7.6 $H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2 - qEx$

$a = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega x + ip)$

$H|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle \qquad n = 0, 1, 2 \cdots$

$(a + a^\dagger) = \frac{2m\omega x}{\sqrt{2m\omega\hbar}} = \sqrt{\frac{2m\omega}{\hbar}}\, x$

$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$

a) $\langle n | -qEx | n \rangle = -qE\sqrt{\frac{\hbar}{2m\omega}} \langle n | a + a^\dagger | n \rangle$

$= -qE\sqrt{\frac{\hbar}{2m\omega}} \sqrt{n+1} \left( \langle n+1 | n \rangle + \langle n | n+1 \rangle \right) = 0$

For the second order correction

$\Delta E_n^2 = \sum \frac{|\langle k | (-qE\sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)) | n \rangle|^2}{E_n - E_k} =$

$= \frac{q^2E^2\hbar}{2m\omega} \left( \frac{n}{E_n - E_{n-1}} + \frac{(n+1)}{E_n - E_{n+1}} \right)$ $\quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ $\quad a|n\rangle = \sqrt{n}\,|n-1\rangle$

$= \frac{q^2E^2\hbar}{2m\omega} \left( \frac{n}{\hbar\omega} - \frac{n+1}{\hbar\omega} \right)$

$= -\frac{q^2E^2}{2m\omega^2}$

b) Let $\frac{1}{2} m\omega^2 x^2 - qEx = \frac{1}{2} m\omega^2 \underbrace{\left( x - \frac{qE}{m\omega^2} \right)}_{x'}{}^2 - \frac{1}{2} m\omega^2 \frac{q^2E^2}{m^2\omega^4}$

$E_n' = \hbar\omega(n + \frac{1}{2}) - \frac{q^2E^2}{2m\omega^2}$

here we see that the second order correction agrees with the exact eigenvalues

7.11

$$\psi_{n_x n_y n_z}(x,y,z) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right)$$

$$E_{n_x n_y n_z} = \frac{\hbar^2}{2m} \frac{\pi^2}{a^2} (n_x^2 + n_y^2 + n_z^2) \qquad n_x, n_y, n_z = 1, 2, 3 \cdots$$

the ground state has $n_x = n_y = n_z = 1$ $(1,1,1)$

the first excited states are $(1,1,2)$ $(1,2,1)$ $(2,1,1)$

$$\langle n_x' n_y' n_z' | \Delta H | n_x n_y n_z \rangle =$$

$$V_0 2^3 \sin\left(\frac{n_x' \pi}{4}\right) \sin\left(\frac{n_x \pi}{4}\right) \sin\left(\frac{n_y' \pi}{2}\right) \sin\left(\frac{n_y \pi}{2}\right) \times \sin\left(\frac{n_z' \pi 3}{4}\right) \sin\left(\frac{n_z \pi 3}{4}\right)$$

for the ground state

$$\langle 1,1,1 | \Delta H | 1,1,1 \rangle = V_0 2^3 \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = 2V_0$$

for the first excited state $\langle n_x' n_y' n_z' | H | (n_x n_y n_z) \rangle$ is the $3 \times 3$ matrix

| | 112 | 121 | 211 |
|---|---|---|---|
| 112 | $6\frac{\hbar^2}{2m}\frac{\pi^2}{a^2} + V_0 2^3 \cdot \frac{1}{2}$ | $0$ | $V_0 2^3 \cdot \frac{1}{\sqrt{2}} \cdot 1 \cdot 1 \cdot (-1)(-\frac{1}{\sqrt{2}})$ |
| 121 | $0$ | $6\frac{\hbar^2}{2m}\frac{\pi^2}{a^2} + V_0 2^3 \frac{1}{2} 0 \cdot \frac{1}{2}$ | $0$ |
| 211 | $V_0 2^3 \cdot \frac{1}{\sqrt{2}} 1 \cdot 1 (-1)(-\frac{1}{\sqrt{2}})$ | $0$ | $6\frac{\hbar^2}{2m}\frac{\pi^2}{a^2} + V_0 2^3 \cdot 1 \cdot 1 \frac{1}{2}$ |

$$= \begin{pmatrix} \frac{3\hbar^2\pi^2}{ma^2} + 4V_0 & 0 & +4V_0 \\ 0 & \frac{3\hbar^2\pi^2}{ma^2} & 0 \\ 4V_0 & 0 & \frac{3\hbar^2\pi^2}{ma^2} + 4V_0 \end{pmatrix}$$

me $\det(E - M) = 0$ ($M$ = above matrix

$$\left(E - \frac{3\hbar^2\pi^2}{ma^2}\right)\left[\left(E - \left(\frac{3\hbar^2\pi^2}{ma^2} + 4V_0\right)\right)^2 - (4V_0)^2\right]$$

