The Dirac equation

① relativistic generalization of Schrodinger equation

② first order in time derivatives with Hermitian Hamiltonian for probability conservation

③ symmetric treatment of space and time derivatives

$$E = i\hbar \frac{\partial}{\partial t} \qquad \bar{p} = -i\hbar \bar{\nabla}$$

relation

$$E^2 = m^2c^4 + \bar{p}^2c^2$$

$$i\hbar \frac{\partial \psi}{\partial t} = c\alpha_x\left(-i\hbar \frac{\partial \psi}{\partial x}\right) + c\alpha_y\left(-i\hbar \frac{\partial \psi}{\partial y}\right) + c\alpha_z\left(-i\hbar \frac{\partial \psi}{\partial z}\right) + \beta mc^2 \psi$$

$$\frac{i\hbar}{c} \frac{\partial}{\partial t} \psi = -i\hbar \sum \alpha_i \frac{\partial \psi}{\partial x_i} + \beta mc \psi$$

to check to see that it satisfies $E^2 = m^2c^4 + \bar{p}^2c^2$

$$\left(\frac{i\hbar}{c} \frac{\partial}{\partial t}\right)\left(\frac{i\hbar}{c} \frac{\partial}{\partial t}\right) =$$

$$\frac{i\hbar}{c} \frac{\partial}{\partial t}\left(-i\hbar \sum \alpha_i \frac{\partial \psi}{\partial x_i} + \beta mc \psi\right) =$$

$$-i\hbar \sum \alpha_i \frac{\partial}{\partial x_i}\left(i\frac{\hbar}{c} \frac{\partial \psi}{\partial t}\right) + \beta mc\left(i\frac{\hbar}{c} \frac{\partial \psi}{\partial t}\right) =$$

$$-i\hbar \sum \alpha_i \frac{\partial}{\partial x_i}\left(-i\hbar \sum_j \alpha_j \frac{\partial \psi}{\partial x_j} + \beta mc\psi\right)$$
$$+\beta mc\left(-i\hbar \sum \alpha_j \frac{\partial \psi}{\partial x_j} + \beta mc\psi\right) =$$

$$-\hbar^2\left(\alpha_i \frac{\partial}{\partial x_i}\alpha_j \frac{\partial}{\partial x_j}\right) - i\hbar mc \sum\left(\alpha_i \frac{\partial}{\partial x_i}\beta + \beta \alpha_i \frac{\partial}{\partial x_i}\right)\psi + m^2c^2\beta^2\psi$$

in order for this to reduce to

$$-\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\hbar^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi + m^2c^2\psi.$$

we require

$$\frac{1}{2}\left(\alpha_i \frac{\partial}{\partial x_i}\alpha_j \frac{\partial}{\partial x_j} + \alpha_j \frac{\partial}{\partial x_j}\alpha_i \frac{\partial}{\partial x_i}\right) = \sum \frac{\partial^2}{\partial x_i^2}$$

$$\alpha_i \frac{\partial}{\partial x_i}\beta + \beta \alpha_i \frac{\partial}{\partial x_i} = 0$$

$$\beta^2 = 1$$

* mere equations cannot be satisfied if α_i and β are ordinary complex numbers
* this simplest choice is to assume α_i and β are x,t independent matrices.

$$(\alpha_i \alpha_j + \alpha_j \alpha_i) = \{\alpha_i \alpha_j\} = 2\delta_{ij}$$

$$(\alpha_i \beta + \beta \alpha_i) = 0$$

$$\beta^2 = 1$$

general comments: since

$$\alpha_i^2 = I \quad \beta^2 = I$$

this means that ± 1 are eigenvalues of α_i β.

$$Tr(\alpha_i) = Tr(\alpha_i \alpha_j^2) = Tr(\alpha_j \alpha_i \alpha_j) = -Tr(\alpha_i \alpha_j^2) = -Tr(\alpha_i)$$

so

$$Tr(\alpha_i) = Tr(-\alpha_i) = 0$$

since the trace is the sum of the eigenvalues, in order to get 0 α_i β must have the same number of positive and negative eigenvalues

