GENERAL QUANTUM THEORY OF COLLISION PROCESSES

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INTRODUCTION

The two central problems of quantum physics are (1) Description of the properties, particularly of the possible energy values, of a single compact physical system (elementary particle, atom, general particle); (2) Description of collisions between such particles.

In this paper we are concerned with the second type of problem. The fundamental points of view concerning it date back to the first years of quantum mechanics and are treated in the book of Mott and Massey.[1] A number of important contributions, however, originated somewhat later. They were suggested partly by the problems of the quantum theory of fields, which necessitated a more abstract and general formulation of the collision process, and partly by the problem in nuclear physics, to order the experimental data on nuclear reactions without any exact knowledge of nuclear forces. Such contributions are the following: The introduction of the concept of the S matrix,[2] the general formulation of the mathematical problem of determining S,[3] the implications of S for the development of processes and the qualitative restrictions of S which result from this (see Section 3), the behavior of cross sections in the vicinity of resonances and thresholds (Section 4), analytic properties of S and related quantities,[4] determination of the potential from the S matrix.[4]

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1. N.F. Mott and H.S.W. Massey, *The Theory of Atomic Collisions* (Clarendon Press. Oxford, 1949).
2. J.A. Wheeler, Phys. Rev. **52**, 1107 (1937). W. Heisenberg, Z.f. Physik, 120, 513 (1943).
3. There are a large number of papers on this "formal theory of scattering." Here we forgo giving even an only approximately complete bibliography; the following three papers (all included in this book) contain the important parts of the conventional formalism: C. Møller, K. Danske Vidensk. Selsk. **23**, Nr. 1 (1945), B.A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950), M. Gell-Mann and M.L. Goldberger, Phys. Rev. **91**, 398 (1953). This formalism does not suffice for rearrangement collisions. See Section 2 of the present paper, and references given there (and in the introduction to this book).
4. We do not discuss these questions in detail in order to keep the size of this report within limits. See, however, R.G. Newton, J. Math. Phys. **1**, 319 (1960). (Included in this book.)

Section 1. Collision Theory

We first discuss a model which will serve to illustrate the concepts and methods of the papers mentioned in reference 3. The generalization of these methods is restricted in only one respect: we use in a fundamental way the assumption that the Hamiltonian operator can be split up in the form

$$H = H_0 + V, \tag{1.1}$$

where H_0 is to describe the kinetic energy in unperturbed motion of isolated particles, and V the interaction between them. If this assumption is not fulfilled (as in field theory, or even nuclear reactions with several channels), the argument has to be modified (see Section 2).

In order to be able to express ourselves as simply possible, we consider the case of scattering of a particle by a potential:

$$H_0 = -\nabla^2/2m;\ V = V(\mathbf{r}). \tag{1.2}$$

We shall assume that, in addition to the scattering states, there also exist a number of bound states (discrete eigenvalues of H). It is evident that the argument is not restricted to this simple example, but

only by the validity of assumption (1.1) and by some conditions on the interaction operator V, which will be formulated generally at the beginning of section 2.

Initial and Final Configuration. The Operators $\Omega^{(-)}$ and $\Omega^{(+)}$

Let the probability amplitude of a normalizable state Ψ for position $\mathbf{r}$ and time t be given by $\psi_t(\mathbf{r})$ (Schrödinger wave function). The asymptotic behavior of Ψ_t can be surveyed in a simple fashion. We split up Ψ into

$$\Psi = \Psi^c + \Psi^c, \tag{1.3}$$

where $\Psi^{\cdot}$ denotes the contribution from the discrete eigenstates of H, and Ψ^c the contribution of the continuum. The ψ_t^0 will give, for any time t, a probability distribution, which is concentrated in the vicinity of the origin (bound states). ψ_t^c, on the other hand, represents a wave packet whose centroid is, for large values of t, far from the origin and which is spread over a large region. In the limit $|t| \to \infty$ the probability density at a fixed point $\mathbf{r}$ decreases as t^{-3}. Now if the interaction is confined to the vicinity of the origin,[5] it no longer influences the behavior of such a wave packet. We can therefore find a solution $\varphi_t(\mathbf{r})$ of the Schrödinger equation without interaction such that

$$\lim_{t\to\infty} \psi_t^c(\mathbf{r}) - \varphi_t(\mathbf{r}) = 0. \tag{1.4}$$

It is important to state the exact meaning of asymptotic relations like (1.4). Let us denote the difference on the left side of (1.4) by $\delta_t(\mathbf{r})$. Then, not only the function δ_t vanishes in the limit for every fixed point

5. As we shall see, it suffices that $V(r)$ decrease faster than $1/r$ for large r. We do not here consider the limiting case of Coulomb scattering.

$\mathbf{r}$; this would already be true for ψ_t^c and φ_t separately. Rather, we have

$$\lim \int |\delta_t(\mathbf{r})|^2 d\mathbf{r} = 0. \tag{1.4a}$$

In order to generalize to more complex systems it is useful to employ a more abstract notation. We consider the function $\delta_t(\mathbf{r})$ for each fixed value of t as a vector in Hilbert space and consider the sequence of vectors δ_t as t increases to ∞. For such a sequence of vectors the formalism of Hilbert spaces defines two different types of convergence, which both play a role in our considerations. A sequence Φ_t is said to converge strongly to zero if the length (norm) of the vectors becomes smaller and smaller and vanishes in the limit. We denote the strong convergence by a double arrow, thus

$$\Phi_t \twoheadrightarrow 0 \text{ if } \lim \|\Phi_t\| = 0. \tag{1.5}$$

A sequence converges weakly to zero (single arrow) if the scalar product of Φ_t with every arbitrary, but fixed, vector vanish in the limit:

$$\Phi_t \to 0 \text{ if } \lim (\Psi, \Phi_t) = 0 \text{ for every fixed } \Psi. \tag{1.6}$$

In the same way we also distinguish between strong and weak convergence of a sequence of operators:

$$U_t \twoheadrightarrow U \quad \text{if for every fixed vector } \Psi\text{: } U_t\Psi \twoheadrightarrow U\Psi \tag{1.7}$$

$$U_t \to U \quad \text{if for every fixed vector } \Psi\text{: } U_t\Psi \to U\Psi \tag{1.8}$$

or, in other words, if the matrix elements of $U_t - U$ between arbitrary, fixed vectors Ψ and Φ vanish in the limit:

$$\lim \langle \Psi | U_t | \Phi \rangle = \langle \Psi | U | \Phi \rangle. \tag{1.8a}$$

Hence the theorem which was mentioned before is as follows:
To every solution ψ_t of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi, \tag{1.9}$$

which is orthogonal to all bound states, there corresponds a unique solution φ_t of the free-particle Schrödinger equation,

$$i\frac{\partial \varphi}{\partial t} = H_0\varphi \tag{1.10}$$

by the requirement

$$\lim_{t\to\infty}(\psi_t - \varphi_t) \to 0. \tag{1.4b}$$

Conversely, (1.4b) also establishes a unique correspondence of some solution of (1.9) to each solution of (1.10).

Here we shall show only the more easily proved second part of the statement. A mathematically rigorous proof of the statement under suitable conditions on the potential V is given in the paper by C. Cook.[6]

Let φ_t be a given solution of (1.10). We call $\psi_t^{t_0}$ that solution of (1.9) which agrees with φ_t for $t = t_0$, and $\chi_t^{t_0} = \psi_t^{t_0} - \varphi_t$. Then

$$\frac{\partial}{\partial t}\|\chi\|^2 = 2\mathrm{Im}\langle\chi|V|\varphi\rangle$$

therefore

$$\frac{\partial}{\partial t}\|\chi\| < \|V\varphi\| \tag{1.11}$$

for $t > t_0$ we then have

$$\|\chi\| < \int_{t_0}^{\infty}\|V\varphi_t\|dt. \tag{1.11a}$$

But

$$\varphi_t(\mathbf{r}) = (2\pi)^{-3/2}\int\tilde{\varphi}(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{r}-p^2t/2m)}d\mathbf{p},$$

6. J. Cook, Jour. of Math. and Phys., 36, 82 (1957). See also M.N. Hack, Phys. Rev. 96, 196 (1954); H. Moses, Nuovo Cim. 1, 104 (1955); J.M.Jauch, Helv. Phys. Acta. 31, 127, 661 (1958).

where $\tilde{\varphi}(\mathbf{p})$ denotes the wave function in momentum space. We may confine our attention to wave functions which have "sufficiently many" continuous derivatives; they form a dense set, i.e., any other state can be approximated arbitrarily accurately by such special states. If $\tilde{\varphi}(\mathbf{p})$ is sufficiently smooth, then the stationary phase method[7] gives the asymptotic form of $\varphi_t(\mathbf{r})$:

$$\varphi_t(\mathbf{r}) = e^{im(r^2/2t-\pi/4)}\left[\left(\frac{m}{t}\right)^{3/2}\tilde{\varphi}\left(\frac{m\mathbf{r}}{t}\right) - \frac{i}{2}\left(\frac{m}{t}\right)^{5/2}\nabla_p^2\tilde{\varphi}\left(\frac{m\mathbf{r}}{t}\right) + \ldots\right]. \tag{1.12}$$

Therefore, up to terms of higher order in $1/t$

$$\|V\varphi\|^2 = \left(\frac{m}{t}\right)^3\int V^2(r)\left|\tilde{\varphi}\left(\frac{m\mathbf{r}}{t}\right)\right|^2 d\mathbf{r}. \tag{1.13}$$

We assume, that $V(r)$ becomes singular only at the origin, if at all, and not more strongly than $1/r$, and that $V(r)$ vanishes for large distances faster than $r^{-(1+\epsilon)}$ $(\epsilon > 0)$. It then follows that $t^{2+2\epsilon}\|V\varphi_t\|^2 \to 0$ i.e., the norm of $V\varphi_t$ vanishes faster than $1/t$ for large values of the time. Consequently, the integration in (1.11a) can be performed and the integral tends toward zero with increasing t_0. This is what we set out to prove.

We now fix our attention specifically on the two functions $\psi_0(\mathbf{r})$ and $\varphi_0(\mathbf{r})$. The former yields the real probability distribution (corresponding to the state Ψ) at time $t = 0$. The latter will be called the final configuration (run back to $t = 0$); it is obtained from the results of observations at a very large time T ("after completion of the collision") and calculated back to time $t = 0$, assuming no interaction between $t = 0$ and $t = T$. We express the relationship between ψ_0 and φ_0 by means of the operators H and H_0:

$$\psi_t = e^{-iHt}\psi_0;\quad \varphi_t = e^{-iH_0t}\varphi_0. \tag{1.14}$$

7 See, e.g., G Braun, Acta Physica Austriaca, 10, 18 (1956).

Hence

$$\psi_0 = \Omega^{(+)}\varphi_0 \tag{1.15}$$

where

$$U(t) = e^{iHt}e^{-iH_0t} \underset{t\to+\infty}{\Rightarrow} \Omega^{(+)}. \tag{1.16}$$

The state of affairs discussed above can now be formulated as follows: The operator $\Omega^{(+)}$ maps the whole Hilbert space $\mathfrak{h}$ on to the subspace $\mathfrak{h}^c$ of scattering states", i.e. states which contain no components in the discrete spectrum of H. $\Omega^{(+)}$ is the limit (in the sense of strong convergence of operators) of a sequence of unitary operators U(t). If bound states exist, $\Omega^{(+)}$ itself is no longer unitary, but certainly still isometric, i.e., for arbitrary vectors φ_1, φ_2, we have

$$(\Omega^{(+)}\varphi_1, \Omega^{(+)}\varphi_2) = (\varphi_1, \varphi_2) \tag{1.17}$$

or[8]

$$\Omega^{(+)\dagger}\Omega^{(+)} = 1. \tag{1.17a}$$

To prove (1.17) we imagine $\Omega^{(+)}$ replaced by U(t). The relation is then trivially true, because U is unitary, and it is preserved in the limit because U(t) converges strongly toward $\Omega^{(+)}$.

The Hermitian adjoint operator $\Omega^{(+)\dagger}$ is defined by

$$\langle\psi|\Omega^{(+)\dagger}|\varphi\rangle = \langle\varphi|\Omega^{(+)}|\psi\rangle^*. \tag{1.18}$$

From the properties of $\Omega^{(+)}$which were mentioned, it follows that $\Omega^{(+)\dagger}$ annihilates the subspace $\mathfrak{h}^D$ of bound states, and that it maps $\mathfrak{h}^c$ back on the whole $\mathfrak{h}$. The inverse of (1.17a) is therefore

$$\Omega^{(+)}\Omega^{(+)\dagger} = P^c, \tag{1.19}$$

where P^c denotes the projection operator of the scattering states.

8. We use the notation $A^\dagger$ for the Hermitian adjoint of the operator A and a^* for the complex conjugate of the number a.

It may seem strange at first sight, that $\Omega^{(+)}$, a strong limit of unitary operators is not itself unitary. But this can easily be understood. If we start with any fixed vector φ, its image vector $U(t)\varphi$ will have a smaller and smaller component in the subspace of bound states as t becomes larger and larger. Conversely, the inverse operator $U^\dagger(t)$ does converge strongly toward $\Omega^{(+)\dagger}$ in the subspace $\mathfrak{h}^c$, but only weakly in the subspace $\mathfrak{h}^D$. For if $U^\dagger(t)$ is applied to a bound state of energy E, the result is $e^{-iEt}e^{iH_0t}\psi$, that is, a more and more widely dispersed wave packet with constant norm. This sequence of states converges weakly toward zero because the scalar product with any fixed vector becomes smaller and smaller.

All the arguments to this point may be repeated word by word with respect to the behavior of ψ_t for $t \to -\infty$. Again the Schrödinger equation without interaction has a solution which asymptotically agrees with ψ_t in this region. Its space dependence, i.e., the function $\varphi_0^1(\mathbf{r})$, will be called the initial configuration (run back to $t = 0$). Thus, in analogy to (1.15), an isometric operator $\Omega^{(-)}$ is defined by

$$\psi_0 = \Omega^{(-)}\varphi_0^1, \tag{1.20}$$

$$U(t) \underset{t\to+\infty}{\Rightarrow} \Omega^{(-)}. \tag{1.21}$$

From (1.16) it follows that for every finite τ

$$e^{iE\tau}\Omega^{(\pm)}e^{-iH_0\tau} = \Omega^{(\omega)} \tag{1.22}$$

and hence the important relation

$$H\Omega^{(\pm)} = \Omega^{(\pm)}H_0. \tag{1.23}$$

The S matrix in the Heisenberg Picture

The fundamental concepts introduced below may at first sight appear to be, to some extent, cumbersome and inappropriate to the collision problem. On the

other hand, they have the widest domain of application, since they are not related—as the interaction picture is—to the decomposition (1.1); furthermore, they do not destroy the explicit Lorentz-invariance of a relativistic theory.

The notion of "state" here does not refer to the time. Rather, to a given apparatus which prepared the physical situation under discussion, there corresponds a unique well-defined state and, mathematically, a fixed vector in the Hilbert space. If we want to characterize a particular state physically, we could, for example, give the expected results of measurements made on the state at an arbitrary time t. Let us call such a statement, for brevity ξ, and consider ξ as an index of the state Ψ_ξ. We are still free to choose any mathematical representation of the corresponding vector in the Hilbert space.

Example: Suppose ξ consists of the statement that we have at time t_1 a Gaussian wave packet, with $\mathbf{r}_0$ its center of mass, $\mathbf{p}_0$ the mean value of the momentum, and $\mathbf{x}$ the spread of the momentum distribution. To give a mathematical characterization, we use the position representation at the time t_2 i.e., we characterize the state by means of a function $\psi(\mathbf{r})$. Consequently, this function represents the probability amplitude, at the place $\mathbf{r}$ and at the time t_2, for that state which appeared at the time t_1, as the gaussian wave packet described above.

