Phys 5742
Homework 1 - Due Friday 1/26

Problem 1: Show that that if $U(\lambda)$ satisfies $U(\lambda_1 + \lambda_2) = U(\lambda_1)U(\lambda_2)$ then $U(\lambda)$ cannot be antiunitary.

Problem 2: Assume a vector $\mathbf{V}$ is rotated about the z axis (active rotation).to Show that the components of the rotated $\mathbf{V}$ are related to the components of the original $\mathbf{V}$ by

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}. \tag{1}$$

Problem 3 Differentiate

$$\mathbf{V}' = U(\theta)\mathbf{V}U^\dagger(\theta) =$$

$$e^{-iG\theta}\mathbf{V}e^{iG\theta} = R^t\mathbf{V} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{2}$$

with respect to θ and then set θ to zero to show:

$$-[iG, \mathbf{V}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}. \tag{3}$$

Component by component this gives

$$[G, V_x] = iV_y, \qquad [G, V_x] = -iV_x, \qquad [G, V_z] = 0 \tag{4}$$

where G is the infinitesimal generator of rotations about the z axis.

Problem 4: Show that a general $SU(2)$ matrix can be expressed in the form

$$\boxed{W = e^{-i\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \cos(\frac{\theta}{2})I - i\hat{\boldsymbol{\theta}}\sin(\frac{\theta}{2})}. \tag{5}$$

Problem 5: Use (5) to show that for

$$W = e^{-i\frac{1}{2}\theta\sigma_x} \tag{6}$$

that

$$R_{lj} = \frac{1}{2}\mathrm{Tr}(\sigma_l W \sigma_j W^\dagger) \tag{7}$$

gives R corresponding to a rotation about the $\hat{\mathbf{z}}$ axis:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 6: Show that if a Hamiltonian is rotationally invariant, $U(\theta)HU^{\dagger}(\theta) = H$ then $[G, H] = 0$ and G is a conserved quantity.

Homework 1 solutions

(1) $|\psi'\rangle = T(\lambda/2)|\psi\rangle$

$|\psi''\rangle = T(\lambda/2)|\psi'\rangle = T(\lambda/2)T(\lambda/2)|\psi\rangle$

$= T(\lambda)|\psi\rangle$

similarly

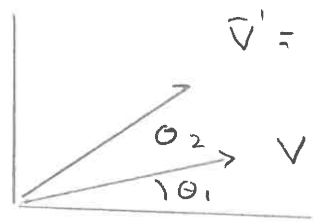
$|\phi'\rangle = T(\lambda/2)|\phi\rangle$

$|\phi''\rangle = T(\lambda/2)|\phi'\rangle = T(\lambda)|\psi\rangle$

$\langle\psi''|\phi''\rangle = \langle\phi'|\psi'\rangle = \langle\psi|\phi\rangle$

This shows that $T(\lambda)$ is unitary

(2) consider



$\bar{V}' = V\cos(\theta_2+\theta_1)\hat{x} + V\sin(\theta_2+\theta_1)\hat{y}$

$V\cos\theta_1\hat{x} + V\sin\theta_1\hat{y} = \bar{V}$

use trig identities

$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1$

$V' = (V\cos\theta_1\cos\theta_2 - V\sin\theta_1\sin\theta_2)\hat{x} +$

$V\sin\theta_1\cos\theta_2 + V\sin\theta_2\cos\theta_1)\hat{y}$

$= (\cos\theta_2 V_x - \sin\theta_2 V_y)\hat{x} +$

$(V_y \cos\theta_2 + V_x \sin\theta_1)\hat{y}$

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

③ $$-i\sigma \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} - \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}(-i\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$-i(\sigma V_x - V_x\sigma) = V_y$

$-i(\sigma V_y - V_y\sigma) = -V_x$

$-i(\sigma V_z - V_z\sigma) = 0$

$[\sigma, V_x] = i V_y$

$[\sigma, V_y] = -i V_x$

$[\sigma, V_z] = 0$

④ $M = e^{i(c + \bar{b}\cdot\bar{\sigma})}$ general unitary matrix

$1 = \det M = e^{Tr(cI + b\cdot\bar{\sigma})} = e^{2c + \bar{b}\cdot Tr(\bar{\sigma})}$

