Phys 5742
Homework 2 - Due Friday 2/24

Problem 1: Starting from

$$[J_i, J_j] = i\sum_{k=1}^{3} \epsilon_{ijk} J_k$$

and

$$J_\pm = J_x \pm iJ_y = J_1 \pm J_2$$

show

$$[\mathbf{J}^2, J_i] = 0 \qquad [J_3, J_\pm] = \pm J_\pm \qquad J_\mp J_\pm = \mathbf{J}^2 - J_z^2 \mp J_z$$

Problem 2: Using

$$[\mathbf{J}_a, \mathbf{J}_b] = 0 \qquad [J_{ai}, J_{aj}] = i\sum_{k=1}^{3} \epsilon_{ijk} J_{ak} \qquad [J_{bi}, J_{bj}] = i\sum_{k=1}^{3} \epsilon_{ijk} J_{bk}$$

Show for $\mathbf{J}_{ab} = \mathbf{J}_a + \mathbf{J}_b$ that

$$[J_{abi}, J_{abj}] = i\sum_{k=1}^{3} \epsilon_{ijk} J_{abk}$$

Problem 3 Using the definitions

$$a_+|n_+, n_-\rangle = |n_+ - 1, n_-\rangle\sqrt{n_+} \qquad a_-|n_+, n_-\rangle = |n_+, n_- - 1\rangle\sqrt{n_-}$$

show

$$[a_+, a_+^\dagger] = 1 \qquad [a_-, a_-^\dagger] = 1$$

and all other commutators involving the operators $a_+, a_+^\dagger, a_-, a_-^\dagger$ vanish.

Problem 4: Show

$$\sum_{j=|j_a-j_b|}^{j_a+j_b} (2j+1) = (2j_a+1)(2j_b+1)$$

Problem 5: Using the expression for $D^j_{\mu\nu}(R)$ in either the class notes or the notes on rotations show that

$$D^{\frac{2}{2}}_{\mu\nu}(R) = R_{\mu\nu}$$

Problem 6: Find the non-zero Clebsch-Gordan coefficients $C^{j\frac{1}{2}\frac{1}{2}}_{\mu\mu_1\mu_2}$ for $j = 0$ and $j = 1$

$$|j, \mu\rangle = \sum_{\mu_1=-j_1}^{j_1} \sum_{\mu_2=-j_2}^{j_2} |\frac{1}{2}, \mu_1\rangle|\frac{1}{2}, \mu_2\rangle C^{j\frac{1}{2}\frac{1}{2}}_{\mu\mu_1\mu_2}$$

Solutions Homework #2

①

a) $[J^2, J_i] = [\sum J_j J_j, J_i] =$

$\sum_j (J_j [J_j, J_i] + [J_j, J_i] J_j =$

$i \sum_{jk} \varepsilon_{ijk} (J_j J_k + J_k J_j) =$

Let $k \to j$ $j \to k$ in the second term

$i \sum \varepsilon_{ijk} J_j J_k + i \sum \varepsilon_{ikj} J_j J_k =$

$i \sum \varepsilon_{ijk} (J_j J_k - J_j J_k) = 0$

b) $[J_3 \, J_\pm] = [J_3, J_1 \pm i J_2]$

$= i J_2 \pm (-i J_1)$

$= \pm (J_1 \pm i J_2)$

$= \pm J_\pm$

c) $J_\mp J_\pm = (J_1 \mp i J_2)(J_1 \pm i J_2) =$

$= J_1^2 + J_2^2 + J_3^2 - J_3^2 \pm i [J_1 J_2]$

$= J^2 - J_3^2 \mp J_3$

② $[J_a^i + J_b^i, J_a^j + J_b^j] =$

$[J_a^i J_a^j] + [J_b^i J_b^j] + \underbrace{[J_a^i, J_b^j]}_{0}$ $\underbrace{[J_b^i J_a^j]}_{0}$

$\sum i \varepsilon_{ijk} (J_a^k + J_b^k)$

$= i \sum_k \varepsilon_{ijk} J_{ab}^k$

③ $(a_+ a_+^\dagger - a_+^\dagger a_+)|n_+\rangle =$

$a_+ |n_+ + 1\rangle \sqrt{n_+ + 1} - a_+^\dagger |n_+ - 1\rangle \sqrt{n_+} =$

$(n_+ + 1)|n_+\rangle - |n_+\rangle n_+ = |n_+\rangle$

since $|n_+\rangle$ is arbitrary

$[a_+, a_+^\dagger] = 1$

The proof on a_- is identical – just replace $+$ by $-$

The vanishing of the other commutators is obvious.