We now use this point of view in characterizing scattering states by means of their initial or final configuration. Let $\Psi_\varphi^{(-)}$ be the state of initial configuration φ, $\Psi_\varphi^{(+)}$ the one with final configuration φ. For each complete orthonormal system of functions $\varphi_k(\mathbf{r})$, we have in $\mathfrak{h}^c$ two complete orthonormal systems, the $\Psi_k^{(-)}$ and the $\Psi_k^{(+)}$. Therefore, if we confine ourselves to $\mathfrak{h}^c$ we can define in this space a unitary operator S as the mapping of the basic vectors $\Psi_k^{(+)}$ onto the corresponding $\Psi_k^{(-)}$

$$\Psi_k^{(-)} = S\Psi_k^{(+)}. \tag{1.24}$$

The probability amplitude for a state of incoming configuration φ_k to have the outgoing configuration φ_l is:

$$\begin{aligned} S_{lk} &= (\Psi_l^{(+)}, \Psi_k^{(-)}) = \langle \Psi_l^{(+)} | S | \Psi_k^{(+)} \rangle \\ &= \langle \Psi_l^{(-)} | S | \Psi_k^{(-)} \rangle. \end{aligned} \tag{1.25}$$

To obtain the last expression one uses the unitary property of S.

S can be expressed in terms of $\Omega^{(+)}$ and $\Omega^{(-)}$. From (1.15) and (1.20) we have[9]

$$\Psi_k^{(+)} = \Omega^{(+)}\varphi_k;\ \Psi_k^{(-)} = \Omega^{(-)}\varphi_k \tag{1.26}$$

$$S = \Omega^{(-)}(\Omega^{(+)})^\dagger. \tag{1.27}$$

The matrix elements of S (that is, the physically interesting quantities) may also be written.

$$\begin{aligned} S_{lk} &= \langle \Psi_l^{(+)}, \Psi_k^{(-)} \rangle = \langle \varphi_l | \Omega^{(+)\dagger} \Omega^{(-)} | \varphi_k \rangle \\ &= \langle \varphi_l | S' | \varphi_k \rangle \end{aligned} \tag{1.28}$$

where

$$S' = \Omega^{(+)\dagger}\Omega^{(-)}. \tag{1.29}$$

Although S is defined only in $\mathfrak{h}^c$, S' is defined and unitary in the whole $\mathfrak{h}$. Eq. (1.23) shows that S commutes with H, S' with H_0:

$$[H, S] = 0, \tag{1.30}$$

$$[H_0, S'] = 0. \tag{1.31}$$

9. For reasons of conceptual clarity one should note that in (1.26) the "states" Ψ and the "configuration functions" φ are considered as elements of the same Hilbert space. This is possible in our simple model (1.2), but not (at least, not uniquely) in reactions with more than one channel.

Interaction Picture, Integral Equations for Collision Problems and Bound States

The notion of "state" has a different meaning in the interaction picture. We speak of a "state at time t" and mean, roughly speaking, the probability distribution, which has been determined by observations at the time t, and then has been transformed to the time 0 using the free Hamiltonian. In particular, in this picture the "state at $t = -\infty$" is nothing else than the previously described incoming configuration, the one at $t = +\infty$ nothing else than the outgoing configuration. For this reason, the interaction picture, in the cases in which it can be used, gives a very elegant formulation of the collision problem. The relation between the state vector χ_t in this picture and the corresponding Heisenberg-vector Ψ is

$$\chi_t = e^{\frac{i}{\hbar}H_0 t} e^{-\frac{i}{\hbar}Ht}\Psi = U^{\dagger}(t)\Psi. \tag{1.32}$$

Therefore

$$\chi_{t_2} = U(t_2, t_1)\chi_{t_1} \tag{1.33}$$

with

$$U(t_2, t_1) = U^{\dagger}(t_2)U(t_1). \tag{1.34}$$

It is easily shown that the transformation operator $U(t_2, t_1)$ is a solution of the equation of motion

$$i\frac{\partial U}{\partial t_2}(t_2, t_1) = \bar{V}(t_2)U(t_2, t_1), \tag{1.35}$$

where

$$\bar{V}(t) = e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H_0 t}. \tag{1.36}$$

If we put $t_1 = -\infty$ and remember the boundary condition $\lim_{t\to-\infty} U(t, -\infty) \Rightarrow 1$, we obtain the integral equation

$$U(t, -\infty) = 1 - i\int_{-\infty}^{t} \bar{V}(t')U(t', -\infty)dt'. \tag{1.37}$$

This integral equation[10] provides a useful starting point for the calculation of $U(t, -\infty)$, and thus of S', by means of perturbation theory.[11]

If we use $U(t, -\infty) = 1$ as our zero order approximat we obtain by means of iteration the higher order co tributions

$$U^{(1)}(t, -\infty) = -i\int_{-\infty}^{t} \bar{V}(t')dt',$$

$$U^{(2)}(t, -\infty) = (-i)^2 \int_{t>t''>t'>-\infty} \bar{V}(t'')\,\bar{V}(t')dt''dt' \tag{1.38}$$

and so on.

Remark: Let us apply (1 37) to a vector φ' and call $U(t, -\infty)\,\varphi' = \varphi_t$.

We obtain

$$\varphi_t = \varphi' - i\int_{-\infty}^{t} \bar{V}(t')\varphi_{t'}dt'. \tag{1.37a}$$

Obviously, it is here possible to think of φ' as the incoming configuration of a Heisenberg state, φ_0 as the true probability amplitude at t = 0, and of $\varphi_{+\infty}$ as the outgoing configuration. The integral converges strongly for $t \to -\infty$, i.e.,

$$\lim_{t\to-\infty} \left\| \int_{-\infty}^{t} \bar{V}(t')\varphi_{t'}dt' \right\| \to 0. \tag{1.39}$$

Now if there exist bound states, then for each such state ψ we have

$$\Omega^{(-)\dagger}\psi = 0 \tag{1.40}$$

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10. S. Tomonaga, Progr. Theor. Phys. 1, 27 (1946); J. Schwinger, Phys. Rev. 74, 1439 (1948).
11. For application to quantum electrodynamics see, e.g., F.J. Dyson, Phys. Rev. 75, 1736 (1949).

(see p. 19). If we put $t_1 = 0$ in (1.35) and observe that $U(t)$ converges weakly to $\Omega^{(-)\dagger}$, we obtain the integral equation

$$U(t,0) = \Omega^{(-)\dagger} - i\int_{-\infty}^{t} \bar{V}(t')U(t',0)dt' \quad \text{(weak convergence for } t \to -\infty\text{)}.$$

Applying this equation to the bound state ψ one obtains according to (1.40)

$$\varphi_t = -i\int_{-\infty}^{t} \bar{V}(t')\varphi_{t'}dt' \quad \text{(weak convergence for } t \to -\infty\text{)}, \qquad (1.41)$$

where

$$\varphi_t = U(t,0)\psi.$$

Integral equations of this kind have been used in field theory to determine bound states.[12] A comparison with (1.37a) shows that, if bound states exist, the solution of (1.37a) is uniquely determined by the incoming configuration only if one requires strong convergence, i.e., if one postulates condition (1.39).

The Adiabatic Theorem

The whole discussion so far has been based on the fact that the effective interaction vanishes for (large) positive or negative times. This happens even if V does not depend explicitly on time, because for $|t| \to \infty$ the particle is with highly predominant probability far away from the scattering center. Therefore, our results remain unchanged if we suppose that the interaction is "switched on and off adiabatically," say by means of the formal substitution of $Ve^{-\gamma|t|}$ for V ($\gamma \to 0$ at the end of the calculation). For each (normalizable) state there is a γ sufficiently small, so that $e^{-\gamma|t|}$ is appreciably different from 1 only for those values of the time for which the effective interaction vanishes anyway.

As long as $\gamma \neq 0$, the equation (1.4) now holds also for non-normalizable wave functions, for example if one

12. H.A. Bethe and E.E. Salpeter, Phys. Rev. 84, 1232 (1951).

chooses for φ a plane wave. Thus one obtains, from (1.15) and (1.23), the following theorem.[13,14]

If the time dependent Schrödinger equation is solved with the interaction $V(r)e^{-\gamma|t|}$ and the initial condition $\psi_t(r) \to e^{i(p\cdot r - \varepsilon t)}$ for $t \to -\infty$, then the limit $\gamma \to 0$ gives for each finite time a solution of the stationary Schrödinger equation $H\psi = E\psi$. Therefore, the interaction switched on adiabatically carries any eigenfunction of H_0 into an eigenfunction of H with the same eigenvalue.

The method of adiabatic switching on has been important at a certain stage of the development of field theory, for reasons which will become apparent in the next section. This method can always be replaced by a sequence of states; physically speaking, by the notion that, in reality, one never works with infinite plane waves, but only with limited wave packets. For purposes of practical calculations, one may go over to plane waves also in this approach and we shall do this in the following, frequently and without explicit warning. We then only have to bear in mind that we are dealing with improper functions, which become physically meaningful only if integrated over a test function (wave packet). The most important formulae needed in calculations involving these improper functions are assembled in the appendix.

Stationary Calculation

It follows, from the definition of $U(t,-\infty)$ and Equation (1.23),

$$U(t,-\infty) = e^{iH_0t}\Omega^{(-)}e^{-iH_0t}. \qquad (1.42)$$

13. M. Born and V. Fock, Zeits. fur Physik 51, 165 (1928).
14. It is necessary to prove something more, i.e.: If one starts with the incoming configuration

$$\varphi(r) = \begin{cases} e^{ip\cdot r} & \text{for } |r| \lessgtr R \\ 0 & \end{cases}$$

and the interaction $V(r)\,e^{-\gamma|t|}$, and one calculates the state Ψ which corresponds to this in the sense of equation (1.4), then it is possible to interchange the order in the limiting procedures $\gamma \to 0$ and $R \to \infty$.

By substituting this expression into the integral equation (1.37), we obtain an equation which determines $\Omega^{(-)}$. To make this equation practically useful, we carry out the calculations in a representation in which H_0 is diagonal, for example in momentum representation. Thus we obtain (see appendix)

$$\langle \mathbf{p}'|\Omega^{(-)}|\mathbf{p}\rangle = \delta(\mathbf{p}' - \mathbf{p}) - \frac{\langle \mathbf{p}'|V\Omega^{(-)}|\mathbf{p}\rangle}{E' - E - i\gamma} \cdot \tag{1.43}$$

If we introduce the abbreviation

$$T^{(-)} = V\Omega^{(-)}, \tag{1.44}$$

(1.43) becomes

$$\langle \mathbf{p}'|\Omega^{(-)} - 1|\mathbf{p}\rangle = -\frac{\langle \mathbf{p}'|T^{(-)}|\mathbf{p}\rangle}{E' - E - i\gamma} \cdot \tag{1.45}$$

For the matrix elements of $S' = U(+\infty, -\infty)$ we find from (1.42)

$$\begin{aligned}\langle \mathbf{p}'|S'|\mathbf{p}\rangle &= \lim_{t\to+\infty} \langle \mathbf{p}'|\Omega^{(-)}|\mathbf{p}\rangle e^{i(E'-E)t} \\ &= \delta(\mathbf{p}' - \mathbf{p}) - 2\pi i\langle \mathbf{p}'|T^{(-)}|\mathbf{p}\rangle\delta(E' - E).\end{aligned} \tag{1.46}$$

In the last transformation equation (A. 5) has been used.

This shows that the S matrix elements can be derived from the matrix elements of $\Omega^{(-)}$, or of $T^{(-)}$, respectively. Only the values of $\langle \mathbf{p}'|T^{(-)}|\mathbf{p}\rangle$ in the energy shell $E = E'$ are needed. The equations (1.44), (1.45) may be used instead of (1.37) as the starting point for a perturbation calculation. If we insert the zeroth order approximation $\Omega^{(-)} = 1$ into (1.44), we obtain for T the first Born approximation

$$T^{(-)(1)} = V. \tag{1.47}$$

It is possible to give to the relations discussed above another somewhat different interpretation. The vector

$$|\mathbf{p}\rangle^{(-)} = \Omega^{(-)}|\mathbf{p}\rangle \tag{1.48}$$

is the Heisenberg state vector whose incoming configuration is a plane wave of momentum p. The matrix $\langle \mathbf{p}'|\Omega^{(-)}|\mathbf{p}\rangle$ considered for fixed $\mathbf{p}$ as a function of $\mathbf{p}'$, is the wave function of this state in the momentum representation. To evaluate its elements we can recall equation (1.23), which follows immediately from (1.44), (1.45). It tells us that $|\mathbf{p}\rangle^{(-)}$ is an eigenvector of H corresponding to the eigenvalue $E = p^2/2\mu$:

$$H|\mathbf{p}\rangle^{(-)} = E|\mathbf{p}\rangle^{(-)}. \tag{1.49}$$

At the same time, we obtain by means of (1.45) the boundary condition, which enables us to choose $|\mathbf{p}\rangle^{(-)}$ among the manifold of the (degenerate) eigenvectors. Namely, the singularity of the wave function in the neighborhood of $E' = E$ must be of the form[15]

$$\delta\langle \mathbf{p}' - \mathbf{p}\rangle - \frac{T}{E' - E - i\gamma}.$$

In the position representation, this boundary condition becomes Sommerfeld's radiation condition for outgoing waves:

$$\langle \mathbf{r}|\mathbf{p}\rangle^{(-)} = (2\pi)^{-3/2}e^{i\mathbf{p}\cdot\mathbf{r}} + \psi_s(\mathbf{r})$$

$$\psi_s(\mathbf{r}) \to f\left(\frac{\mathbf{r}}{r}\right)\frac{e^{ip\cdot r}}{r} \text{ for } r\to\infty. \tag{1.50}$$

Explicit Formal Solution

Since $|\mathbf{p}\rangle^{(+)}$ and $|\mathbf{p}\rangle$ are eigenstates of H, and H_0, respectively,

$$(H-E)\{|\mathbf{p}\rangle^{(\pm)} - |\mathbf{p}\rangle\} = -V|\mathbf{p}\rangle.$$

15. P.A.M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1949), Chapter VIII.

Moreover

$$\lim_{t\to\pm\infty} e^{i(H-E)t}(|\mathbf{p}\rangle^{(\pm)} - |\mathbf{p}\rangle) = |\mathbf{p}\rangle^{(\pm)} - \lim_{t\to\pm\infty} U(t)|\mathbf{p}\rangle = 0$$

It follows, from $t \to \infty$ in equation (A.5)

$$|\mathbf{p}\rangle^{(\pm)} = |\mathbf{p}\rangle - \frac{1}{H - E \pm i\gamma} V|\mathbf{p}\rangle. \tag{1.51}$$

This formal solution of the problem is sometimes useful. If we form its scalar product with $|\mathbf{p}'\rangle^{(\pm)}$, we find, due to the orthogonality relations of the $|\mathbf{p}\rangle^{(\pm)}$,

$$^{(\pm)}\langle \mathbf{p}'|\mathbf{p}\rangle = \delta(\mathbf{p}' - \mathbf{p}) + \frac{1}{E' - E \pm i\gamma}\,{}^{(\pm)}\langle \mathbf{p}'|V|\mathbf{p}\rangle . \tag{1.52}$$

One of these formulae is the complex conjugate of (1.43). For the S matrix, we have

$$S_{\mathbf{p}'\mathbf{p}} = {}^{(+)}\langle \mathbf{p}'|\mathbf{p}\rangle^{(-)} = \lim_{t\to-\infty} {}^{(+)}\langle \mathbf{p}'|\mathbf{p}\rangle e^{i(E'-E)t}$$

$$= \lim_{t\to+\infty} {}^{(-)}\langle \mathbf{p}'|\mathbf{p}\rangle e^{i(E'-E)t},$$

and thus, if we insert (1.52)

$$(S-1)_{\mathbf{p}'\mathbf{p}} = -2\pi i\delta(E' - E)\,{}^{(+)}\langle \mathbf{p}'|V|\mathbf{p}\rangle$$

$$= -2\pi i\delta(E' - E)\langle \mathbf{p}'|V|\mathbf{p}\rangle^{(-)} \tag{1.53}$$

The two matrices

$$\langle \mathbf{p}'|T^{(-)}|\mathbf{p}\rangle = \langle \mathbf{p}'|V\Omega^{(-)}|\mathbf{p}\rangle = \langle \mathbf{p}'|V|\mathbf{p}\rangle^{(-)} \tag{1.54a}$$

$$\langle \mathbf{p}'|T^{(+)}|\mathbf{p}\rangle = \langle \mathbf{p}'\,\Omega^{(+)\dagger}V|\mathbf{p}\rangle = {}^{(+)}\langle \mathbf{p}'|V|\mathbf{p}\rangle \tag{1.54b}$$

are therefore identical on the energy shell, where they determine the "regular part" of the S matrix. A comparison with (1.47) indicates that the exact expression for the S matrix results from the first Born approximation, if one substitutes $\langle \mathbf{p}'|V|\mathbf{p}\rangle^{(-)}$ or ${}^{(+)}\langle \mathbf{p}'|V|\mathbf{p}\rangle$ in place of $\langle \mathbf{p}'|V|\mathbf{p}\rangle$.