$= e^{2c}$

this require $c = 0, i\pi$ $\quad e^{i\pi} = -1$

this means

$M = \pm e^{i\bar{b}\cdot\bar{\sigma}}$

$= \pm\left(1 + \sum \frac{(i)^n}{n!}(\bar{b}\cdot\bar{\sigma})^n\right)$

note $(\bar{b}\cdot\bar{\sigma})^2 = \sum_{ij} b_i b_j \sigma_i \sigma_j = b^2 + i\,\bar{b}\times\bar{b} = b^2$

$$\mathcal{M} = \pm\left(1+\sum_{n=1}^{\infty}\frac{(-)^n}{(2n)!}\,b^{2n}\right) +$$

$$\pm\left(i\,\hat{b}\cdot\bar{\sigma}\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)!}\right)$$

$$= \pm\left(\cos b\; I + i\,\hat{b}\cdot\bar{\sigma}\,\sin b\right)$$

choose $\bar{b} = -i\,\frac{\bar{\Theta}}{2}$

$$\mathcal{M} = \pm\left(\cos\frac{\theta}{2}\, I - i\,\hat{\theta}\cdot\bar{\sigma}\,\sin\left(\frac{\theta}{2}\right)\right)$$

(5) For $W = e^{-\frac{i}{2}\theta\sigma_z} = \cos\left(\frac{\theta}{2}\right) - i\,\sigma_z \sin\left(\frac{\theta}{2}\right)$

$$R_{ij} = \frac{1}{2}\,Tr\left(\sigma_i\left(\cos\left(\frac{\theta}{2}\right) - i\sigma_z \sin\left(\frac{\theta}{2}\right)\right)\sigma_j\left(\cos\left(\frac{\theta}{2}\right) + i\,\sigma_z \sin\frac{\theta}{2}\right)\right)$$

$$= \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right) Tr\left(\sigma_i \sigma_j\right)$$

$$\frac{1}{2}\sin^2\left(\frac{\theta}{2}\right) Tr\left(\sigma_i\,\sigma_z\,\sigma_j\,\sigma_z\right)$$

$$\frac{i}{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) Tr\left(\sigma_i\,\sigma_j\,\sigma_z - \sigma_i\sigma_z\sigma_j\right)$$

$$= \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right) 2\,\delta_{ij}$$

$$\frac{1}{2}\sin^2\left(\frac{\theta}{2}\right) Tr\left(\delta_{iz} + i\,\varepsilon_{izk}\sigma_k\right)\left(\delta_{iz} + i\,\varepsilon_{izn}\sigma_n\right)$$

$$\frac{i}{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) Tr\left(\sigma_i\, 2\,i\,\varepsilon_{jzk}\sigma_k\right)$$

note: even powers of σ are traceless

$$\mathrm{Tr}\,(\delta_{iz} + i\,\varepsilon_{izk}\sigma_k)(\delta_{jz} + i\,\varepsilon_{jzk}\sigma_k)$$

$$= \delta_{ij}\delta_{jz}\cdot 2 - \delta_{ij}\delta_{i\neq z}\cdot 2$$

$$\mathrm{Tr}\,(2i\,\sigma_i\,\varepsilon_{jzk}\sigma_k) = 4i\,\varepsilon_{jzi} = 4i\,\varepsilon_{ijz}$$

$$\therefore R_{ij} = \cos^2\left(\tfrac{\theta}{2}\right)\delta_{ij} + \sin^2\left(\tfrac{\theta}{2}\right)\delta_{ij}\delta_{iz} - \delta_{ij}\delta_{i\neq z}\sin^2\left(\tfrac{\theta}{2}\right)$$

$$- 2\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)\varepsilon_{ijz}$$

$$\begin{pmatrix} \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) & -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & 0 \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) & 0 \\ 0 & 0 & \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) $\dfrac{d\sigma}{dt} = i\,[H\,\sigma] = 0 \quad \therefore \sigma$ is conserved