④ $$\sum_{j=|j_a - j_b|}^{j_a + j_b} (2j+1) =$$

note $2j$ increases linearly with j

$$\sum_{j=|j_a - j_b|}^{j_a + j_b} 2j = 2(\text{average of } j)(\#\text{ terms})$$

$(j_a \geq j_b)$

$$= 2\,\frac{j_a + j_b + j_a - j_b}{2} \times (j_a + j_b - j_a + j_b + 1)$$

$$= 2 j_a (2 j_b + 1)$$

add $\sum_{j=|j_a - j_b|}^{j_a + j_b} 1 = (2 j_b + 1) = N$

$$\sum_{j=|j_a-j_b|}^{j_a+j_b} (2j+1) = (2j_b+1)(2j_a+1)$$

(5) when $j=\frac{1}{2}$ $D^{j}_{\mu\nu}(R)$ is a homogeneous polynomial of degree 1 - This means for each factor of μ,ν only 1 term contributes.

The expression for $D^{j}_{\mu\nu}(R)$ is

$$\sum_{k=0}^{j+\nu} \frac{\sqrt{(j+\mu)!\,(j-\mu)!\,(j+\nu)!\,(j-\nu)!}}{k!\,(j+\mu-k)!\,(j+\nu-k)!\,(k-\mu-\nu)} R_{++}^{k}\, R_{-+}^{j+\nu-k}\, R_{+-}^{j+\mu-k}\, R_{--}^{k-\mu-\nu}$$

note $\left(\frac{1}{2}+\frac{1}{2}\right)! = 1$

If $k=1 \Rightarrow j+\mu=1 \quad j+\nu=1 \Rightarrow \mu=\nu=\frac{1}{2} \quad k-\mu-\nu=0$

$\therefore$ $k=1 \Rightarrow R^{++}\cdot 1\cdot 1\cdot 1$

$k=0 \quad \mu=\frac{1}{2} \Rightarrow \nu=-\frac{1}{2}$ $\qquad 1\; R_{+-}\; 1\cdot 1$

$k=0 \quad \nu=\frac{1}{2} \quad \mu=-\frac{1}{2}$ $\qquad 1\; R_{-+}\; 1\; 1$

$k=0 \quad \mu=-\frac{1}{2} \quad \nu=-\frac{1}{2}$ $\qquad 1\cdot 1\cdot 1\; R_{--}$

This shows $D^{\frac{1}{2}}_{\mu\nu}(R) = R$

⑥

$$|1\,1\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$J^-|1\,1\rangle = |1\,0\rangle\sqrt{(1+1)(1-1+1)}$$

$$= |1\,0\rangle\sqrt{2}$$

$$J^-|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle =$$

$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}$$

$$+\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}$$

$$= |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\therefore\ |1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)$$

$$J^-|1\,0\rangle = |1\,-1\rangle\sqrt{(1+0)(1-0+1)} = |1\,-1\rangle\sqrt{2}$$

$$= \frac{1}{\sqrt{2}}\,J^-\left(|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}$$

$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\Big)$$

$$\therefore\ |1\,-1\rangle = \frac{1}{2}\left(|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)$$

$$= |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

since $|00\rangle \perp |10\rangle$

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}-\tfrac{1}{2}\rangle + |\tfrac{1}{2}-\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right)$$

$\Rightarrow$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}-\tfrac{1}{2}\rangle - |\tfrac{1}{2}-\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle\right)$$

$\uparrow$

Condon Shortly convention

$$C^{1\frac{1}{2}\frac{1}{2}}_{1\frac{1}{2}\frac{1}{2}} = 1 \qquad C^{1\frac{1}{2}\frac{1}{2}}_{-1-\frac{1}{2}-\frac{1}{2}} = 1$$

$$C^{1\frac{1}{2}\frac{1}{2}}_{0\frac{1}{2}-\frac{1}{2}} = C^{1\frac{1}{2}\frac{1}{2}}_{0-\frac{1}{2}\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$C^{0\frac{1}{2}\frac{1}{2}}_{0\frac{1}{2}-\frac{1}{2}} = -C^{0\frac{1}{2}\frac{1}{2}}_{0-\frac{1}{2}\frac{1}{2}} = \frac{1}{\sqrt{2}}$$