Section 2. General Collision Processes

All the relevant quantities of Section 1 may be expressed in terms of the operators H and H_0 alone. Thus, from a formal point of view one could regard the relations of Section 1 as an algorithm which yields an S matrix as soon as a splitting of the Hamiltonian in the form (1.1) is given. But, of course, the splitting cannot be arbitrary: it must be such that the interaction V vanishes asymptotically. More exactly

$$\lim_{t\to\pm\infty} \| Ve^{-iHt}\Phi\| \to 0 \tag{2.1}$$

stronger than $1/t$ for a dense set of state vectors Φ.

Apart from elastic scattering there is hardly a problem of physical interest in which the requirement (2.1) is satisfied. Even in Section 1, (2.1) is satisfied only for the states of $\mathfrak{h}^c$. There the violation of (2.1) is not serious because no transitions are possible between the bound states and the scattering states. This is different, however, if we have to deal with genuine reactions in which bound states (composite particles) may form or disintegrate.

More frequently one meets problems in which at least

$$\lim_{t\to\pm\infty} \| Ve^{-iH_0t}\Phi\| \to 0 \tag{2.2}$$

stronger than $1/t$ for every vector Φ

In these cases at least part of the statements of Section 1 remain true.

In quantum field theory we know of no splitting of H which satisfies either (2.1) or (2.2). Therefore, it is good to generalize the considerations of Section 1 in such a manner that no use will be made of the concept of an "unperturbed Hamiltonian" H_0. Before doing this, let us consider shortly in a qualitative way the situation in three typical examples.

(1) Nuclear Reactions: Specifically a system of 2 neutrons and a proton.

We can decompose the space of states into three orthogonal subspaces: $\overset{(+)}{\mathfrak{h}}{}^{(3)}$ contains the states (in the Heisenberg picture) which have a final configuration of three isolated nucleons. $\overset{(+)}{\mathfrak{h}}{}^{(2)}$ contains the states with a deuteron and a neutron in the final configuration, $\overset{(+)}{\mathfrak{h}}{}^{(1)}$ the triton states. The latter are irrelevant to the discussion, they play exactly the same role as the states from $\mathfrak{h}^0$ in Section 1. We will disregard them, therefore. If we classify according to initial configurations, we obtain an analogous decomposition of the space into $\overset{(-)}{\mathfrak{h}}{}^{(3)}$, $\overset{(-)}{\mathfrak{h}}{}^{(2)}$. Of course, $\overset{(+)}{\mathfrak{h}}{}^{(3)} \neq \overset{(-)}{\mathfrak{h}}{}^{(3)}$ since capture processes are possible.

In the Hamiltonian

$$H = -\Sigma\nabla_i^2 + \Sigma V_{ik} \tag{2.3}$$

we first make the splitting

$$H_0 = -\Sigma\nabla_i^2; \quad V = \Sigma V_{ik}. \tag{2.4}$$

Then (2.2) is satisfied. But instead of (2.1) we have e.g.

$$\lim_{+\infty} \| V e^{-iHt}\Phi \| \begin{cases} = 0 \text{ if } \Phi \text{ is in } \overset{(+)}{\mathfrak{h}}{}^{(3)} \\ \neq 0 \text{ if } \Phi \text{ is } \perp \overset{(+)}{\mathfrak{h}}{}^{(3)}. \end{cases} \tag{2.4}$$

Because of (2.2) isometric operators $\Omega^{(+)}$ exist which are the strong limits of U(t) as in (1.16), (1.21), for

$$\frac{\partial}{\partial t} U(t)\Phi = ie^{iHt} V e^{-iH_0 t}\Phi. \tag{2.5}$$

The norm of the right hand side is the expression (2.2). Since it decreases stronger than $1/t$, we can integrate (2.5) until $|t| \to \infty$.

The analogous discussion for $U^\dagger(t)$ would have to make use of the assumption (2.1). Since this does not hold, $U^\dagger(t)$ does not converge strongly. The matrix elements $\langle \Phi_1 | U^\dagger(t) | \Phi_2 \rangle$ between arbitrary but fixed states Φ_1, Φ_2 must still converge, of course, because they can also be expressed in terms of U. Thus $U^\dagger(t)$ has weak limits at $t \to \pm\infty$. Altogether

$$U(t) \Rightarrow \Omega^{(+)}; \quad U\dagger(t) \to \Omega^{(+)\dagger}$$

$$\Omega^{(+)}, \Omega^{(-)} \text{ isometric.} \tag{2.6}$$

What is the significance of the operators $\Omega^{(+)}$, $\Omega^{(+)\dagger}$? Similar to Section 1, $\Omega^{(+)}$ maps the whole space $\mathfrak{h}$ isometrically on a subspace, namely on $\overset{(+)}{\mathfrak{h}}{}^{(3)}$. $\Omega^{(+)\dagger}$ does the reverse: it maps $\overset{(+)}{\mathfrak{h}}{}^{(3)}$ on $\mathfrak{h}$ and annihilates all vectors orthogonal to $\overset{(+)}{\mathfrak{h}}{}^{(3)}$. The matrix elements $\langle \mathbf{p}'_1\mathbf{p}'_2\mathbf{p}'_3 | S' | \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 \rangle$ of the operator $S' = \Omega^{(+)\dagger}\Omega^{(-)}$ give the correct probability amplitudes for the transitions from a 3-particle initial configuration to a 3-particle final configuration. S' is not isometric, which is obvious from the physical point of view since there is a non-zero probability that an initial configuration of three isolated nucleons ends up as a deuteron plus a neutron. Mathematically, the isometry is destroyed in the following way: The vectors of $\overset{(-)}{\mathfrak{h}}{}^{(3)}$ which result first by application of $\Omega^{(-)}$ contain in general a component in $\overset{(+)}{\mathfrak{h}}{}^{(2)}$. The contribution of this component in the norm is then annihilated by the application $\Omega^{(+)\dagger}$.

To obtain the states from $h^{(+)}(2)$ we should of course include the interaction between the proton and the neutron which are bound together in the deuteron in H_0. In general, one would have to introduce a different splitting of H for every channel if one wanted to apply the methods of Section 1 to many channel reactions. This has some disadvantages. On the one hand it introduces superfluous quantities because of the operators Ω_α (α = channel index) only certain matrix elements are of interest. On the other hand in such a scheme the Pauli principle can be taken into account only at the end of the calculation (in our example the operators H_0 and V cannot be chosen symmetrical in the two neutrons if we want to include the binding forces of one neutron with the proton in H_0). This prevents the transfer of this method to quantum field theory because there the symmetry principle is intrinsic in the formalism.

(2) Meson Theory with Fixed Source

If we confine ourselves to considering only states with one nucleon, which is supposed to be infinitely heavy and nailed down at the origin, then we obtain a model of the following kind[16]

$$H_0 = \int \omega a^\dagger(\mathbf{k}) a(\mathbf{k}) d\mathbf{k} \tag{2.7}$$

$$V = g \int u(\mathbf{k})(\mathbf{k}\cdot\sigma)[a(\mathbf{k}) + a^\dagger(\mathbf{k})]d\mathbf{k}. \tag{2.8}$$

Here $\omega = \sqrt{k^2 + m^2}$, $a(k)$ is the annihilation operator, $a^\dagger(\mathbf{k})$ the creation operator of a meson with momentum $\mathbf{k}$; σ are the spin matrices of the nucleon and, $u(\mathbf{k})$ is a given source function whose Fourier transform $\tilde{u}(\mathbf{r})$ differs from zero essentially only in a

16. For the sake of simplicity, we take a neutral meson theory in which only the spin of the nucleon, not the distinction proton-neutron matters. Even the spin we consider only because without it the model would contain no effective interaction.

small region around the origin (the "extension" of the nucleon). The possible processes in the model are the elastic scattering of mesons by the nucleon and meson creation by meson-nucleon collision. The Hilbert space is characterized as follows: it contains the two ground states of H_0 called Φ_0 and Φ_0'. They differ by their spin orientation and are both annihilated by $a(\mathbf{k})$. We call them states of a bare nucleon. From them we obtain a complete basis by repeated application of the operators $a^\dagger(\mathbf{k})$.

One recognizes quickly that neither (2.1) nor (2.2) is satisfied. Condition (2.2) is easiest tested in a basis in which H_0 is diagonal. Let us consider, for instance, a state of the form

$$\Phi = \int \varphi(\mathbf{k}) a^\dagger(\mathbf{k}) d\mathbf{k} \Phi_0,$$

which contains a bare nucleon and a meson with momentum wave function $\varphi(\mathbf{k})$. Then

$$e^{-iH_0 t}\Phi = \int \varphi(\mathbf{k}) e^{-i\omega t} a^\dagger(\mathbf{k}) d\mathbf{k} \Phi_0.$$

For large times the meson wave function in position space (Fourier transform of $\varphi(k)e^{-i\omega t}$) will be practically zero in the neighborhood of the origin. Applying V, we can therefore commute it with the meson creation operators in the limit $|t| \to \infty$. Thus

$$Ve^{-iH_0 t}\Phi \to \int \varphi(\mathbf{k}) e^{-i\omega t} a^\dagger(\mathbf{k}) d\mathbf{k} V\Phi_0.$$

But

$$V\Phi_0 \neq 0 \tag{2.9}$$

since V itself contains creation operators and therefore (2.2) is not satisfied. The reason is, of course, that the states of a physical nucleon Φ_a, Φ_a' (i.e., the lowest eigenstates of H) differ from Φ_0, Φ_0' and that

V does not describe the actual interaction between physical particles. In particular also

$$V\Phi_k \neq 0 \qquad (2.10)$$

and this means, by a similar reasoning as above, that (2.1) does not hold. The fact that (2.2) does not hold does not necessarily imply that a perturbation expansion with respect to V is impossible. Roughly speaking, it means that V manifests itself everywhere in space but still V may be a small disturbance if considered locally. Such a situation has been called "a persistent interaction" by van Hove. He has developed a method of perturbation expansion which may replace (1.38) in such cases.[17, 18]

(3) Relativistic Field Theory.

Conventional relativistic field theories started from a scheme like (1.1) in which H_0 was taken as the Hamiltonian of a system of uncoupled field. Again V is a "persistent interaction." However, here the physical particles are infinitely different from the "bare particles." This shall mean: the connection between the physical particles can be discussed in a meaningful manner only if one cuts off the interaction at some high energy E. Then one finds that the expansion coefficients of a state of a physical particle in the basis in which H_0 is diagonal are spread out far and are very small. In the limit $E \to \infty$ they all tend to zero. One can then no longer use the basis system of eigenfunctions of H_0 (nor H_0 itself for that matter) without encountering infinite quantities.

Collision Theory without Use of the Operator H_0[19]

To fix the ideas, let us think again of nuclear reactions. We do not want, however, to restrict ourselves

17. L. van Hove, Physica **21**, 901 (1955); **22**, 343 (1956).
18. N. Hugenholtz, Physica **23**, 481 (1957).
19. The basic idea of the method here described is found in the paper by H. Ekstein, Phys. Rev. **101**, 880 (1956).

to a fixed nucleon number. For the general formulation which we have in mind this would be only a bothersome auxiliary condition. Rather we consider states with arbitrary nucleon number and take H to be defined in the subspace corresponding to nucleon number A by (2.3) with sums running from 1 to A.

The various stable particles[20] we distinguish by indices $\alpha, \beta, \ldots$. Let E_α be the energy at rest (negative binding energy), A_α the nucleon number, M_α the mass of such a particle and Φ_p^α its wave function in the state with momentum p. In the position representation Φ_p^α is of course a product of a plane wave $e^{ip \cdot r_s}$ (r_s center of mass coordinate) and a function of the $A_\alpha - 1$ relative coordinates. Also

$$H\Phi_p^\alpha = (E_\alpha + \epsilon_\alpha)\Phi_p^\alpha ; \quad \epsilon_\alpha = \frac{p^2}{2M_\alpha}$$

We consider a wave packet with respect to the center of mass motion formed from such states:

$$|\xi\rangle = \int \varphi_\xi(\mathbf{p})\Phi_p^\alpha d\mathbf{p}. \qquad (2.12)$$

The index ξ now combines the information about the type of particle (α), the probability distribution of its center of mass (φ_ξ) and, if necessary, also its spin orientation. Because of (2.11) we have

$$e^{-iHt}|\xi\rangle = \int \varphi_\xi(\mathbf{p})e^{-i(E_\alpha + \epsilon_\alpha)t}\Phi_p^\alpha d\mathbf{p}. \qquad (2.13)$$

The quantities Φ_p^α and their linear combinations $|\xi\rangle$ are wave functions of A_α variables. Now we consider products of such wave functions. For the

20. Among these may be included many of the excited states of the nuclei which decay by virtue of non-nuclear interactions (β, γ transitions). These are stable with respect to the Hamiltonian (2.3) and we will restrict ourselves for simplicity here to cases in which the duration of the nuclear interaction in the collision is short as compared to the times of a γ-transition.

product formation we use the symbol $\wedge$ to make it clearly recognizable. The exact definition of the product shall be: $\Phi_1 \wedge \Phi_2$ is a function of $A_\alpha + A_\beta$ coordinates which arises from the product $\Phi_1(r_1, \ldots r_{A_\alpha}) \Phi_2 (r_{A_\alpha+1}, \ldots r_{A_\alpha+A_\beta})$ if one antisymmetrizes with respect to the coordinates which refer to identical nucleons. Our aim is to express the Heisenberg states which correspond to a particular initial or final configuration. This is now easily done as follows: Let, e.g., $|\xi\eta\rangle^{(+)}$ be the state with a final configuration of two particles which have respectively the asymptotic probability distributions (reduced to t = 0) of their centers of mass given by φ_ξ and φ_η. We maintain that

$$|\xi\eta\rangle^{(+)} = \lim_{+\infty} e^{iHt}[(e^{-iHt}|\xi\rangle) \wedge (e^{-iHt}|\eta\rangle)]. \quad (2.14)$$

The generalization to arbitrary initial or final configurations is obvious. The S matrix is then given by (1.25).

The argument which leads to (2.14) is essentially identical with the one in Section 1 with respect to (1.4). To see this, one only needs to insert (2.13) on the right hand side of (2.14) and pass to the position representation. The only difference of (2.14) as compared to (1.16), (1.26) is the following: in generalization of the transformation e^{-iH_0t} we have to insert the motion of the isolated particles, which is expressed with the help of the product symbol $\wedge$ and the actual Hamiltonian H. In the special case of a pure scattering problem H_0 is defined just so that the square bracket in (2.14) may be replaced by $e^{-iH_0t}[|\xi\rangle \wedge |\eta\rangle]$. In more general problems there will be no operator H_0 which would do this service for arbitrary product states.