This has roots

$$E = \frac{3\hbar^2\pi^2}{ma^2}$$

$$E_\pm = \left(\frac{3\hbar^2\pi^2}{ma^2} + 4V_0\right) \pm 4V_0 = \begin{cases} \frac{3\hbar^2\pi^2}{ma^2} + 8V_0 \\ \frac{3\hbar^2\pi^2}{ma^2} \end{cases}$$

7.12

$$H = V_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \epsilon \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} V_0$$

a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $E = V_0$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $E = V_0$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $E = 2V_0$

b) $\det(E - H) = \det \begin{pmatrix} E - (1-\epsilon)V_0 & 0 & 0 \\ 0 & (E - V_0) & \epsilon V_0 \\ 0 & \epsilon V_0 & (E - 2V_0) \end{pmatrix}$

$$(E - (1-\epsilon)V_0)\left((E - V_0)(E - 2V_0) - \epsilon^2 V_0^2\right) = 0$$

$$(E - (1-\epsilon)V_0)\left(E^2 - 3V_0 E + 2V_0^2 - \epsilon^2 V_0^2\right)$$

the roots are

$$E_1 = (1-\epsilon)V_0 \qquad \checkmark$$

$$E_\pm = \frac{3V_0 \pm \sqrt{9V_0^2 - 8V_0^2 + 4\epsilon^2 V_0^2}}{2}$$

$$E_\pm = V_0\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{1 + 4\epsilon^2}\right) \simeq$$

$$V_0\left(\frac{3}{2} \pm \frac{1}{2}(1 + 2\epsilon^2 + \cdots)\right) + \cdots$$

$$V_0 \begin{cases} 2 + \epsilon^2 + \cdots & \checkmark \\ 1 - \epsilon^2 + \cdots & \checkmark \end{cases}$$

c) For the $E = 2V_0$ state

$$\Delta E^1 = (0\,0\,1)\,\epsilon \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\Delta E^2 = \frac{1}{E_3 - E_1} |\langle 3|\Delta E|1\rangle|^2 + \frac{1}{E_3 - E_2} |\langle 3|\Delta E|2\rangle|^2$$

$$= \frac{V_0^2}{V_0} \epsilon^2 \left| (0\,0\,1) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right|^2$$

$$+ \frac{V_0^2}{V_0} \epsilon^2 \left| (0\,0\,1) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2$$

$$= 0 + \epsilon^2 V_0 \cdot 1$$

which gives the correct first and second terms

d) For the degenerate case we need to diagonalize

$$\det \begin{pmatrix} E - V_0 & \epsilon V_0 \\ \epsilon V_0 & E - 2V_0 \end{pmatrix} = E^2 - 3V_0 E + 2V_0^2 - \epsilon^2 V_0$$

This is the same determinant as in part b)

$$E_\pm = V_0 \left( \frac{3}{2} \pm \frac{1}{2} \sqrt{1 + 4\epsilon^2} \right)$$

7.15 Virial theorem

Assume $H = \frac{\vec{p}^2}{2m} + V(\vec{x})$

Let $|\psi\rangle$ be an eigenstate of $H$

$$\frac{d}{dt}\langle\psi(t)|\,\vec{x}\cdot\vec{p}\,|\psi(t)\rangle =$$

$$\frac{i}{\hbar}\langle\psi(t)|\left(H(\vec{x}\cdot\vec{p}) - (\vec{x}\cdot\vec{p})H\right)|\psi(t)\rangle = \frac{i\Xi}{\hbar}\langle\psi(t)|(\vec{x}\cdot\vec{p}-\vec{x}\cdot\vec{p})|\psi(t)\rangle = 0$$

on the other hand

$$\frac{d}{dt}\langle\psi(t)|\,x\cdot p\,|\psi(t)\rangle = \frac{d}{dt}\langle\psi(0)|U^\dagger(t)\vec{x}U(t)\cdot U^\dagger(t)\vec{p}\,U(t)|\psi(0)\rangle$$

$$\frac{\hbar}{i}\langle\psi(t)|[H,\vec{x}]\cdot\vec{p} + \vec{x}\cdot[H\,\vec{p}]\,|\psi(t)\rangle$$

$$[H\,\vec{x}] = \left[\frac{p^2}{2m},\vec{x}\right] = \frac{p}{2m}(-i\hbar) + \frac{(-i\hbar)}{2m}\vec{p} = -i\frac{\hbar}{m}\vec{p}$$

$$[H\,\vec{p}] = [V(x),\vec{p}] = -\frac{\hbar}{i}\vec{\nabla}V$$

using this in the above

$$0 = \frac{\hbar}{i}\left(\langle -i\frac{\hbar}{m}\vec{p}\cdot\vec{p} + \frac{\hbar}{i}\vec{x}\cdot\vec{\nabla}V\rangle\right)$$

$$2\left\langle\frac{p^2}{2m}\right\rangle = \langle\vec{x}\cdot\vec{\nabla}V\rangle$$

