Phys 5742
Homework 3 - Due Friday 2/9

Problem 1: Use the relation

$$Y_l^m(\hat{n}) = \sqrt{\frac{2l+1}{4\pi}} D^{*l}_{m0}(R)$$

where $\hat{\mathbf{n}} = R\hat{\mathbf{z}}$ to integtate a product of three spherical harmonics

$$\int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi \, Y_{l_a}^{m_a}(\theta,\phi) Y_{l_b}^{m_b}(\theta,\phi) Y_{l_c}^{m_c}(\theta,\phi)$$

Express your answer in terms of Clebsch Gordan coefficients.

Problem 2: Assume that a Hamiltonian is invariant with respect to rotations about the x and y axes. Show that it must alos be invariant with respect to rotations about the z axis.

Problem 3: Express the spherical harmonics $Y^2_m(\theta,\phi)$ in term of the Cartesian coordinates, $x/r, y/r, z/r$. Convince yourself that each Y^2_m is a homogeneous polynomial of degree 2 in these quantities.

Problem 4: The spherical harmonics $Y^1_m(\hat{\mathbf{r}})$ are simultaneous eigenstates of L^2 and L_z with the eigenvalues $1(1+1) = 2$ and m. Use properties of rotations to find linear combinations of these states that are simultaneous eigenstates of L^2 and L_y with eigenvalues 2 and m.

Problem 5: Assume that a spinless particle is bound to a rotationally invariant potential and assume that it is in an eigenstate of L^2 with eigenvalue $2(2+1) = 6$. Show that this state must be degenerate. (this means that there is more than one eigenstate with the same energy eigenvalue).

Problem 6: Using the $|n_+, n_-\rangle$ basis for the angular momentum states find operators (in terms of $a_\pm$ and $a^\dagger_\pm$ that raise and lower the eigenvalue j without changing m?.

Homework 3 – solutions

① recall $Y_\ell^m(\theta\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, D^{*\ell}_{m0}(R)$

$$\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, Y_{\ell_a}^{m_a}(\theta\phi)\, Y_{\ell_b}^{m_b}(\theta\phi)\, Y_{\ell_c}^{m_c}(\theta\phi) =$$

$$\sqrt{\frac{2\ell_a+1}{4\pi}}\sqrt{\frac{2\ell_b+1}{4\pi}}\sqrt{\frac{2\ell_c+1}{4\pi}} \int dR \times 4\pi$$

$$D^{*\ell_a}_{m_a0}(R)\, D^{*\ell_b}_{m_b0}(R)\, D^{*\ell_c}_{m_c0}(R) =$$

$$\sqrt{\frac{2\ell_a+1}{4\pi}}\sqrt{\frac{2\ell_b+1}{4\pi}}\sqrt{\frac{2\ell_c+1}{4\pi}} \cdot 4\pi \int dR$$

$$\sum C^{\ell\,\ell_a\,\ell_b}_{m\,m_a\,m_b}\, C^{\ell\,\ell_a\,\ell_b}_{0\,0\,0}\, C^{\ell'\,\ell\,\ell_c}_{m'\,m\,m_c}$$

$$C^{\ell'\,\ell\,\ell_c}_{0\,0\,0}\, D^{\ell'*}_{m'0}(R) =$$

$$\boxed{\sqrt{\frac{(2\ell_a+1)(2\ell_b+1)(2\ell_c+1)}{4\pi}} \times C^{\ell_c\,\ell_a\,\ell_b}_{-m_c\,m_a\,m_b}\, C^{\ell_c\,\ell_a\,\ell_b}_{0\,0\,0}\, C^{0\,\ell_c\,\ell_c}_{0\,-m_c\,m_c}\, C^{0\,\ell_c\,\ell_c}_{0\,0\,0}}$$

(2) $[J_z, H] = i[[J_x, J_y], H] =$

$-i[H[J_x, J_y]] =$

$-i[\underbrace{[H, J_x]}_{0}, J_y] - i[J_x \underbrace{[H, J_y]}_{0}] = 0$

here we used

$[A[BC]] + [B[AC]] + [C[BA]] = 0$

you can check this - it is called the Jacobi identity

(3) $Y_2^{-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} e^{-2i\phi} \sin^2\theta =$