The virtue of (2.14) lies in the fact that it is easily applied to more complex collision problems (e.g., quantum field theory with bound states) in which our intuition is not as well developed yet as in nuclear physics. The only thing needed on the side of the

formalism is a suitable definition of the product operation and this is not difficult to find since it has to satisfy only some qualitative criteria and one therefore has a large amount of arbitrariness in the choice. The product operation shall express the following physical situation: Let Φ_1 and Φ_2 be two states which are localized at the time of reference (t = 0) within two volumes V_1 and V_2 respectively and let V_1 and V_2 be far separated. Then $\Phi_1 \wedge \Phi_2$ shall describe the "doubly localized" state in which we have within V_1 the situation described by Φ_1 and within V_2 the situation described by Φ_2. Evidently the expression $\Phi_1 \wedge \Phi_2$ has only a clearcut physical meaning if the localization volumes of the two factor states are far apart. The definition of the product for neighboring states is arbitrary. Conversely, in (2.14) only the asymptotic product enters. To see this, let us divide space into cells and write $e^{-iHt}|\xi\rangle = \sum_i \Phi_\xi^{(i)}(t)$ where $\Phi_\xi^{(i)}(t)$ denotes the part for which the center of mass lies in cell number i. Then the square bracket in (2.14) becomes $\sum_{i,k} \Phi_\xi^{(i)}(t) \wedge \Phi_\xi^{(k)}(t)$. If one omits in the double sum the terms in which i = k (or even those where i and k are neighbors) then this does not affect the expression in the limit $|t| \to \infty$. Namely: the probability of a particle to be in some particular cell (i.e., $\|\Phi_\xi^{(i)}(t)\|^2$) decreases as t^{-3}; the number of occupied cells increases as t^3; if the cells are chosen large enough $\Phi_\xi^{(i)} \wedge \Phi_\xi^{(i)}$ and $\Phi_\xi^{(k)} \wedge \Phi_\xi^{(k)}$ are practically orthogonal for $i \neq k$. Therefore[21]

$$\|\sum_i \Phi_\xi^{(i)}(t) \wedge \Phi_\eta^{(i)}(t)\|^2 \sim \sum_i \|\Phi_\xi^{(i)}(t) \wedge \Phi_\eta^{(i)}(t)\|^2 \sim t^{-3}$$

21. It is perhaps not necessary to point out that this argument is valid even if the states $|\xi\rangle$ and $|\eta\rangle$ are identical. The total probability of finding the particles within a finite distance of each other vanishes also in that case for $t \to \infty$. The argument depends on the evanescence of the wave packet, not on the difference of the mean momenta.

Therefore, the simple properties of the asymptotic product with respect to metric, transformation under rotations, etc., yield similar simple properties of the state $|\xi\eta\rangle^{+}$.

Orthogonality and Completeness of the $|a\rangle^{(+)}$. Unitarity of S

Let Φ_1 and Φ_2 be again a pair of far separated single particle states, Φ_1' and Φ_2' another such pair. From the physical significance of the asymptotic product it follows

$$\Phi_1 \wedge \Phi_2 = \pm \Phi_2 \wedge \Phi_1 \tag{2.15}$$

(the – sign in the case of Fermi particles).

Furthermore, for the scalar product

$$\langle \Phi_1' \wedge \Phi_2' | \Phi_1 \wedge \Phi_2 \rangle = \langle \Phi_1' | \Phi_1 \rangle \langle \Phi_2' | \Phi_2 \rangle \pm \langle \Phi_1' | \Phi_2 \rangle \langle \Phi_2' | \Phi_1 \rangle \tag{2.16}$$

In wave mechanics a formal check of these relations can of course be made easily from the definition of $\wedge$. From (2.14) we find, observing that e^{iHt} is a unitary transformation,

$$^{(+)}\langle \xi' \eta' | \xi \eta \rangle^{(+)} = \lim_{+\infty} \langle (e^{-iHt}|\xi'\rangle) \wedge (e^{-iHt}|\eta'\rangle) \rangle \langle (e^{-iHt}|\xi\rangle) \wedge (e^{-iHt}|\eta\rangle) \rangle$$

Since for large $|t|$ only the asymptotic product enters we can apply (2.16) and obtain

$$^{(+)}\langle \xi' \eta' | \xi \eta \rangle^{(+)} = \langle \xi' | \xi \rangle \langle \eta' | \eta \rangle \pm \langle \xi' | \eta \rangle \langle \eta' | \xi \rangle \tag{2.17}$$

Analogously for the states $|\xi\eta\rangle^{(-)}$. To write the orthogonality relations in a general and transparent fashion

we imagine as in Section 1 to every particle type α some complete orthonormal basis system of states $|\alpha k\rangle$ and abbreviate a configuration $\alpha k, \beta_l, \ldots$ by a single roman letter a. If, in the case of Bosons, the αk appears $n_{\alpha k}$ times in the configuration we define

$$|a\rangle^{(+)} = (n_{\alpha k}!\, n_{\beta l}! \ldots)^{-\frac{1}{2}} |\alpha k \ldots \beta_l \ldots\rangle^{(+)}.$$

Then we have in obvious generalization of (2.14)

$$^{(+)}\langle a \mid b \rangle^{(+)} = {}^{(-)}\langle a \mid b \rangle^{(-)} = \delta_{ab}. \tag{2.18}$$

Thus the states $|a\rangle^{(+)}$ as well as the $|a\rangle^{(-)}$ form an orthonormal system.

Since by definition S maps each $|a\rangle^{(+)}$ on the corresponding $|a\rangle^{(-)}$ (2.18) tells that S is isometric. The unitarity of S follows if one can show still that the system $|a\rangle^{(+)}$ as well as the system $|a\rangle^{(-)}$ is complete. To prove the completeness we must have some general knowledge about the spatial form of $e^{iHt}\varphi$ for an arbitrary state φ at large times. While this is evident from the physical point of view in most cases, we do not know at the moment of a general and transparent formal discussion starting from the properties of H. Note that for the results (2.14) and (2.18) we needed only the asymptotic form of the free motion of a single particle.

Invariances and Conservation Laws

Most invariance properties which play a role in quantum physics may be described in the following terms: A unitary operator R is known which has the properties

$$R(\Phi_1 \wedge \Phi_2) = (R\Phi_1) \wedge (R\Phi_2), \tag{2.19}$$

if Φ_1 and Φ_2 are localized far apart; and

$$[R, H] = 0. \tag{2.20}$$

Examples in this category are rotations in space, translations in space and time, space reflection, the isotopic spin group (charge independence). For every such operator we find from (2.14)

$$R|\xi\eta\rangle^{(+)} = |R\xi, R\eta\rangle^{(+)} \tag{2.21}$$

or more generally,

$$R|a\rangle^{(\pm)} = |a'\rangle^{(\pm)}, \tag{2.21a}$$

where a' is the configuration obtained from a if all the single particle states of which a is composed are individually subjected to the transformation R.

There are some important invariance properties for which (2.20) is not true, most notably the proper Lorentz transformations. We should note here, therefore, that (2.20) is too narrow a characterization of the notion of invariance and that it is not necessary for the derivation of (2.21a). What we need instead is rather an extension of (2.19). It is convenient to introduce a time dependent product by

$$\Phi_1 \overset{t}{\wedge} \Phi_2 = e^{iHt}[(e^{-iHt}\Phi_1) \wedge (e^{-iHt}\Phi_2)].$$

It has the same physical significance as our original symbol if we replace the reference time 0 by the reference time t.

If R is a unitary operator which instead of (2.20) satisfies:

$$R(\Phi_1 \overset{t}{\wedge} \Phi_2) \approx R\Phi_1 \overset{t}{\wedge} R\Phi_2, \tag{2.22}$$

whenever the two states Φ_1, Φ_2 have far separated localization volumes at the arbitrary time t, then we can associate R with an invariance property. It is evident that (2.21a) holds for such an operator R because (2.14) becomes in this notation simply

$$|\xi\eta\rangle^{(+)} = \lim_{t\to+\infty} |\xi\rangle \overset{t}{\wedge} |\eta\rangle. \tag{2.14a}$$

From (2.21a) we obtain

$$RS|a\rangle^{(-)} = R|a\rangle^{(-)} = |a'\rangle^{(-)} = S|a'\rangle^{(-)} = SR|a\rangle^{(+)}$$

i.e.

$$[R,S] = 0; \qquad RSR^{-1} = S. \tag{2.23}$$

In words: S is invariant under the transformation or alternatively the quantity R is conserved in the collision.

Finally, we consider briefly the transformation of "time reflection." In wave mechanics this corresponds simply to the transition from the wave function $\psi(\mathbf{r}_1 \ldots \mathbf{r}_n)$ to the complex conjugate $\psi^*(\mathbf{r}_1 \ldots \mathbf{r}_n)$.[23] This transformation is not linear. Rather we have, if we denote it by T

$$T(\alpha\Phi_1 + \beta\Phi_2) = \alpha^* T\Phi_1 + \beta^* T\Phi_2, \tag{2.24}$$

$$(T\Phi_1, T\Phi_2) = (\Phi_1, \Phi_2)^*. \tag{2.25}$$

An operator with these properties is called antiunitary. We see that (in wave mechanics) T satisfies again (2.19) and (2.20). From the latter equation, however, we get now

$$Te^{iHt} = e^{-iHt}T \tag{2.26}$$

because T is antiunitary. Therefore (2.14) gives

$$T|\xi\eta\rangle^{(+)} = |T\xi, T\eta\rangle^{(-)}$$

or, more generally

$$T|a\rangle^{(\pm)} = |a_T\rangle^{(\mp)}, \tag{2.27}$$

23. For a more detailed discussion see E. P. Wigner, *Group Theory and Quantum Mechanics*. Academic Press, New York (1959).

where a_τ results from a by subjecting all single particle states contained in a to the time reversal operation.

If the theory is invariant under time reversal, then

$$S_{ba} = {}^{(+)}\langle b|a\rangle^{(-)} = {}^{(+)}\langle Tb|Ta\rangle^{(-)*}$$
$$= {}^{(+)}\langle a_\tau|b_\tau\rangle^{(-)} = S_{a_\tau b_\tau}. \qquad (2.28)$$

Thus the transition probability $a \to b$ is equal to that for the (almost) inverse process $b_\tau \to a_\tau$ (principle of detailed balancing). The coordination of the two "inverse" processes becomes slightly simpler if one uses instead of T the operation of space-time-reflection (usually denoted by PT; P for "parity"). Namely, if b is a momentum configuration $\mathbf{p}_1 \ldots \mathbf{p}_n$ then b_τ is the configuration with the reversed momenta $-\mathbf{p}_1 \ldots -\mathbf{p}_n$ and $b_{p\tau}$ is again the original configuration $\mathbf{p}_1 \ldots \mathbf{p}_n$.

Integral Equations

For the practical calculation of S Eq. (2.14) is, of course, just as unsuited as the analogous Eq. (1.16). We shall reformulate the mathematical problem as in Section 1. Let the wave functions $\varphi_\xi^{(p)}$ again degenerate to δ-functions, i.e. let us consider (with due caution) the improper states corresponding to momentum configurations. Then (2.14) becomes

$$|a\rangle^{(+)} = \lim_{t\to+\infty} e^{i(H-E_a)t}\Phi_a \qquad (2.29)$$

with

$$E_a = \Sigma E_\alpha + \epsilon_\alpha \qquad \Phi_a = \Phi^\alpha_{p_1} \wedge \Phi^\beta_{p_2} \ldots \quad \text{see (2.11))}$$

We have to remember that in general the Φ_a will not

be mutually orthogonal and not even linearly independent (in contrast to the $|a\rangle^{(+)}$).[24]

We rewrite (2.29) in the form

$$\lim_{t\to+\infty} e^{i(H-E_a)t}(|a\rangle^{(+)} - \Phi_a) = 0 \qquad (2.30)$$

and introduce the (improper) vector T_a.

$$T_a = (H - E_a)\,\Phi_a. \qquad (2.31)$$

Note that if we do not take account of the Pauli principle then $T_a = V_a\Phi_a$ where V_a is the interaction which is characteristic for the channel a. From the boundary condition (2.30) follows the formal solution (see (A.5))

$$|a\rangle^{(+)} - \Phi_a = -\frac{1}{H - E_a + i\gamma}\,T_a. \qquad (2.32)$$

Similarly

$$|a\rangle^{(-)} - \Phi_a = -\frac{1}{H - E_a - i\gamma}\,T_a. \qquad (2.33)$$

If we take the scalar product with $|b\rangle^{(+)}$ and $|b\rangle^{(-)}$ respectively we get by virtue of the orthogonality relations (2.18)

$${}^{(\pm)}\langle b|\Phi_a\rangle = \delta_{ab} + \frac{1}{E_b - E_a \pm i\gamma}\,{}^{(\pm)}\langle b|T_a\rangle. \qquad (2.34)$$

Further, because of (2.29) and

$${}^{(+)}\langle b|e^{i(H-E_a)t}|\Phi_a\rangle = e^{i(E_b-E_a)t}\,{}^{(+)}\langle b|\Phi_a\rangle,$$

we find for the S matrix

$$S_{ba} = \delta_{ba} - 2\pi i\delta(E_b .. E_a)\,{}^{(+)}\langle b|T_a\rangle$$
$$= \delta_{ba} - 2\pi i\delta(E_b - E_a)\,\langle T_b|a\rangle^{(-)} \qquad (2.35)$$

24. The Φ_a are called "basic functions" by H. Ekstein and "asymptotically stationary states" by L. van Hove.

Equation (2.34) is together with (2.31) the generalization of (1.52). It contains again the complete formulation of the mathematical problem of the stationary collision theory: $|b\rangle^{(+)}$ is the solution of the stationary Schrödinger equation $H|b\rangle^{(+)} = E_b|b\rangle^{(+)}$ with singularities as prescribed by (2.34). The system (2.34), (2.31) is, however, much more disagreeable than the special case (1.52). For instance, a systematic perturbation expansion analogous to Section 1 is not possible. The mathematical reason for this is that the Φ_a are not linearly independent. It is not sufficient to use (2.34) only for a linearly independent subset of the Φ_a because this will not determine $|b\rangle^{(+)}$ uniquely.[25]

For the sake of completeness we shall still write down a system of equations which is occasionally useful.[26]

We have

$$^{(+)}\langle b|T_a\rangle = \langle\Phi_b|T_a\rangle + {}^{(+)}\langle b - \Phi_b|T_a\rangle$$
$$= \langle\Phi_b|T_a\rangle - \left\langle\frac{1}{H - E_b + i\gamma}T_b|T_a\right\rangle;$$

Therefore

$$^{(+)}\langle b|T_a\rangle = \langle\Phi_b|H - E_a|\Phi_a\rangle - \sum\frac{^{(+)}\langle c|T_b\rangle^{*(+)}\langle c|T_a\rangle}{E_c - E_b - i\gamma}. \qquad (2.36)$$

Field Theory. Renormalization, Asymptotic Condition

In a field theory let Φ_r^α be the state which describes one (real) particle of the kind α localized at the position $\mathbf{r}$ at time $t = 0$.[27] We want to assign to this state a creation operator q_r^α so that

$$\Phi_r^\alpha = q_r^\alpha|0\rangle \qquad (2.37)$$

where $|0\rangle$ denotes the vacuum state. This equation does not, of course, define q_r^α because, roughly speaking, it only fixes one column of the infinite matrix q. It would seem natural to define the product of single particle states which we need in order to express the S matrix in the following way

$$\Phi_r^\alpha \wedge \Phi_{r'}^\beta = q_r^\alpha q_{r'}^\beta|0\rangle. \qquad (2.38)$$

It is however necessary first to remove the mentioned ambiguity in the definition of the creation operators q_r^α sufficiently so that (2.38) will be in accord with the asymptotic physical significance of the product. This is done if we admit for q_r^α only such an operator which is expressible in terms of the basic field quantities of a small space-time environment of the point $\mathbf{r}$, $t = 0$. Such an operator we will call "almost local."[28]

We discuss once more the example of fixed source meson theory. To make the comparison with relativistic field theories easier we introduce the operators $\psi(\mathbf{r})$ of the nucleon field and write instead of (2.7, 2.8).

$$H_0 = \int\omega a^\dagger(\mathbf{k})a(\mathbf{k})d\mathbf{k} + M\int\psi(\mathbf{r})\psi^\dagger(\mathbf{r})d\mathbf{r} = H_0^{\text{Mes}} + H_0^{\text{Nucl}} \qquad (2.39)$$

$$V = g\int u(\mathbf{k})\left(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\right)\psi^\dagger(\mathbf{r})\sigma\psi(\mathbf{r})d\mathbf{k}d\mathbf{r}. \qquad (2.40)$$

The previously mentioned states of a "bare nucleon" are now expressed by $\psi^\dagger(0)|0\rangle$. Since V does not

25. See, e.g., L. L. Foldy and Toboeman, Phys. Rev: **105**, 1099 (1957).
26. See, e.g., M. Gell-Mann and M. Goldberger, reference 3, H. Ekstein, reference 9. These equations form the starting-point for the Low equation in meson-nucleon scattering. See G. C. Wick, Rev. Mod. Phys. **27**, 339 (1955).