$\therefore$ dim of vector space is even

$\{\alpha_i, \beta$ are linearly independent

$$\sum a_i \alpha_i + b\beta = 0$$

ie multiply by β or α_i

$$\sum a_i \beta \alpha_i + b = 0$$

$$\mathrm{Tr}(\alpha_i\beta) = \mathrm{Tr}(\beta\alpha_i) = -\mathrm{Tr}(\alpha_i\beta) = 0$$

$$\Rightarrow b\,\mathrm{Tr}(I) = 0 \Rightarrow b = 0$$

this is true for the other 3 a_i

For dim = 2 there are only 3 independent traceless Hermitian matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

∴ this means that the smallest dimension for the matrices α, β is 4

Dirac chose

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

check

$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{pmatrix} + \begin{pmatrix} \sigma_2\sigma_1 & 0 \\ 0 & \sigma_2\sigma_1 \end{pmatrix} = \begin{pmatrix} i(\sigma_3-\sigma_3) & 0 \\ 0 & i(\sigma_3-\sigma_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix} = I$$

$$\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If we define

$x^0 = ct$ (so it has the same dimension $c \times x$

$$\boxed{\left(i\hbar \frac{\partial}{\partial x^0} + i\hbar \sum \alpha_i \frac{\partial}{\partial x_i} - \beta mc\right)\Psi = 0}$$

This is the Dirac equation. In this case since α_i β are 4×4 matrices

$\Psi(x) \rightarrow \Psi_i(x) \quad i = 1,2,3,4.$

If we try a plane wave solution

$$\vec{\Psi}(x,t) = \vec{\Psi}(0)\, e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x} - \frac{i}{\hbar}Et}$$

using this in the dirac equation give

$$\left(i\hbar \frac{1}{c}\frac{i}{\hbar}E + i\hbar\,\alpha_i \frac{i}{\hbar}p_i - \beta mc\right)\vec{\Psi}(0)\, e^{\frac{i}{\hbar}p\cdot x - \frac{i}{\hbar}Et} = 0$$

This means that

$$\boxed{\left(\frac{E}{c} - \vec{\alpha}\cdot\vec{p} - \beta mc\right)\vec{\Psi}(0) = 0}$$

Modern usage multiplies the Dirac equation β

$$\gamma^0 \equiv \beta \quad \gamma^i \equiv \beta\alpha_i$$

Then the equation takes on the form

$$\left(i\hbar \sum_{\mu=0}^{3} \gamma^\mu \frac{\partial}{\partial x^\mu} - mc \right) \psi = 0$$

or equivalently

$$\left(\frac{E}{c}\gamma^0 - \sum \gamma^i p^i - mc \right) \psi = 0$$

If we define $p_0 = \frac{E}{c}$ $p_i = -p^i$ this becomes

$$\boxed{\left(\sum p_\mu \gamma^\mu - mc \right) \psi = 0}$$

Note

$$\gamma^0\gamma^i = \beta\beta\alpha_i = -\beta\alpha_i\beta = -\gamma^i\gamma^0$$

$$\{\gamma^0\gamma^i\} = 0$$

$$\gamma^i\gamma^j = \beta\alpha_i\beta\alpha_j = -\beta\alpha_i\alpha_j\beta = \beta\alpha_j\alpha_i\beta = -\beta\alpha_j\beta\alpha_i = -\gamma^j\gamma^i$$

$$\{\gamma^i\gamma^j\} = 0 \quad i \neq j$$

$$\gamma^{i2} = \beta\alpha_i\beta\alpha_i = -\beta^2\alpha_i^2 = -1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

to solve for the plane wave solution note

$$\left(\sum p_\mu \gamma^\mu - mc\right)\left(\sum p_\nu \gamma^\nu + mc\right)\psi =$$

$$\left(\sum p_\mu p_\nu \tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) - m^2c^2 1\right)\psi =$$

$$\left(p_0^2 - \bar{p}^2 - m^2c^2\right)\psi$$

$$\left(\frac{E^2}{c^2} - \bar{p}^2 - m^2c^2\right)\psi = 0$$

which vanishes since $E^2 = c^2p^2 + m^2c^4$

this means that if

$$\left(\sum p_\nu \gamma^\nu + mc\right)\psi \neq 0$$

then it is a plane wave solution of the Dirac equation. the matrix has the structure.