$\frac{1}{4}\sqrt{\frac{15}{2\pi}} (\sin\theta(\cos\phi - i\sin\phi))^2$

$\frac{1}{4}\sqrt{\frac{15}{2\pi}} \left(\frac{x - iy}{r}\right)^2$

$Y_2^{-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}} e^{-i\phi} \sin\theta\cos\theta =$

$\frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta(\cos\phi - i\sin\phi)\cos\theta =$

$\frac{1}{2}\sqrt{\frac{15}{2\pi}} \frac{(x - iy)z}{r^2}$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta - 1)$$

$$= \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(\frac{3z^2 - r^2}{r^2}\right)$$

$$Y_2^1 = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\,e^{i\phi}\sin\theta\cos\theta =$$

$$= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\,\sin\theta(\cos\phi + i\sin\phi)\cos\theta$$

$$= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\,\frac{(x+iy)z}{r^2}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,e^{2i\phi}\sin^2\theta$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,(\sin\theta(\cos\phi + i\sin\phi))^2$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\frac{(x+iy)^2}{r^2}$$

These are homogeneous polynomials in $\vec{r}/r$ of degree $l=2$

④

④ consider

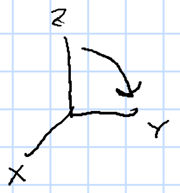
$$J_y \mathcal{U}^\dagger |j\mu\rangle = \mathcal{U}^\dagger J_z |j\mu\rangle$$
$$= \mu\, \mathcal{U}^\dagger |j\mu\rangle$$

multiply by $\mathcal{U}$

$$\mathcal{U} J_y \mathcal{U}^\dagger |j\mu\rangle = \mu |j\mu\rangle$$
$$= J_z |j\mu\rangle$$

we need to find $\mathcal{U}(R)$ satisfying

$$\mathcal{U}(R) J_y \mathcal{U}^\dagger(R) = \sum J_i R_{iy} = J_z$$



This corresponds to a rotation about the x axis by $-\pi/2$

$$R_x(-\tfrac{\pi}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$|j\mu\rangle_y = \mathcal{U}(R_y(-\tfrac{\pi}{2}))\,|j\mu_z\rangle$$

⑤ Assume

$$0 = [H, \vec{J}] \qquad \vec{J} = \vec{L} + \vec{S} = \vec{L} + \vec{0} = \vec{L}$$

$$= [H, \vec{L}]$$

It follows that $[H, \vec{L}^2] = 0$

$$L^2 |\psi\rangle = 2(2+1) |\psi\rangle$$

This means

$$\begin{aligned} |\psi\rangle = \; & |2\,{-2}\rangle\langle 2\,{-2}|\psi\rangle + \\ & |2\,{-1}\rangle\langle 2\,{-1}|\psi\rangle + \\ & |2\,0\rangle\langle 2\,0|\psi\rangle + \\ & |2\,1\rangle\langle 2\,1|\psi\rangle + \\ & |2\,2\rangle\langle 2\,2|\psi\rangle \end{aligned}$$

It follows that $L^+|\psi\rangle$ or $L^-|\psi\rangle$ is non 0, independent of $|\psi\rangle$ an an eigenstate of L^2 with eigenvalue

$$2(2+1) = 6$$

(6) Recall $n_t = j \pm u$

To increase j and not change u

$n_+ \to n_+ + 1$ $\quad n_+ + n_- \to n_+ + n_- + 2$

$n_- \to n_- + 1$ $\quad n_+ - n_- \to n_+ - n_-$

This means

$|n_+ + 1 \; n_- + 1\rangle =$

$a_+^\dagger a_-^\dagger |n_+ n_-\rangle \sqrt{(n_+ + 1)(n_- + 1)}$

$$\boxed{O = a_+^\dagger a_-^\dagger \sqrt{(n_+ + 1)(n_- + 1)}}$$