27. A definition of the term "localized state" in a relativistic theory has been given by T. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
28. R. Haag, Colloque Intern. Lille (1957), and Phys. Rev. **112**, 669 (1958).

act on the states $a^\dagger(\mathbf{k})|0\rangle$, there is no difference in this model between "physical" and "bare" mesons. Therefore, we may simply put in the case of the mesons

$$q_r^{Mes} = a^\dagger(\mathbf{r}) = (2\pi)^{-3/2}\int a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{k}. \qquad (2.41)$$

For the nucleons, on the other hand, we must distinguish between the creation operator of a physical nucleon and the field operator ψ. Let us call the former Q(for $\mathbf{r} = 0$). Then the state whose final configuration is a nucleon at rest at the origin and a meson of momentum $\mathbf{k}$ will be given according to (2.29), (2.38)

$$|\mathbf{k}, N\rangle^{(+)} = \lim_{t\to+\infty} e^{i(H-\omega-E_N)t}a^\dagger(\mathbf{k})Q|0\rangle. \qquad (2.42)$$

In the field theoretical literature one finds in general instead of (2.42) a different relation which has been developed from an adaptation of (1.16). The uncritical application of (1.16) would yield the senseless expression

$$\lim e^{iHt}e^{-iH_0t}\,a^\dagger(\mathbf{k})\psi^\dagger(0)|0\rangle. \qquad (2.43)$$

To get from this to $|\mathbf{k}N\rangle^{(+)}$ as defined by (2.42) the following changes are necessary: first, a different splitting of H (mass renormalization)

$$H_0' = H_0^{Mes} + E_N\int\psi^\dagger(\mathbf{r})\psi(\mathbf{r})d\mathbf{r}$$

$$V' = V + (M - E_N)\int\psi^\dagger(\mathbf{r})\psi(\mathbf{r})d\mathbf{r}. \qquad (2.44)$$

By this one achieves that the spectrum of H_0' and that of H become identical. This gives then instead of (2.43)

$$\lim e^{i(H-\omega-E_N)t}a^\dagger(\mathbf{k})\psi(0)|0\rangle. \qquad (2.45)$$

This differs still from (2.42) because $\psi^\dagger(0)$ stands instead of Q. Now we imagine the bare nucleon state expanded according to the eigenstates of H:

$$\psi^\dagger(0)|0\rangle = \alpha Q|0\rangle + \Phi^c \qquad (2.46)$$

where Φ^c combines all the components which contain apart from a physical nucleon also (real) mesons in the final configuration. If we insert this in (2.45) the first term gives just $\alpha|\mathbf{k},N\rangle^{(+)}$ according to (2.42). The second term converges, however, weakly toward zero because the motion of the mesons which are contained in Φ^c is not compensated and therefore the scalar product of this term with any fixed vector vanishes in the limit $t \to \infty$. Thus we have

$$e^{iHt}e^{-iH_0't}a^\dagger(\mathbf{k})\psi^\dagger(0)|0\rangle \to \alpha|\mathbf{k}N\rangle^{(+)} \quad \text{(weakly)}$$

$$\alpha = \langle\Phi_N|\Phi_0\rangle\ ;\ 0<\alpha<1. \qquad (2.47)$$

The second change to make in (2.43) is therefore a "renormalization of the nucleon field," i.e. to replace ψ by (2.48).

$$\psi'(\mathbf{r}) = \frac{1}{\alpha}\,\psi(\mathbf{r}). \qquad (2.48)$$

Thus one may use (2.43) if one inserts the renormalized quantities and observes that the convergence is only a weak one.[29]

The renormalizations just discussed are finite. They arise from the particular method of calculation, i.e. from historical reasons, and they have no meaning independent of this particular approach. In a local, relativistic field theory the situation is different in so far as the renormalization constants $1/\alpha$, M are infinite and thus the unrenormalized quantities do not exist at all (if the renormalized ones do). In addition there is at least one further divergence difficulty in these theories which is counteracted by a third renormalization. (True coupling constant renormalization.)

29. The remark that the usual limit relations for large times, i.e. those of Section 1, are valid only in the sense of weak convergence in quantum field theory was first made by H. Lehman, K. Symanzik and W. Zimmermann, Nuovo Cim. 1, 205 (1955). For a thorough discussion see also O. W. Greenberg, Princeton Thesis (1957).

Section 3. The Physical Content of S

Cross Sections

Let the initial configuration consist of 2 particles (types α, β) with almost sharp momenta $\mathbf{p}_1^{(0)}, \mathbf{p}_2^{(0)}$. We ask for the probability of a final configuration of n particles (types $\alpha', \beta' \ldots$) with momenta which lie in a small region around $\mathbf{p}_1' \ldots \mathbf{p}_n'$. We suppress for the sake of clarity the indices α, etc., which characterize the particle species and also spin indices. These may be added at the end again without difficulty. The resulting formulae (3.10), (3.12) are valid both for relativistic and for non-relativistic kinematics. We shall discuss the relativistic transformation properties of the quantities in question briefly at the end.

We denote by $|\mathbf{p}\rangle$ a single particle state with momentum $\mathbf{p}$, with the normalization

$$\langle \mathbf{p}_2 | \mathbf{p}_1 \rangle = \delta(\mathbf{p}_2 - \mathbf{p}_1). \tag{3.1}$$

The S matrix will be understood to mean

$$\langle \mathbf{p}_1' \ldots \mathbf{p}_n' | S | \mathbf{p}_1 \mathbf{p}_2 \rangle = {}^{(+)}\langle \mathbf{p}_1' \ldots \mathbf{p}_n' | \mathbf{p}_1 \mathbf{p}_2 \rangle^{(-)} \tag{3.2}$$

where $|\mathbf{p}_1 \mathbf{p}_2\rangle^{(-)}$ and $|\mathbf{p}_1' \ldots \mathbf{p}_n'\rangle^{(+)}$ are obtained according to (2.14) starting from single particle states which are normalized by (3.1). Because of the conservation of energy and momentum S has the form

$$\langle \mathbf{p}_1' \ldots \mathbf{p}_n' | S - 1 | \mathbf{p}_1 \mathbf{p}_2 \rangle = \delta(\Sigma \mathbf{p}_1' - \Sigma \mathbf{p}) \delta(E' - E) \langle \Omega | T | \mathbf{p}_1 \mathbf{p}_2 \rangle \tag{3.3}$$

where T is a nonsingular function of $\mathbf{p}_1$, $\mathbf{p}_2$ and the $2n-4$ variables Ω. We imagine here that instead of $\mathbf{p}_1' \ldots \mathbf{p}_n'$ the variables $\Sigma \mathbf{p}'$, $\Sigma E'$ and Ω are introduced and write[30]

$$d\mathbf{p}_1' \ldots d\mathbf{p}_n' = d(\Sigma \mathbf{p}') d(\Sigma E') d\Omega. \tag{3.4}$$

30. The choice of the variables Ω is largely a matter of convenience. Thus in the case of elastic scattering one usually takes for Ω either the scattering angle or the momentum transfer. In order to keep the formulae generally applicable, we define by (3.4) the symbol $d\Omega$ which means in practice, of course, the calculation of the functional determinant for the change of variables from $\mathbf{p}_1' \ldots \mathbf{p}_n'$ to $\Sigma \mathbf{p}'$, $\Sigma E'$ and Ω. This corresponds to the calculation of the "density of final states" in most textbooks.

Let now $\varphi_1(\mathbf{p}_1)$, $\varphi_2(\mathbf{p}_2)$ be the wave functions of the particles in the initial configuration. They shall be essentially different from zero only for momenta in the neighborhood of $\mathbf{p}_1^{(0)}$, $\mathbf{p}_2^{(0)}$. Then the n-particle component of the final configuration is

$$\begin{aligned} \varphi^{(n)}(\mathbf{p}_1' \ldots \mathbf{p}_n') &= \int \langle \mathbf{p}_1' \ldots \mathbf{p}_n' | S | \mathbf{p}_1 \mathbf{p}_2 \rangle \, \varphi_1(\mathbf{p}_1) \varphi_2(\mathbf{p}_2) d\mathbf{p}_1 d\mathbf{p}_2 \\ &\approx \langle \Omega | T | \mathbf{p}_1^{(0)} \mathbf{p}_2^{(0)} \rangle \int \delta(\mathbf{p}_1 + \mathbf{p}_2 - \Sigma \mathbf{p}') \\ &\quad \delta(E_1 + E_2 - \Sigma E') \varphi_1(\mathbf{p}_1) \varphi_2(\mathbf{p}_2) d\mathbf{p}_1 d\mathbf{p}_2. \end{aligned} \tag{3.5}$$

We have replaced here S by $S-1$ which makes no difference unless we want to consider elastic scattering angle. Further, we have put T in front of the integral because it is regular and may thus be considered as approximately constant in the small momentum regions in which φ_1 and φ_2 are different from zero. The probability to observe n final particles with momenta in the ranges $d\mathbf{p}'$ around $\mathbf{p}'$ is

$$dW = |\varphi^{(n)}|^2 d\mathbf{p}_1' \ldots d\mathbf{p}_n' = |\varphi^{(n)}|^2 d(\Sigma \mathbf{p}') d(\Sigma E') d\Omega \tag{3.6}$$

The distribution with respect to the total momentum $\Sigma \mathbf{p}'$ and the total energy $\Sigma E'$ is not of interest. Because of the conservation laws this reflects only the corresponding distribution in the initial configuration. We are interested only in the distribution with respect to the variables Ω. This is the integral of dW over the total energy and momentum and we will call it dw. From (3.5) we get

$$dw = |\langle \Omega | T | \mathbf{p}_1^{(0)} \mathbf{p}_2^{(0)} \rangle|^2 B d\Omega. \tag{3.7}$$

The quantity B is most easily expressed if we use the initial wave functions in r-space and time:

$$\varphi_{1,2}(\mathbf{r}, t) = (2\pi)^{-3/2} \int \varphi_{1,2}(\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{r} - Et)} d\mathbf{p}. \tag{3.8}$$

Then B is simply

$$B = (2\pi)^2 \int |\varphi_1(\mathbf{r}, t)|^2 |\varphi_2(\mathbf{r}, t)|^2 \, d\mathbf{r} \, dt. \tag{3.9}$$

Here it is perhaps useful to remark that independent of whether we have the energy-momentum relation in non-relativistic or in relativistic form, $|\varphi(\mathbf{r},t)|^2\,d\mathbf{r}$ always means the probability of meeting the particle in the volume element $d\mathbf{r}$ at time t. This is by virtue of our normalization (3.1).[27] Thus, the contribution of the space-time region $\Delta V dt$ to the reaction is

$$dw = (2\pi)^2|\langle T\rangle|^2\rho_1\rho_2\,\Delta V\Delta t d\Omega \tag{3.10}$$

if $\rho_1\rho_2$ are the spatial particle densities of the initial configuration.

In analogy to the kinetic theory of gases the concept of collision cross section dQ is introduced as follows: If particles 1 are shot in with constant current density $I_1 = \rho_1 v_1$ and particles 2 are at rest, then the number of reactions per unit space-time cell which lead into the region $d\Omega$ is written in the form

$$dN = I_1\rho_2 d\Omega. \tag{3.11}$$

The comparison with (3.10) shows

$$dQ = \frac{1}{v_1}\,(2\pi)^2|\langle T\rangle|^2 d\Omega. \tag{3.12}$$

If one defines the cross section in the center of mass system, then in (3.11) I_1 is replaced by the current density of the relative motion and therefore $(v_1 + v_2)$ replaces v_1 in (3.12).

We discuss briefly the relativistic transformation properties of the quantities used. If we pass from an energy momentum vector $(\mathbf{p}, E)$ by Lorentz transformation to $(\mathbf{p}^\tau, E^\tau)$ then the surface element of the hyperbola $E^2 - \mathbf{p}^2c^2 = m^2c^4$ stays invariant:

$$\frac{d\mathbf{p}}{E} = \frac{d\mathbf{p}^\tau}{E^\tau}. \tag{3.13}$$

Therefore, according to (3.4)

$$d\Omega = (\Pi E_i')dL \tag{3.14}$$

where dL is invariant. Also, it follows that the momentum states normalized according to (3.1) are changed by the Lorentz transformation to

$$|\mathbf{p}\rangle \to \sqrt{\frac{E^\tau}{E}}\,|\mathbf{p}^\tau\rangle. \tag{3.15}$$

Therefore, according to (3.2), (3.3)

$$(\Pi\sqrt{E_i^\tau})\,\sqrt{E_1E_2}\,\langle\Omega|T|\mathbf{p}_1\mathbf{p}_2\rangle = \langle\Omega|M|\mathbf{p}_1\mathbf{p}_2\rangle \tag{3.16}$$

is an invariant function. Instead of (3.10) we can write

$$dw = (2\pi)^2|\langle M\rangle|^2\frac{\rho_1\rho_2}{E_1E_2}\,dLd^4\mathbf{x} \tag{3.17}$$

as the ratio of the time-components of two parallel 4-vectors and therefore invariant. This makes the relativistic invariance of (3.17) explicit. To define also the concept of cross section in invariant form one has to write

$$dQ = (2\pi)^2\frac{|\langle M\rangle|^2 dL}{F} = (2\pi)^2|\langle T\rangle|^2 d\Omega\frac{E_1E_2}{F} \tag{3.18}$$

where F is a Lorentz invariant quantity which equals $v_1E_1m_2c^2$ in the rest system of particle 2 (see (3.12)). One finds[31]

$$F = c\left|(E_1E_2 - c^2\mathbf{p}_1\cdot\mathbf{p}_2)^2 - m_1^2m_2^2c^8\right|^{\frac{1}{2}} \tag{3.19}$$

In the center of mass system the factor $\frac{E_1E_2}{F}$ reduces to $\frac{1}{v_1 + v_2}$ so that the definition (3.18) agrees with the customary one also in the center of mass system.

31. C. Møller, reference 3.

Statements about the Space-time Development of the Collision, which are derivable from S

Only the absolute values of the S matrix-elements in the momentum representation enter into the calculation of cross sections. The phase of $\langle \mathbf{p}_1' \dots \mathbf{p}_n' | S | \mathbf{p}_1 \mathbf{p}_2 \rangle$ is not used. Therefore, the physical statements derivable from the knowledge of S are not exhausted by giving the cross sections. The additional information contained in S relates to the evolution of the collision process in space and time. Two typical examples of questions of this kind concern the range of the interaction and the delay time of the partners of the reaction in the interaction region. We consider again an initial configuration of two particles and imagine the center of mass motion to be separated off.