$$\boxed{2\langle T\rangle = \langle\vec{x}\cdot\vec{\nabla}V\rangle}$$

For the Hydrogen atom

$$\vec{x}\cdot\vec{\nabla}V = -\frac{e^2}{4\pi\epsilon_0}\left(r\frac{\partial}{\partial r}\right)\left(\frac{1}{r}\right) = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

$$\boxed{-\langle V\rangle = 2\langle T\rangle} \rightarrow 2\langle T+V\rangle = \langle 2V - V\rangle = \langle V\rangle$$

mit means

$$\langle V \rangle = 2 E_{nem}$$

$$\left\langle -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \right\rangle = 2E_{nem}$$

$$\left\langle \frac{1}{r} \right\rangle = -2\left(\frac{4\pi\epsilon_0}{e^2}\right) E_{nem}$$

7.17

as in the case of the Hydrogen

$$E^2 = m^2c^4 + p^2c^2$$

$$E = mc^2\sqrt{1+\frac{p^2}{m^2c^2}}$$

$$T = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} - mc^2$$

$$= \frac{1}{2}mc^2\frac{p^2}{m^2c^2} + \frac{1}{2}\frac{1}{2}\left(-\frac{1}{2}\right)mc^2\left(\frac{p^2}{m^2c^2}\right)^2 + \cdots$$

$$= \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2} + \cdots$$

the correction is $-\frac{1}{8}\frac{p^4}{m^3c^2}$

$$p = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left(a^+ - a\right)$$

$$p^4 = \frac{1}{\hbar^4}\frac{\hbar^2m^2\omega^2}{4}\left(a^{+2} - aa^+ - a^+a + a^2\right)^2 =$$

$$= \frac{1}{\hbar^4}\frac{\hbar^2m^2\omega^2}{4}\left(a^{+2} - 2a^+a + 1 + a^2\right)^2$$

$$\Delta E = -\frac{1}{8m^3c^2}\frac{m^2\omega^2}{4\hbar^2}\left|\left(a^{+2} - 2a^+a + 1 + a^2\right)|n\rangle\right|^2$$

$$= -\frac{\omega^2}{32\hbar^2mc^2}\left|\sqrt{n+1}\sqrt{n+2}\,|n+2\rangle + (2n+1)|n\rangle + \sqrt{n}\sqrt{n-1}\,|n-2\rangle\right|^2$$

$$= -\frac{\omega^2}{32\hbar^2mc^2}\left((n+1)(n+2) + (2n+1)^2 + n(n-1)\right)$$

$$= -\frac{\omega^2}{32\hbar^2mc^2}\left(6n^2 + 6n + 3\right)$$

7.25

From 6.59 - 6.61

$$\langle n'\ell'm' | V_x + iV_y | n\ell m\rangle = -\sqrt{2}\, C^{\ell 1 \ell'}_{m m'} \langle n'\ell' \| V \| n\ell\rangle$$

$$\langle n'\ell'm' | V_x - iV_y | n\ell m\rangle = \sqrt{2}\, C^{\ell 1 \ell'}_{m m'} \langle n'\ell' \| V \| n\ell\rangle$$

$$\langle n'\ell'm' | V_z | n\ell m\rangle = C^{\ell 1 \ell'}_{m m'} \langle n'\ell' \| V \| n\ell\rangle$$

This holds for any $\bar{V}$, If we let

$$\alpha = \langle n'\ell' \| V \| n\ell\rangle / \langle n'\ell' \| W \| n\ell\rangle$$

then

$$\frac{\langle n'\ell'm' | V_+ | n\ell m\rangle}{\langle n'\ell'm' | W_+ | n\ell m\rangle} = \alpha$$

$$\frac{\langle n'\ell'm' | V_- | n\ell m\rangle}{\langle n'\ell'm' | W_- | n\ell m\rangle} = \alpha$$

$$\frac{\langle n'\ell'm' | V_z | n\ell m\rangle}{\langle n'\ell'm' | W_z | n\ell m\rangle} = \alpha$$

$$\langle n'\ell'm' | V_x | n\ell m\rangle = \langle n'\ell'm' | \tfrac{1}{2}(V_+ + V_-) | n\ell m\rangle =$$

$$\alpha\left(\langle n'\ell'm' | \tfrac{1}{2}(W_+ + W_-) | n\ell m\rangle\right) = \alpha \langle n\ell'm' | W_x | n\ell m\rangle$$

$$\langle n'\ell'm' | V_y | n\ell m\rangle = \langle n'\ell'm' | \tfrac{i}{2}(V_- - V_+) | n\ell m\rangle =$$

$$\alpha\left(\langle n'\ell'm' | \tfrac{i}{2}(W_- - W_+) | n\ell m\rangle\right) = \alpha \langle n'\ell'm' | W_y | n\ell m\rangle.$$