $u = \sqrt{2\alpha}\, x$ $\quad du = \sqrt{2\alpha}\, dx$

$$N^2 \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-u^2} \left(\alpha - 2\alpha^2 \frac{u^2}{2\alpha} + \frac{1}{2} \frac{u^2}{2\alpha} + \lambda \frac{u^4}{4\alpha^2} \right)$$

$$N^2 \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-u^2} \left(\alpha - \alpha u^2 + \frac{1}{4\alpha} u^2 + \lambda \frac{u^4}{4\alpha^2} \right) =$$

$$\frac{1}{\sqrt{\pi}} \left(\alpha \int e^{-u^2} du + \left(\frac{1}{4\alpha} - \alpha\right) \int e^{-u^2} u^2 du + \lambda \frac{1}{4\alpha^2} \int e^{-u^2} u^4 du \right.$$

To calculate the integrals

note

$$\int e^{-\alpha u^2} du = \sqrt{\frac{\pi}{\alpha}} \to \sqrt{\pi}$$

$$\int u^2 e^{-u^2} = -\frac{d}{d\alpha}\left(\sqrt{\frac{\pi}{\alpha}}\right)_{\alpha=1} = \frac{1}{2}\sqrt{\pi}\, \alpha^{-3/2} = \frac{\sqrt{\pi}}{2}$$

$$\int u^4 e^{-u^2} = \left(-\frac{d}{d\alpha}\right)^2 \sqrt{\frac{\pi}{\alpha}}_{\alpha=1} = \frac{1}{2}\left(\frac{3}{2}\right)\sqrt{\pi}\, \alpha^{-5/2} = \frac{3}{4}\sqrt{\pi}$$

$$\langle\Psi|H|\Psi\rangle = \alpha + \frac{1}{2}\left(\frac{1}{4\alpha} - \alpha\right) + \frac{3\lambda}{16\alpha^2}$$

To find the minimum

evaluate

$$0 = \frac{d}{d\alpha}\langle\psi|H|\psi\rangle = 1 - \frac{1}{2} - \frac{1}{8\alpha^2} - \frac{6\lambda}{16\alpha^3}$$

$$= \frac{1}{2} - \frac{1}{8\alpha^2} - \frac{3}{8}\frac{\lambda}{\alpha^3}$$

multipy by $8\alpha^3$ to get

$$\boxed{0 = 4\alpha^3 - \alpha - 3\lambda}$$

we want roots that satisfy

$$\frac{d^2}{d\alpha^2}\langle\psi|H|\psi\rangle\Big|_{root} \geq 0$$

$$= \frac{1}{4\alpha^3} + \frac{9}{8}\frac{\lambda}{\alpha^4}$$

since we are only interested in positive roots for a wave function that falls off — we need to positive roots of $4\alpha^3 - \alpha - 3\lambda = 0$

This is negative and decreasing at $\alpha = 0$

The general structure is either the red or black line, it is clear that there is one positive root α_+

$$\boxed{\begin{gathered} \langle\psi|H|\psi\rangle = \frac{1}{2} - \frac{1}{8\alpha_+^2} - \frac{3\lambda}{8\alpha_+^3} \\ \alpha_+ \quad \text{positive root of} \\ 4\alpha^3 - \alpha + 3\lambda = 0 \end{gathered}}$$

This can be computed numerically for a given value of λ