Range of the Interaction

The relative motion of the two particles in the initial configuration is described by a solution φ_t of the force free Schrödinger equation. To probe the range of the interaction we use for φ a Gaussian wave packet whose center moves along the line

$$\bar{\mathbf{r}}(t) = \mathbf{r}_0 - \frac{\mathbf{p}_0}{m} t \,. \tag{3.20}$$

We take $\mathbf{r}_0$ orthogonal to $\mathbf{p}_0$ so that the closest approach is reached at $t = 0$. Also we will choose $t = 0$ as the time at which the packet is most narrowly concentrated. Thus

$$\varphi_0(\mathbf{r}) = (\pi b^2)^{-3/4} e^{-(\mathbf{r}-\mathbf{r}_0)^2/2b^2 + i\mathbf{p}_0\cdot\mathbf{r}}; \tag{3.21}$$

Then

$$|\varphi_t(\mathbf{r})| = (\pi b_t^2)^{-3/4} e^{-(\mathbf{r}-\bar{\mathbf{r}}(t))^2/2b_t^2}. \tag{3.22}$$

with

$$b_t^2 = b^2 + \frac{t^2}{m^2 b^2} \,. \tag{3.23}$$

Certainly for sufficiently large $r_0 = |\mathbf{r}_0|$

$$S\varphi \approx \varphi \,.$$

We want to estimate how well this is satisfied. First we have, as in (1.11)

$$\|(S-1)\varphi\| < \int_{-\infty}^{\infty} \|\chi_t\| dt \tag{3.24}$$

$$\|\chi_t\| = \left\| i\frac{\partial \varphi_t}{\partial t} - H\varphi_t \right\| . \tag{3.25}$$

In the case of elastic scattering by a square well potential of depth V_0 and range a this gives

$$\|\chi_t\| = V_0 \left[\int_{r<a} |\varphi_t(\mathbf{r})|^2 d\mathbf{r} \right]^{\frac{1}{2}} \approx \left(\frac{a}{b_t}\right)^{3/2} \exp -\frac{1}{2b_t^2}\left[\sqrt{r_0^2 + \frac{p_0^2 t^2}{m^2}} - a \right]^2 . \tag{3.26}$$

We have omitted numerical factors of order of magnitude 1. The form of (3.26) seems to be always suited as an estimate of $\|\dot{\chi}\|$ as long as one is dealing with an interaction which has a fairly well defined range a. (3.26) merely makes the qualitative statement that $\|\dot{\chi}\|$ is proportional to the square root of the probability for the initial configuration to be within the interaction region at time t. The relevant quantity in (3.26) is the exponent, i.e. the function

$$f(t) = \frac{\left[\left(r_0^2 + \frac{p_0^2 t^2}{m^2} \right)^{\frac{1}{2}} - a \right]^2}{2\left(b^2 + \frac{t^2}{m^2 b^2} \right)} \,. \tag{3.27}$$

In the neighborhood of $t = 0$, $f(t)$ is approximately constant and equal to $f_0 = \frac{(r_0 - a)^2}{2b^2}$ and for $t \to \infty$ it goes toward the constant value $f_\infty = \frac{p_0^2 b^2}{2}$. For the

transition from f_0 to f_r the following two characteristic times are relevant if $r_0 > a$:

$$t_p = \frac{mr_0}{p_0}; \qquad t_r = mb^2. \tag{3.28}$$

t_r is the time for the evanescence of the wave packet, the meaning of t_p is obvious. To get a simple expression let us choose the position uncertainty b such that $f_0 = f_r$ i.e.

$$b^2 = \frac{r_0 - a}{p_0}. \tag{3.29}$$

The $f(t)$ will be approximately constant and therefore this choice minimizes (3.24) for fixed r_0 and p_0. We get from (3.24), (3.29)

$$\|(S-1)\varphi\| < V_0\left(\frac{a}{b}\right)^{3/2} e^{-\frac{1}{2}(r_0-a)p_0} \int \frac{dt}{(1+\frac{t}{t_2})^{3/2}}$$

$$\|(S-1)\varphi\| < V_0 t_2 \left(\frac{a}{b}\right)^{3/2} e^{-\frac{1}{2}(r_0-a)p_0}$$

$$= mV_0 a^{3/2}\left(\frac{r_0-a}{p_0}\right)^{1/4} e^{-\frac{1}{2}(r_0-a)p_0}.$$

Since $r_0 p_0$ is the mean relative angular momentum of the two initial particles, this estimate tells us that $S_l - 1$ must decrease at least as $e^{-l/2}$ for large angular momentum quantum numbers l. (S_l being that part of S which acts on states with angular momentum l.)

A rigorous discussion of the behavior of S_l for large values of l was given by D.S. Carter in the case of scattering by a potential.[32]

He starts from the integral equation for the collision problem and finds

$$|S_l - 1| = \sin\delta_l < \text{const} \int r V(r) J^2_{l+\frac{1}{2}}(pr)dr. \tag{3.31}$$

32. D.S. Carter, Princeton Thesis (1952).

If one uses the Debye estimate for the Bessel functions with large index

$$J_\nu(z) \sim \frac{\exp -\nu(\sigma - \tan\sigma)}{(\nu\tan\sigma)^{1/2}} \sim \left(\frac{\nu}{z}\right)^{-\nu}$$

where

$$\frac{\nu}{z} = \cos\sigma \sim \frac{1}{2}e^{\circ},$$

then

$$\sin\delta_l \sim l^{-1}\int r|V(r)|(pr)^{2l+1}dr.$$

If V vanishes exactly for $r > a$ this gives

$$\sin\delta_l \sim \left(\frac{l}{pa}\right)^{-2l}. \tag{3.32}$$

If V drops exponentially $\sim e^{-r/a}$ for large r then δ_l decreases much slower than (3.32) because the integral then gives an additional factor $(2l+1)!$ in the numerator. The Debye estimate is adequate in this case only if $pa \ll 1$. More generally one finds for $V \sim e^{-r/a}$:[33]

$$\sin\delta_l \sim (2x)^{-l}; \qquad x = 1 + \frac{2}{2p^2a^2}, \tag{3.33}$$

which agrees in the worst case ($pa \gg 1$) approximately with our estimate based on (3.30). The reason why (3.30) gives in general a poorer estimate than (3.31) lies in the use of a Gaussian packet for φ. This contains many angular momenta which have roughly a Gaussian distribution around the mean value $\bar{l} = r_0 p_0$ with width $\sqrt{l}$. Therefore, already the components of φ with low angular momentum give a contribution to (3.30) proportional to $e^{-l/2}$.

33. N. Khuri, Phys. Rev., 107, 1148 (1957).

Delay Time

Let $\varphi_t(\mathbf{r})$ again be a solution of the free Schrödinger equation of a particle with mass m. The asymptotic form for $|t| \to \infty$ is given by (1.12). The expectation value of $r = |\mathbf{r}|$[34] has the form

$$\langle r \rangle = vt + c + \text{times of order } t^{-1}. \tag{3.34}$$

We get the first term if we take for φ the leading term in (1.12), i.e. the one in which is proportional to $t^{-3/2}$. This gives

$$v = \frac{1}{m}\int |\tilde{\varphi}(\mathbf{p})|^2 |\mathbf{p}| d\mathbf{p} = \frac{\langle p \rangle}{m}. \tag{3.35}$$

The time independent term in (3.34) arises from the interference between the first and the second terms of φ_t in (1.12):

$$\begin{aligned} c &= -\frac{i}{2}\int [\tilde{\varphi}(\mathbf{p})^* \nabla^2 \tilde{\varphi}(\mathbf{p}) - \tilde{\varphi}(\mathbf{p}) \nabla^2 \tilde{\varphi}(\mathbf{p})^*] d\mathbf{p} \\ &= \frac{i}{2}\int \left(\tilde{\varphi}^* \frac{\partial \tilde{\varphi}}{\partial p} - \tilde{\varphi} \frac{\partial \tilde{\varphi}^*}{\partial p} \right) d\mathbf{p} \\ &= \frac{i}{2}\int \varphi^{*2} \frac{\partial}{\partial p}\left(\frac{\tilde{\varphi}}{\tilde{\varphi}^*} \right) d\mathbf{p}. \end{aligned} \tag{3.36}$$

If η is the phase of $\tilde{\varphi}$ i.e.,

$$\tilde{\varphi}(p) = |\tilde{\varphi}(p)| e^{i\eta(p)}, \tag{3.37}$$

then

$$c = -\int |\tilde{\varphi}|^2 \frac{\partial \eta}{\partial p} d\mathbf{p} = -\left\langle \frac{\partial n}{\partial p} \right\rangle = -\left\langle \frac{\partial \eta}{\partial E} \frac{p}{m} \right\rangle. \tag{3.38}$$

If the wave packet has a narrow spread in energy we

34. We consider the absolute value rather than the vector r itself because we want to apply the formulae later on the scattered wave which runs in all directions. Then the expectation value of the vector r is not an interesting quantity.

can also summarize the above formulae by

$$\langle r \rangle = v(t - \tau); \quad v = \frac{\langle |p| \rangle}{m}; \quad \tau = \left\langle \frac{\partial \eta}{\partial E} \right\rangle. \tag{3.39}$$

We consider now the quantity τ for the wave packets φ_i and φ_f (φ_i shall describe the relative motion in the initial configuration, φ_f that of the final configuration in some 2-particle channel). Also we will take $\tilde{\varphi}_i(\mathbf{p})$ to have only a small spread around the center $\mathbf{p}_0 = \Omega_0, E$, and a phase which does not strongly depend on the direction of p. Then in the notation of (3.3), (3.4)

$$\begin{aligned} \varphi_f(E, \Omega) &= \int \langle \Omega | T | \Omega', E \rangle \varphi_i(\Omega', E) d\Omega' \\ &\approx \langle \Omega | T | \Omega_0 E \rangle \int \varphi_i(E, \Omega') d\Omega' \end{aligned} \tag{3.40}$$

and we obtain

$$\tau_f = \tau_i + \tau_D$$

$$\tau_D = \frac{1}{4\pi}\int d\Omega \left(\frac{\partial \eta_T}{\partial E} \right) \tag{3.41}$$

where η_T is the phase of $\langle \Omega | T | \Omega_0 E \rangle$.
Using (3.39) and (3.40), τ_D may be interpreted as the delay of the particles in the region of interaction.[35] Under what assumptions is τ_D measurable? Clearly it is necessary that (3.37) be sufficiently accurate for a finite time t of the order of magnitude of τ_D. Now the basic equation (1.12) is an expansion in powers of t_s/t (t_s is the spreading time of the

35. The discovery that one can read off the delay time according to (3.41) from the S matrix is due to L. Eisenbud, Princeton Thesis (1948). The formal identification of metastable states with poles of the S matrix in the complex plane is related to (3.41) See below.

wave packets). Therefore $t \gg t_s$. Further, no noticeable interaction may occur after time t. This means on the one hand that $t \gg a/v$ (passage time) and on the other hand that we cannot choose t_s arbitrarily small, for the effective interaction disappears sufficiently rapidly for large times only when $\Delta p/p \ll 1$, and therefore $t_s \gg m/p^2$ (p is the momentum in the final state, Δp is its uncertainty). In summary: τ_D is measurable if

$$\tau_D > \frac{a}{v} \tag{3.42}$$

$$\tau_D > \frac{m}{p^2} \tag{3.43}$$

E. P. Wigner[36] has made (3.41) the starting point of a causality criterion by observing that while an arbitrarily large, positive delay time is indeed permissible (metastable states), τ_D is not allowed to assume much larger negative values than $-2a/v$.

Section 4. Resonance Theory

There are a few statements about the cross section which can be made without explicit knowledge of the solution of scattering problems and independently of perturbation calculations. They concern the behavior in the neighborhood of certain special points (resonances, thresholds of reactions, very small energies). The S matrix at these points is constructed approximately from simple expressions with only a few parameters, whose relationship with the corresponding parameters of interaction (range, strength) can be given in some cases. This is important for the analysis of experimental scattering data.

36. E. P. Wigner, Phys. Rev. 98, 145 (1955).

Quasi-stationary States, Resonances, Scattering Maxima

Although in the continuous spectrum of H every value of the energy is possible, certain values are distinguished when one of the following quantities has a maximum:

i) The delay time of a wave packet of mean energy E in the region of interaction (Section 3).

ii) The probability of the particles being contained in the region of interaction (in the solution of the stationary Schrödinger equation).

iii) The cross section for an elastic scattering or for a reaction.

When the maximum is sufficiently sharp, we speak of a quasi-stationary or metastable state (i) or a resonance (ii). The relationship of the criteria (i) and (ii) is immediately clear (see also equation (4.36)). The relation between scattering maxima (iii) and the metastable states is not quite so direct. Let us consider elastic scattering in a partial wave with a certain angular momentum. Here $S(E) = e^{2i\delta(E)}$. Criterion (i) selects those values of the energy for which $\delta'(E)$ is a maximum, hence $\delta'' = 0$, whereas criterion (iii) selects those for which $\delta(E) = (n + \frac{1}{2})\pi$.[37] A relationship between (i) and (iii) would therefore mean that at the points of inflection of the curve $\delta(E)$ the values of δ lie in the vicinity of $(n + \frac{1}{2})\pi$. We will prove below that the variation of δ in the vicinity of a well-marked resonance may be represented by

$$\delta(E) = \eta(E) + \bar{\delta}(E) \tag{4.1}$$

37 Since $\sigma \sim k^{-2} \sin^2 \delta$, the maxima of scattering lie, if one does not consider the slowly varying factor k^2, either at the points $\delta = (n + \frac{1}{2})\pi$ or at the maxima of δ. The latter, however, because of the causality restriction $\delta' \geq -a$, cannot be very narrow and should therefore not be considered.

where η is a slowly varying function, while δ increases by π in a small region of energy and is practically constant outside this region and equal to 0 or π, respectively (Figure 1). Therefore η determines the cross section just below and above

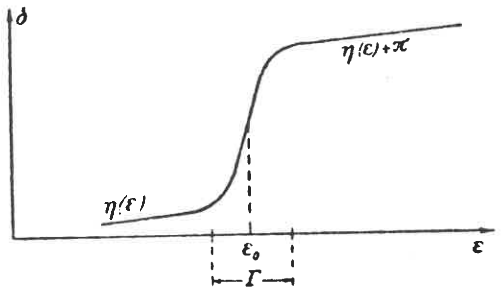

Fig. 1. Behavior of the phase shift in the vicinity of a resonance.

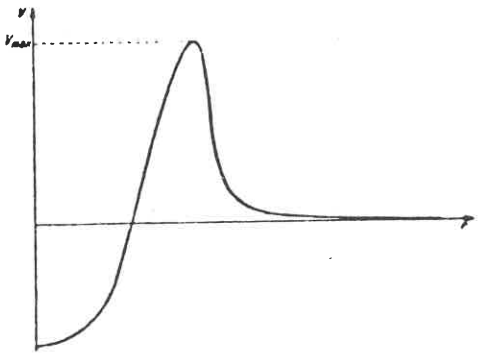

Fig. 2. Model of a potential with a quasi-stationary state.

the resonance. If η is small, the cross section has a maximum in the center of the resonance; that is, the criteria (i) and (iii) are approximately equivalent. For purposes of orientation as to the magnitude of

η, let us take as an example the scattering of a particle by the potential represented in Figure 2 which yields some sharp resonances. Outside the resonances the effect of this potential on an incoming particle is almost the same as that of a rigid sphere if $E \ll V_{max}$. This means that

$$\eta(E) \approx -ka. \tag{4.2}$$

Therefore, as long as $ka \ll 1$, resonances and scattering maxima practically coincide. At higher energy a displacement appears between the two which manifests itself as an asymmetry in the cross section curve near the resonance.

Causes of Resonances

What physical reasons (what properties of the Hamiltonian operators) are responsible for the occurrence of a resonance? We will confine ourselves to three well-known examples:

1) Scattering of light by an atomic shell.

2) α-decay of atomic nuclei, idealized as the motion of a particle in a potential of the form of Figure 2.[38]

3) Resonance scattering by heavy nuclei.

It is easy to see in the first two cases the small change in the model metastable states which will carry them over into exact stable states. In example 1) it consists of the neglect of the interaction between radiation and matter, in example 2) in raising the potential barrier to an impenetrable wall. The following discussion is completely analogous for both examples. We will consider example 2) because of simpler notation.[39]

38. G. Gamov, Z. Phys. 51 204 (1928).
39. A similar calculation can be found in Dirac, reference 15, in the paragraph on resonance scattering.