$$\frac{E}{c}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}+\begin{pmatrix}0&0&-p_3&-p_1+ip_2\\0&0&-p_1+ip_2&p_3\\-p_3&-p_1+ip_2&0&0\\-p_1-ip_2&p_3&0&0\end{pmatrix}+\begin{pmatrix}mc&&&\\&mc&&\\&&mc&\\&&&mc\end{pmatrix}$$

$$\begin{pmatrix}\frac{E}{c}+mc & 0 & mc-p_3 & -p_1+ip_2\\ 0 & \frac{E}{c}+mc & p_1-ip_2 & p_3\\ -p_3 & -p_1+ip_2 & -\frac{E}{c}+mc & 0\\ -p_1-ip_2 & p_3 & 0 & -\frac{E}{c}+mc\end{pmatrix}\begin{pmatrix}\psi\end{pmatrix}=0$$

if we set $\bar{p}=0$ these become

$$\begin{pmatrix}2mc\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\2mc\\0\\c\end{pmatrix} \quad \text{if} \quad E>0$$

$$\begin{pmatrix}0\\0\\2mc\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\2mc\end{pmatrix} \quad \text{if} \quad E<0$$

this means that there are 2 solutions with positive energy - and 2 with negative energy.

* The Dirac equation has 4 independent solutions

  (+) 2 have positive energy

  (-) 2 have negative energy

* It can be shown that the 2 positive energy & 2 negative energy solutions correspond to states that are S_z up & down in the rest frame

  ⇒ solutions of the Dirac equation are spin $\frac{1}{2}$ particles

* electron in an electromagnetic field

  $p^\mu \to p^\mu - \frac{e}{c} A^\mu$

  in the dirac equation

* relativistic covariance

  $(p_\mu \gamma^\mu - mc)\psi = 0$

  $\psi \to \psi' = S(\Lambda)\psi$

  $(p'_\mu \gamma^\mu - mc) S(\Lambda)\psi = 0$

  multiply by $\bar{S}(\Lambda)$

  $p'_\mu \bar{S}(\Lambda) \gamma^\mu S(\Lambda) = p_\mu \gamma^\mu$

  $p'_\mu = \Lambda_\mu{}^\nu p_\nu$

  $\Lambda_\mu{}^\nu p_\nu \bar{S} \gamma^\mu S(\Lambda) = \Lambda_\alpha{}^\mu p_\mu S(\Lambda^{-1}) \gamma^\alpha S(\Lambda) = p_\mu \gamma^\mu$

equating coefficients of P_μ

$$\sum_\alpha S^{-1}(\Lambda)\gamma^\alpha S(\Lambda)\Lambda_\alpha{}^\mu = \gamma^\mu$$

multiply by $\sum_\mu \Lambda^\rho{}_\mu$

$$\sum S^{-1}(\Lambda)\gamma^\alpha S(\Lambda)\underbrace{\Lambda_\alpha{}^\mu \Lambda^\rho{}_\mu} = \Lambda^\rho{}_\mu \gamma^\mu$$

$$\eta_{\mu\nu}\Lambda_\alpha{}^\mu \Lambda_\sigma{}^\nu \eta^{\sigma\rho}$$

$$\eta_{\alpha\sigma}\eta^{\sigma\rho}$$

$$\delta_{\alpha\rho}$$

$$\boxed{S^{-1}(\Lambda)\gamma^\alpha S(\Lambda) = \sum_\mu \Lambda^\alpha{}_\mu \gamma^\mu}$$

The problem is to find the 4×4 matrices representing lorentz transformations

Before we discuss these

* the Dirac equation gives the correct electron magnetic moment
* solutions are spin $\frac{1}{2}$ particles
* Probability is conserved

* It does not fix the problem of negative energy states.

• Dirac's solution.

Because solutions of the equation are fermions, they satisfy the Pauli exclusion principle. Dirac assumed that all negative energy states are filled.