Let H_0 be the Hamiltonian operator for the approximate problem (impenetrable wall). Let $H = H_0 + H'$ be the actual operator, which corresponds to the potential in Figure 2. Let $|n\rangle$ be the discrete eigenstates of H_0 within $r < a$ (energy ϵ_n). Let $|\epsilon\rangle$ be the continuous eigenstates of H_0 for $r > a$. The continuous spectrum goes from 0 to ∞ and therefore overlaps the discrete ϵ_n wherever these are positive. We will now find an eigenfunction of H with an eigenvalue $E \approx \epsilon_0 > 0$ by a perturbation calculation with respect to H'. As the unperturbed eigenvalue ϵ_0 is degenerate, we must first determine the zeroth approximation from a secular equation. Let us exise in the continuous spectrum a small interval I about the point $\epsilon = \epsilon_0$ and try as a zeroth approximation

$$\Psi(0) = c|0\rangle + \int_I \psi(\epsilon)|\epsilon\rangle\, d\epsilon\,. \tag{4.3}$$

We neglect the energy dependence of the matrix elements $\langle 0|H'|\epsilon\rangle$ and $\langle\epsilon'|H'|\epsilon''\rangle$ in the small interval I and write for them simply H'_{01}, or H'_{11}, respectively. Then the secular equation becomes:

$$\begin{aligned}(\epsilon_0 - E + H'_{00})c + H'_{01}\int_I \psi(\epsilon)d\epsilon &= 0\\ (\epsilon - E)\psi(\epsilon) + cH'_{10} + H'_{11}\int \psi(\epsilon')d\epsilon' &= 0.\end{aligned} \tag{4.4}$$

We normalize the solution as follows:

$$\psi(\epsilon) = \delta(\epsilon - E) + \frac{A}{\epsilon - E - i\gamma}, \tag{4.5}$$

Then

$$\int_I \psi(\epsilon)d\epsilon = 1 + \pi i A$$

and

$$c = \frac{H'_{01}}{(E - \epsilon_0 - H'_{00})(1 + \pi i H'_{11}) + \pi i|H'_{01}|^2}\,. \tag{4.6}$$

The perturbation H' insures that the discrete eigenstates will be absorbed into the continuum. When H' is small, however, the point $E = \epsilon_0 + H'_{00}$ is distinguished by a sharp maximum in the interval of the probability c for finding the particle within $r < a$. For S one finds in the zero order approximation[40]

$$S = \frac{(E - \epsilon_0 - H'_{00})(1 - \pi i H'_{11}) - \pi i|H'_{01}|^2}{(E - \epsilon - H'_{00})(1 + \pi i H'_{11}) + \pi i|H'_{01}|^2} e^{-2ika} \tag{4.7}$$

The essential result is that $S(E)$ has a pole in the lower half plane and a zero point in the upper half plane.[41] These remain in an exact calculation. We can also write S in the form

$$S(E) = e^{2i\eta(E)}\frac{(E - E_r) - i\Gamma/2}{(E - E_r) + i\Gamma/2}, \tag{4.8}$$

which agrees with (1.1) if in Figure 1 one sets

$$\bar{\delta} = \arctan\frac{\Gamma}{2(E_r - E)}\,. \tag{4.9}$$

The original method of Gamow for determination of the lifetime of an α-emitter begins from the observation that S has a pole for the complex energy $z = E_r - i\Gamma/2$. Since the solution of the Schrodinger equation for large r has the form

$$u(r) = e^{-ikr} - Se^{ikr};\ k = \frac{1}{\hbar}\sqrt{2mE};\ \mathrm{Re}k > 0 \tag{4.10}$$

one obtains a solution for $E = z$ without an incoming

40. The factor e^{-2ika} originates from the fact that the approximate problem already gives rise to scattering with the phase $-ka$.
41. These statements are to be used with care since $S(E)$ is a double-valued analytic function of E. See Section 6 for a more exact description.

wave. The solution of the time-dependent Schrödinger equation outside of the interval is then

$$u(r,t) \sim e^{i(k_r' r - \varepsilon_r t)} e^{i(k_r'' r - \Gamma t/2)};$$

$$(k_r' - ik_r'') = \frac{1}{\hbar}\sqrt{2mz} \qquad (4.11)$$

and corresponds to a radioactive source of lifetime Γ^{-1}. One can therefore get positions and widths of the resonance levels as the real and imaginary parts of the eigenvalues of a boundary value problem of the Schrödinger equation.[42] The boundary conditions are

$$u(0) = 0; \quad u(r) \to e^{ikr} \text{ for } r \to \infty \qquad (4.12)$$

The complex eigenvalues have physical meaning, of course, only when their imaginary parts are sufficiently small. The relationship of this method with the description of a metastable state by a wave packet is discussed by Casimir and by Breit and Yost.[43] In the case of resonance reactions with heavy nuclei the quasi-stationary states come about because the colliding particle distributes its energy over many degrees of freedom, provided it enters the nucleus at all. It then takes some time before, accidentally, enough energy is concentrated on a single particle again so that it may be emitted. If one desires to correlate the positions and the widths of resonances with the parameters of nuclear forces, one encounters first the difficulty of the formulation of the "Ersatz problem" which leads to the selection of discrete energy values (operator H_0 in the discussion of the first two examples).[44] We will return to this. A generalization of the method of complex

42. This is, of course, not an eigenvalue problem in Hilbert-space. However, compare the method of H. Rollnik, Z. Phys. **145**, 639 (1956).
43. H. Casimir, Physica **1**, 193 (1934); G. Breit and F. L. Yost, Phys. Rev. **48**, 203 (1935).
44. E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

energies for this problem was carried out by Kapur and Peierls.[45]

The General Resonance Formula[46]

We want the generalization of (4.8) for collision processes with several channels (resonance reactions). We designate the eigenstates of H by $|\alpha E)$. Here α is the complete set of quantum numbers, which are needed in addition to E for a complete characterization of the initial and the final configurations. In the case of reactions with only two fragments in both the initial and the final channel it is useful to select for the quantum numbers in α which describe each of the fragments, the angular momentum of their relative motion as well as the total angular momentum and its Z-component. Thus α is a discrete index. The S matrix is

$$^{(+)}(\alpha E | \beta E')^{(-)} = S_{\alpha\beta}^{(E)}\, \delta(E - E'). \qquad (4.13)$$

If we choose the normalization

$$^{(+)}(\alpha E | \beta E')^{(-)} = \delta_{\alpha\beta}\, \delta(E - E'), \qquad (4.14)$$

then the sub-matrix $S_{\alpha\beta}(E)$ is a unitary matrix in the indices α, β for fixed energy E. In analogy to (4.8) S(E) may be constructed in the vicinity of a resonance with the help of a matrix $S^{(r)}(E)$, which exclusively describes the influence of metastable states on the process [$\bar{\delta}$ in (4.1)] and a matrix $S^{(0)}$. The general resonance formula considering $S^{(0)}$ was derived by Wigner[47] from the following considerations: the solution of the stationary Schrödinger equation for an energy in the vicinity of resonance has, in the interaction region, the form $c\psi$, where ψ may be thought of as the wave-function of the

45. P. L. Kapur and R. Peierls, Proc. Roy. Soc. **A166**, 277 (1938).
46. For more detailed review, see A. M. Lane and R. G. Thomas, Revs. Mod. Phys. **30**, 257 (1958), and G. Breit, Encyclopedia of Physics (Springer-Verlag, Berlin, 1959), Vol. **41/1**.
47. E. P. Wigner, Phys. Rev. **70**, 606 (1946).

metastable state. Thus ψ does not change its shape when the energy E passes through the resonance. Only the normalization factor c (probability of excitation, $|c|^2$, of the metastable state) will pass through a sharp maximum.

Here we wish to use the same method for the derivation of the resonance formula as was used for the proof of (4.8). Therefore, let the $|\alpha E\rangle_0$ be a basis of approximate solutions outside the interaction region (eigenstates of H_0), and let $|0\rangle$ be the stable state in the approximate solution inside the interaction region. Without loss of generality we can omit those parts of the coupling matrix H' that only relate the states outside the interaction region, as they may be included in H_0. For the same reason we set $H'_{00} = 0$. With the introduction of the projection operator $Q = 1 - |0\rangle\langle 0|$ we therefore write

$$H_0 = QHQ + \epsilon_0 |0\rangle\langle 0| \tag{4.15}$$

$$\epsilon_0 = \langle 0|H|0\rangle\,;\; H' = H - H_0 \tag{4.16}$$

Then one obtains for the eigenstates of H

$$|\alpha E\rangle^{(+)} = |\alpha E\rangle_0^{(+)} + c_\alpha(E)\left(|0\rangle - \frac{Q}{H_0 - E + i\eta}H'|0\rangle\right) \tag{4.17}$$

if $|\alpha E\rangle_0^{(+)}$ are the corresponding eigenstates of H_0. Again $c_\alpha(E)$ is obtained from the secular equation

$$c_\alpha(E) = \frac{\langle 0|H'|\alpha E\rangle_0^{(+)}}{E - \epsilon_0 + \langle 0|H'\frac{Q}{H_0 - E}H'|0\rangle - i\pi\Sigma_\beta |\langle 0|H'|\beta E\rangle^{(+)}|^2} \tag{4.18}$$

and for $S_{\alpha\beta}(E)$ one obtains

$$S_{\alpha\beta} = S^{(0)}_{\alpha\beta} - 2\pi i c_\alpha^*(E)\langle 0|H'|\beta E\rangle_0^{(-)}. \tag{4.19}$$

Further, with the introduction of the basis

$$|\alpha E\rangle = \sum_\beta (S^{(0)-\frac{1}{2}})_{\beta\alpha}|\beta E\rangle_0^{(-)} = \sum_\beta (S^{(0)\frac{1}{2}})_{\beta\alpha}|\beta E\rangle_0^{(+)} \tag{4.20}$$

as well as the matrix elements

$$H'_{\alpha 0} = \langle \alpha E|H'|0\rangle \tag{4.21}$$

and the abbreviations

$$E_0 = \epsilon_0 - \langle 0|H'\frac{Q}{H_0 - E}H'|0\rangle \tag{4.22}$$

$$\Gamma = 2\pi\Sigma|H'_{\gamma 0}|^2 \tag{4.23}$$

$$\chi_\alpha = H'_{\alpha 0}(\Sigma|H'_{\gamma 0}|^2)^{-\frac{1}{2}};\; P_{\alpha\beta} = \chi_\alpha\chi_\beta^*, \tag{4.24}$$

the S matrix can be written in the form

$$S = S^{(0)\frac{1}{2}}\left(1 - \frac{i\Gamma}{E - E_0 + i\Gamma/2}P\right)S^{(0)\frac{1}{2}}. \tag{4.25}$$

We will further assume that the quantum numbers α, β have been so selected that because of invariance under time reversal $S_{\alpha\beta}$ is a symmetric matrix (compare references 23 and 44). Then

$$S = S^{(0)\frac{1}{2}}S^{(r)}S^{(0)\frac{1}{2}} \tag{4.26}$$

or, in detail

$$S^{(r)}_{\alpha\beta} = \delta_{\alpha\beta} - \frac{i\Gamma}{E - \epsilon_0 + i\Gamma/2}\chi_\alpha\chi_\beta^*;\; \chi_\alpha \text{ real}, \Sigma\chi_\alpha^2 = 1. \tag{4.27}$$

Since the probability that the collision partners will really interpenetrate is slight outside the resonance

one can make the further assumption that $S^{(0)}$ contributes only to elastic scattering, i.e., it is a diagonal matrix. Then for actual reactions the Breit-Wigner formula is obtained:

$$\sigma_{\alpha\beta} \sim \frac{\Gamma_\alpha \Gamma_\beta}{(E - \epsilon_0)^2 + \Gamma^2/4}; \quad \Gamma_\alpha = \Gamma\chi_\alpha^2; \Sigma\Gamma_\alpha = \Gamma. \tag{4.28}$$

whereas for elastic scattering one finds interference between "potential scattering" and "resonance scattering."

$$\sigma_\alpha^{el} \sim \left| e^{2i\eta_\alpha}\left(1 - \frac{i\Gamma_\alpha}{E - \epsilon_0 + i\Gamma/2}\right) - 1\right|^2$$
$$= \left| \frac{\Gamma_\alpha}{E - \epsilon_0 + i\Gamma/2}\right) - 2e^{i\eta_\alpha} \sin \eta_\alpha \Big|^2 \tag{4.29}$$

Equation (4.27) shows that all the eigenvalues of $S^{(r)}$ except one are equal to unity. This last eigenvalue has the form $e^{2i\tilde{\delta}}$, where $\tilde{\delta}$ is given by (4.9). The corresponding eigenvector is χ. Therefore the width Γ is again given by the derivative of $\tilde{\delta}$:

$$2/\Gamma = \tilde{\delta}'. \tag{4.30}$$

Physically (4.28) and (4.29) are of special interest since the reaction cross sections as well as the resonance part of the elastic scattering cross section are made up of products $\sigma_\alpha \omega_\beta$, with $\omega_\beta = \chi_\beta^2$, i.e., $\sum \omega_\beta = 1$. Therefore σ_α can be interpreted as the cross section for formation of the metastable state starting from channel α, and ω_β can be interpreted as the probability that this state decays into the channel β. The reaction is especially clearly described by an incoming wave packet $\int C(E)|\alpha E)^{(-)} dE$ whose uncertainty in the energy is large compared to Γ. A part of this wave packet is reflected at the

edge of the interaction region. A portion, of width Γ and mean energy E_0, penetrates and forms a metastable state. It is observed by the appearance of an "absorption line." The metastable state decomposes with the mean lifetime $\hbar/\Gamma$ and the probabilities ω_β into the various channels β. The emission lines which thus arise all have the same widths and mean energy which furthermore are independent of the process leading to the metastable state, that is, independent of the entrance channel α. The decomposability of nuclear reactions into these two steps was first postulated by Niels Bohr.[48]

Instead of S, the matrix

$$K = i\frac{1 - S}{1 + S}; \text{ Inversion: } S = \frac{1 + iK}{1 - iK} \tag{4.31}$$

is frequently considered. From (4.26) we obtain

$$K = K^{(0)} + K^{(r)} \qquad \Delta = \frac{\Gamma}{2}\chi|K^{(0)}|\chi)$$

$$K^{(r)} = \frac{\Gamma'/2}{E - E_0 + \Delta}\varphi_\alpha\varphi_\beta; \quad \varphi_\alpha = \frac{1}{N}(1 + K_0^2)_{\alpha\rho}\chi_\rho$$

$$K_0 = i\frac{1 - S^{(0)}}{1 + S^{(0)}} \qquad N^2 = (\chi|1 + K_0^2|\chi); \Gamma' = N^2\Gamma. \tag{4.32}$$

Near the resonance the K matrix has a pole on the real axis. It corresponds to the maximum of the cross section and is displaced from the resonance by Δ. $K^{(r)}$ is essentially identical with Wigner's R matrix. Wigner has generalized the resonance formulae for the case in which more than one resonance lies in the energy range considered.[41, 49]

48. N. Bohr, K. Danske Vidensk. Selsk. **14**, Nr. 10 (1937).
49. E. P. Wigner, Phys. Rev. **70**, 606 (1946).

Estimates of the Widths

The width Γ of the resonance will, according to (4.30), be essentially determined by the derivative of the phase with respect to the energy. We therefore turn to the calculation of this quantity. For simplicity we first confine ourselves to pure potential scattering. This has, moreover, the advantage that one can make use of some methods of the theory of differential equations over and above the mathematics of Hilbert space. It is possible to indicate generalizations for nuclear reactions with at most two reaction fragments (see below).

The Schrödinger equation for the radial wave function multiplied by r, with angular momentum l and energy $\epsilon = \hbar^2k^2/2m$ is

$$-u_\epsilon'' + \left\{\frac{l(l+1)}{r^2} + V(r)\right\}u_\epsilon(r) = k^2u_\epsilon(r). \tag{4.33}$$

We normalize u so that for large r

$$u_\epsilon(r) \to w_\epsilon(r) = \sin\left(kr - \frac{l\pi}{2} + \delta\right) \tag{4.34}$$

We multiply (4.33) by u_{ϵ_1} (solution with energy ϵ_1) and subtract the corresponding equation in which ϵ_1 is exchanged with ϵ:

$$\frac{d}{dr}(u_\epsilon u_{\epsilon_1}' - u_\epsilon' u_{\epsilon_1}) = (k^2 - k_1^2)u_\epsilon u_{\epsilon_1}. \tag{4.35}$$

The same equation holds for the ω_ϵ. If they are subtracted from (4.35) one may, on account of (4.34), integrate both sides up to $r \to \infty$ and, because $u(0) = 0$

$$(k^2 - k_1^2)\int_0^\infty (u_\epsilon u_{\epsilon_1}' - w_\epsilon w_{\epsilon_1})dr$$

$$= w_\epsilon(0)w_{\epsilon_1}'(0) - w_\epsilon'(0)w_{\epsilon_1}(0).$$

Letting k_1 go to k

$$\frac{d\delta}{dk} = (-1)^l\frac{\sin 2\delta}{2k} + 2\int_0^\infty (u^2 - w^2)dr. \tag{4.36}$$

For one thing, this equation again gives us Wigner's causality criterion.[36] Since for l not too large, $u \approx w$ beyond the range a of the potential, and since the mean value of w^2 equals $\frac{1}{2}$, the integral cannot be much more negative than $-a/2$. This leads to the estimate

$$\frac{d\delta}{dk} > -\left(a + \frac{1}{2k}\right), \tag{4.37}$$

which for $ka > 1$ agrees with the qualitative argument at the end of Section 3. Of course, for very small k $\frac{d\delta}{dk}$ can approach large negative values according to the estimate (4.37). A more exact analysis (Section 6) shows that just this situation arises when there is a bound state with low binding energy. When no bound states are present

$$\frac{d\delta}{dk} > -a. \tag{4.37a}$$

While $\frac{d\delta}{dk}$ is limited for negative values it can become arbitrarily large positively, when $\int_0^a u^2\,dr$ becomes large. This immediately shows the complete equivalence of the previously formulated criteria i) and ii).

We will assume for the following discussion that $V(r)$ vanishes exactly beyond the range a[50] and use the two linearly independent solutions of the force-free equation ($\rho = kr$)

$$F_l(\rho) = \rho j_l(\rho) = \sqrt{\frac{\pi}{2}}\sqrt{\rho}\,J_{l+\frac{1}{2}}(\rho);$$

$$G_l(\rho) = -\rho n_l(\rho) = -\sqrt{\frac{\pi}{2}}\sqrt{\rho}\,N_{l+\frac{1}{2}}(\rho); \tag{4.38}$$

50. The consideration of a known long-range force (e.g., Coulomb force) in the outer region causes no difficulties in principle (reference 44).

Then

$$F'_l G_l - G'_l F_l = 1 \tag{4.39}$$

and

$$F_l(\rho) = \rho^{l+1}\left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\sin\rho}{\rho} =$$

$$\begin{cases} \dfrac{\rho^{l+1}}{1\cdot 3\ldots(2l+1)} & \text{for } \rho \ll l+1 \\ \sin\left(\rho - \dfrac{l\pi}{2}\right) & \text{for } \rho \gg l+1 \end{cases}$$

$$G_l(\rho) = \rho^{l+1}\left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\cos\rho}{\rho} =$$

$$\begin{cases} \dfrac{1\cdot 3\ldots(2l-1)}{\rho^l} & \text{for } \rho \ll l+1 \\ \cos\left(\rho - \dfrac{l\pi}{2}\right) & \text{for } \rho \gg l+1. \end{cases} \tag{4.40}$$

Outside the interaction region, with the normalization (4.34),

$$u = F\cos\delta + G\sin\delta. \tag{4.41}$$

We now look for the boundary condition that will mark the resonance. From (4.1) $\delta_{res} = \eta + \pi/2$, therefore

$$u_{res} = -F\sin\eta + G\cos\eta \equiv \bar{G}. \tag{4.42}$$

The solution for an energy immediately below the resonance ($\delta = \eta$) is

$$v = F\cos\eta + G\sin\eta = \bar{F}. \tag{4.43}$$

Thus

$$G^2 + \bar{F}^2 = \bar{G}^2 + F^2;\quad \bar{F}'\bar{G} - \bar{F}\bar{G}' = 1. \tag{4.44}$$

One can write the condition (4.42) in two limiting cases as an energy independent homogeneous boundary condition on u:

Case 1) $\rho_0 = ka \ll l+1$.

Here $F(\rho_0) \ll G(\rho_0)$. Therefore if we disregard the abnormal case $\eta \approx \pi/2$, $u_{res} \sim r^{-1}$. The resonance energy is therefore distinguished by the existence of a corresponding solution of (4.33), which satisfied the boundary condition:

$$u(0) = 0;\ \left.\frac{du}{dr}\right|_a + \frac{l}{a}u = 0. \tag{4.45}$$

At first it seems strange that the resonance condition requires no knowledge of the phase η. The reason is that almost any arbitrary value for the logarithmic derivative $u'/u|_a$ will yield a small value of $|\sin\delta|$ according to 4.41. Only when $u'/u|_a$ lies in a narrow region around $-l/a$ can sin δ be noticeably different from zero. This shows that in the energy region considered here the condition (4.45) is not only necessary but also sufficient for a resonance.

Case 2) $\rho_0 \gg l$; for η we choose the scattering phase of a rigid sphere of radius a.

Since for $\rho \gg l$ with $l = 0$ for any ρ), $F' = G$ and $F' = -F$,

$$-u'_{res} = v = \bar{F}.$$

The reflection condition on the rigid sphere demands, however, that $v(\rho_0) = 0$. Therefore, we have in this limiting case the boundary conditions

$$u(0) = 0,\ u'(a) = 0. \tag{4.46}$$

The characterization of resonances by such a boundary value problem is useful in more complicated systems (nuclear reactions), in which many

resonances lie in one energy interval, which fulfill the assumptions 1) or 2). With respect to the wave functions in the region of interaction and in combination with the boundary conditions (4.45) or (4.46) the Hamiltonian operator is Hermitian and possesses a complete, discrete orthogonal system of eigenfunctions $\chi^{(\lambda)}$ with eigenvalues $E^{(\lambda)}$.[44] The $E^{(\lambda)}$ are the resonance energies so long as they fall in the region in which the chosen boundary condition is physically meaningful;[51] the $\chi^{(\lambda)}$ are the corresponding wave function of the metastable states. The boundary value problem, therefore, represents (in a certain energy region) the operation H_0 used earlier.

To determine the width one can in both limiting cases start with the relation

$$\frac{d\delta}{dk} + au^2(a) = 2\int_0^a u^2(r)dr. \tag{4.47}$$

Integrating (4.35) from 0 to a and letting k_1 approach k, yields the following

$$\left(\frac{\partial u}{\partial k}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial k \partial r}u\right)_a = 2k\int_0^a u^2 dr. \tag{4.48}$$

The left-hand side is transformed using (4.41) and the differential equation for the functions F and G:

$$k\frac{d\delta}{dk} + \left[\rho\left(\frac{du}{d\rho}\right)^2 - u\frac{du}{d\rho} + \rho u^2 - \frac{l(l+1)}{\rho}u^2\right]_{\rho=ka} = 2k\int_0^a u^2 dr. \tag{4.49}$$

In both limiting cases considered, boundary condition (4.45) or (4.46), the square bracket reduces approximately to ρu^2 and one has (4.47).

51. In the case $ka \gg 1$; however, an eigenvalue of the boundary value problem (4.46) is by no means always equivalent to a resonance, as is seen from simple examples.

We now use in place of u the normalized eigenfunction of the metastable state χ

$$\int_0^a \chi^2 dr = 1; \quad u_{res}(r) = \frac{u(a)}{\chi(a)}\chi(r). \tag{4.50}$$

Then, using (4.47), one obtains for the width

$$\Gamma = \frac{2}{\frac{d\delta}{d\epsilon}} = \frac{\hbar^2 k}{m}\frac{2}{\frac{d\delta}{dk}} = \begin{cases} \frac{\hbar^2 k}{m}\frac{\chi^2(a)}{G^2(a)}\left(1 - \frac{a^2}{2}\chi^2(a)\right) & \text{Case 1} \\ \frac{\hbar^2 k}{m}\chi^2(a) & \text{Case 2} \end{cases}$$

$$\approx \frac{\hbar^2 k}{m}\frac{\chi^2(a)}{G^2(a) + F^2(a)}.$$

Here the facts were used that $u^2(a)$ equals $G^2(ka)$ in Case 1 and equals unity in Case 2. One has the reasonable result that the narrower the resonance the smaller is the value of the normalized wave function χ at the boundary.

The generalization of the above formulae in the case of nuclear reactions has been given earlier only for processes in which no more than two fragments appeared in the final configuration; in which, therefore, the wave function differs from zero only in an inner region $\mathfrak{G}$ and in certain "channels" of configuration space. $\mathfrak{G}$ is that part of the configuration space in which no group classification of particles leads to separation distances appreciably greater than the range of nuclear forces. A "channel" corresponds to a classification of nucleons into two groups, which are separated from one another by more than the range of nuclear forces, while the nucleons within one group interact with each other. Within such a "channel" the wave function, with its dependence on the relative coordinates of a group of nucleons, is a linear combination of discrete stationary states of the corresponding nuclear fragments.

We use the symbols:

α = The whole of all magnitudes which are necessary to characterize the fragments (number of particles, quantum numbers which determine the internal movement of each fragment). Instead of j_1, m_1, j_2, m_2 (magnitudes and z-components of the intrinsic angular momenta of the individual fragments) it is useful to employ the resultant spin (j_α) and its z-component (m_α) together with j_1, j_2. We will, contrary to usage in the immediately preceding paragraph, call the entirety of numbers α a channel.

i_α = Internal coordinates of the fragments.

r_α, Ω_α = Length and direction of the line connecting their centers of mass.

l = orbital angular momentum of the relative motion of the fragments.

J, M = amount and z-component of the total angular momentum.

a_α = value of r_α at the boundary of the inner region.

The wave function $\varphi_\alpha\ (i_\alpha)$ of the internal motion of the two fragments is combined with the angular component of the relative motion to give a wave function $\varphi_{\alpha l}$ with a definite total angular momentum J and z-component M. It contains all coordinates except the distance r_α:

$$\varphi_{\alpha l}(i_\alpha,\Omega_\alpha) = \frac{1}{a_\alpha}\sum_{m,m_\alpha} C^{j_\alpha l J}_{m_\alpha m M}\varphi_\alpha(i_\alpha)Y_{lm}(\Omega_\alpha) \qquad (4.52)$$

In solving the Schrödinger equation we can choose constant values for J and M, since they are good quantum numbers. Thus we do not explicitly write down these indices in $\varphi_{\alpha l}$.

Two further observations are necessary before we apply (4.45) and (4.51) to nuclear reactions. The first concerns the form of the wave functions in "closed channels" which are those states of the fragments which, on the basis of energy considerations, cannot appear as end products. In these, the wave function decays exponentially with r_α and can be set equal to zero, provided that the energy does not lie too close below the threshold of the reaction. We will disregard this case.[52]

Secondly, one would like to assume that the $\varphi_{\alpha l}$, which belong to various group classifications of the nucleons, do not overlap for all practical purposes, so that to a good approximation

$$\int \varphi_{\alpha l}^{*}\,\varphi_{\beta l'}\,dO = \delta_{\alpha\beta}\delta_{ll'}, \qquad (4.53)$$

where dO is the surface element of the border of the internal region $\mathfrak{B}$. At the starting point of channel α

$$dO = a_\alpha^2\, d\Omega_\alpha\, di_\alpha. \qquad (4.54)$$

According to the preceding assumptions the form of dO will only be needed at such starting points. Both observations mean that one must choose a sufficiently large inner region.

With these assumptions the formulae (4.45) and (4.51) can be transformed for use with nuclear reactions by inserting instead of the boundary value at r = a the corresponding integral over the surface O. That is, the boundary condition which will select out the resonances (for small values of $k_\alpha a_\alpha$) is now[53]

$$\int \varphi_{\alpha l}^{*}\left(\frac{\partial\psi}{\partial n} + \frac{l(l+1)}{a_\alpha}\psi\right)d\Omega = 0 \qquad (4.55)$$

52. E. P. Wigner, Phys. Rev. 73, 1002 (1948).

53. In place of l substitute $l + 1$ in (4.45) because ψ corresponds to u/r there. $\frac{\partial\psi}{\partial n}$ means the normal derivative perpendicular to O and is, therefore, equal to $\frac{\partial\varphi}{\partial r_\alpha}$ at the starting point of channel α.

for all open channels[54] with the given J, M in the region of energy considered. If χ is a normalized solution of the boundary value problem with energy $E^{(\lambda)}$ (wave function of the metastable intermediate nucleus):

$$\int_{\mathfrak{G}} |\chi|^2 d\tau = 1 \tag{4.56}$$

then the partial width $\Gamma_{\alpha l}$ (see (4.28)) where α denotes the channel and l the orbital angular moment, is given by (4.51):

$$\Gamma_{\alpha l} = \frac{\hbar^2 k_\alpha}{m_\alpha} \frac{\gamma_{\alpha l}^2}{G_l^2(k_\alpha a_\alpha) + F_l^2(k_\alpha a_\alpha)}$$

$$\gamma_{\alpha l} = \int \chi^* \varphi_{\alpha l} d\Omega. \tag{4.57}$$

It can be shown that the $\gamma_{\alpha l}$ are real if one chooses the phases of the $\varphi_{\alpha l}$ and χ in a suitable way.[44]

Like (4.51), (4.57) does not yet contain a direct relation between the parameters of interaction and the measurable quantities $\Gamma_{\alpha l}, E^{(\lambda)}$. However, they may be used to make some statistical statements and to derive sum rules which can be tested directly.[55]

For completeness we will mention at this point a more general application of the method of Wigner and Eisenbud.[44] The method which we have partly described here consists of separating out a complete orthogonal system $\chi^{(\lambda)}$ in $\mathfrak{G}$ by means of a suitable homogeneous linear boundary condition and then expressing the S matrix elements by means of the values of the normalized $\chi^{(\lambda)}$ on O (i.e., the $\gamma_{\alpha l}^{(\lambda)}$ and the $E^{(\lambda)}$. Even if the chosen boundary

54. When one has in mind only a limited energy interval, the solution of the Schrödinger equation is automatically almost equal to zero at the entrance of each channel which is closed in the entire interval, so that specification of a boundary condition is unnecessary.
55. T. Teichmann and E. P. Wigner, Phys. Rev. 87, 123 (1952); J. M. Blast and V. F. Weisshopf, *Theorelical Nuclear Physics* (Wiley, New York, 1952).

condition should not be related to a resonance condition, the resulting development of S is occasionally useful in the description of the action of short range forces.

Section 5. The Behavior of S at Low Energies

The behavior of the scattering matrix at low energies is determined by the fact that for interactions with finite range a only the scattering phases for sufficiently small angular momenta $\hbar l$ differ noticeably from zero. The collision parameter $\hbar l/mv = l/k$ increases with increasing angular momentum, and finally exceeds the range of the forces. Quantum mechanically this follows directly from the expansion (4.40). In order to be able to use it we bring $S_{\alpha\beta}$ into a suitable form. First, according to (2.32):

$$|\epsilon\alpha\rangle^{(\pm)} = |\epsilon\alpha\rangle - \frac{1}{H - \epsilon \pm i\eta}(H - \epsilon)|\epsilon\alpha\rangle, \tag{5.1}$$

and (2.35):

$$\begin{aligned} &{}^{(+)}\langle\alpha\epsilon'|\epsilon\beta\rangle^{(-)} - \delta_{\alpha\beta}\delta(\epsilon' - \epsilon) \\ &= -2\pi i {}^{(+)}\langle\alpha\epsilon|H - \epsilon|\epsilon\beta\rangle\delta(\epsilon' - \epsilon) \\ &= -2\pi i\langle\alpha\epsilon|H - \epsilon|\epsilon\beta\rangle^{(-)}\delta(\epsilon' - \epsilon). \end{aligned} \tag{5.2}$$

for $(H - \epsilon)|\epsilon\alpha\rangle$ we will simply write $V_\alpha|\epsilon\alpha\rangle$.[56] By substitution of (5.2) for $S_{\alpha\beta}$ we get

$$S_{\alpha\beta}^{(+)} = \delta_{\alpha\beta} - 2\pi i {}^{(+)}\langle\alpha\epsilon|V_\beta|\epsilon\beta\rangle \tag{5.3a}$$

$$S_{\alpha\beta}^{(-)} = \delta_{\alpha\beta} - 2\pi i \langle\alpha\epsilon|V_\alpha|\epsilon\beta\rangle^{(-)}. \tag{5.3b}$$

The following analysis again refers to the case of

56. Here the Pauli principle must be observed. In general, there is no operator H_0 such that: $\epsilon|\epsilon\alpha\rangle = H_0|\epsilon\alpha\rangle$ is true for all states.