Implication - it is possible to excite a negative energy state to positive energy, leaving a "hole" at the Fermi surface. This hole behaves like an electron with a positive charge

∴ this led Dirac to predict the existence of anti electrons (discovered by Carl Anderson in 1932) (cosmic ray cloud chamber)

× Dirac's treatment of negative energies does not work

It protons also satisfy the dirac equation, neutral atoms have integer spin, which means negative energy atoms do not satisfy the Pauli principle

Mel solution

① The Dirac equation is a good relativistic equation - it gets too many things right

② It is correctly interpreted as a classical field equation, like Maxwells equations

③ Quantizing the Dirac field for an electron and the Maxwell field results in a quantum field theory called quantum electrodynamics - It is the most precisely tested theory known.

This provides the basic structure of the fundamental theories of physics

QED	electron	photon
QCD	quark	gluons
Weak	electron muon tau neutrino	photon W, Z boson
GR		graviton

electron magnetic moment

$$\left[\left(\frac{E}{c} - \frac{e}{c}A^0\right) - \vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc\right]\Psi = 0$$

multiply by

$$\left(\frac{E}{c} - \frac{e}{c}A^0\right) + \vec{\alpha}\cdot\left(p - \frac{e}{c}\vec{A}\right) + \beta mc$$

$$\left(\frac{E}{c} - \frac{e}{c}A^0\right)^2 - \underbrace{\alpha_i\alpha_j\left(p^i - \frac{e}{c}A^i\right)\left(p^j - \frac{e}{c}A^j\right)}_{(A)} - m^2c^2$$

$$\left.\begin{aligned} &+\left(\frac{E}{c} - \frac{e}{c}A^0\right)\vec{\alpha}\cdot\left(p - \frac{e}{c}A\right) \\ &- \vec{\alpha}\cdot\left(p - \frac{e}{c}A\right)\cdot\left(\frac{E}{c} - \frac{e}{c}A^0\right) \end{aligned}\right\} (B)$$

remark

$$\begin{pmatrix} 0 & \sigma\cdot A \\ \sigma\cdot A & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma\cdot B \\ \sigma\cdot B & 0 \end{pmatrix} = \begin{pmatrix} \sigma\cdot A\,\sigma\cdot B & 0 \\ 0 & \sigma\cdot A\,\sigma\cdot B \end{pmatrix} =$$

$$\begin{pmatrix} A\cdot B + i\,A\times B\cdot\sigma & 0 \\ 0 & A\cdot B + i(A\times B)\cdot\sigma \end{pmatrix}$$

$$(A) = \left[-\left(\vec{p} - \frac{e}{c}\vec{A}\right)\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) - i\left(p - \frac{e}{c}A\right)\times\left(\vec{p} - \frac{e}{c}\vec{A}\right)\right]\cdot i\sigma$$

$$\left[-\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\frac{e}{c}\left(\vec{A}\times\vec{p} + \vec{p}\times\vec{A}\right) = -\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\frac{e}{c}\hbar\vec{\nabla}\times\vec{A}\right]\cdot i\sigma$$

$$= \left[-\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\frac{e}{c}\hbar\vec{B}\right]\cdot i\sigma$$

$$(B) = +\alpha\frac{e}{c}(-i\hbar)\nabla\phi = -ie\hbar\frac{1}{c}\vec{\alpha}\cdot\vec{E}$$

$$\left(\frac{E}{c} - \frac{e}{c}A^{\circ}\right)^2 + \frac{e\hbar}{c}\bar{B}\cdot\bar{\sigma} - ic\hbar\,\bar{\alpha}\cdot\bar{E} - \left(p - \frac{e}{c}A\right)^2 - m^2c^2$$

$$\frac{E}{c} = mc + \frac{T}{c}$$

$$\left(mc + \frac{T}{c} - \frac{e}{c}A^{\circ}\right)^2 = m^2c^2 + 2mc\left(\frac{T}{c} - \frac{e}{c}A\right) + \cdots$$

$$2mT - 2meA + \frac{e\hbar}{c}B\cdot\sigma - ic\hbar\,\alpha\cdot E - \left(p - \frac{e}{c}A\right)^2$$

$$T \approx eA - \frac{e\hbar}{2mc}B\cdot\sigma + i\frac{e\hbar}{2mc}\alpha\cdot E - \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2$$

magnetic moment $\frac{e\hbar}{2mc}$