Lecture 23

Last time

Amperes Law

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I$$

$\bar{B}$ magnetic field

direction of magnetic field due to right hand rule from current

Integral is around any closed loop surrounding current

Amperes law is equivalent to the Biot Savart law

For a current density

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I = \int \bar{J} \cdot \hat{n} \, dA$$

where the integral is over the area enclosed by the curve, and the normal is given by the right hand rule

simple cases with symmetry

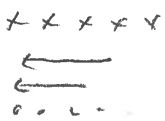
- long straight wire

$$2\pi r B = \mu_0 I \Rightarrow$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$

(same law derived from Biot Savart law

- infinite toroid



(1) no field outside – use superposition to show fields due to xxx cancel fields due to ooo

(2) inside

$$B \cdot l = \mu_0 n l I$$

$$\boxed{B = \mu_0 n I}$$

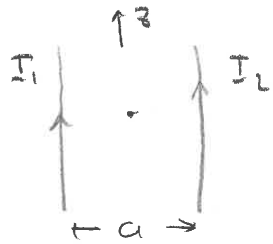
toroid

$$2\pi R B = \mu_0 N I$$

$$\boxed{B = \frac{\mu_0 N I}{2\pi r}}$$

different elements can be combined using the superposition principle

example - 2 wires separated by a



field due to single wire

$$\bar{B} = \frac{\mu_0 I}{2\pi r}\left(\frac{x\hat{y}}{r} - \frac{y\hat{x}}{r}\right)$$

shift origin $x \to x \pm \frac{a}{2}$

$$\bar{B} = \bar{B}_1 + \bar{B}_2$$

$$= \frac{\mu_0}{2\pi}\left(\frac{I_1}{(x+\frac{a}{2})^2+y^2}\left((x+\tfrac{a}{2})\hat{y} - y\hat{x}\right)\right) +$$

$$\frac{\mu_0}{2\pi}\left(\frac{I_1}{(x-\frac{a}{2})^2+y^2}\left((x-\tfrac{a}{2})\hat{y} - y\hat{x}\right)\right)$$

this gives the total magnetic field due to 2 wires

Magnetic Flux

this is defined like electric flux

$$\Phi_B \equiv \int \bar{B}\cdot\hat{n}\,dA$$

where the integral is over a surface

Faradays Law

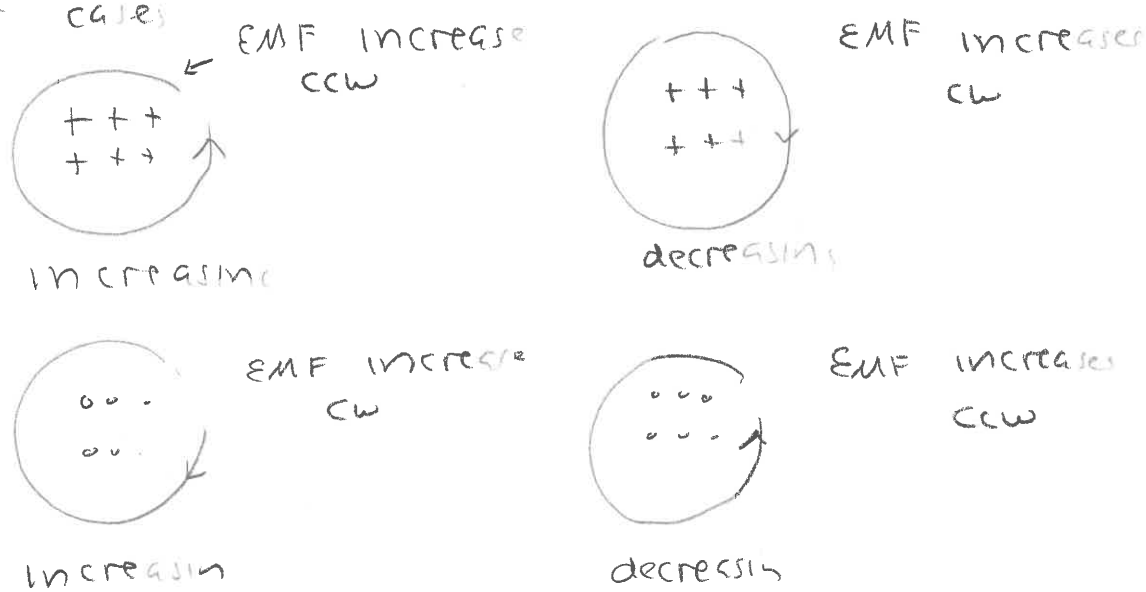
This is the third of the 4 maxwell's equations. It relates the change in magnetic flux with time to an induced EMF around the boundary of the area.

$$\frac{d\Phi}{dt} = -\mathcal{E}$$

$$\frac{d\Phi}{dt} = \int \frac{\partial \bar{B}}{\partial t} \cdot \hat{n}\, dA$$

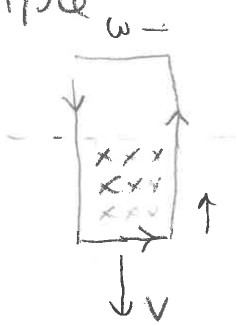
The - sign means that the emf produces a current that produces a field that opposes the change in flux

This is best illustrated in the four cases



The idea that the EMF opposes the change in flux is called Lenz's law

example



The motion of the loop is increasing the flux

Because of this the emf will cause a current to move in the direction of the arrow

$I\,dl \times F$ will result in a force in the upward direction

If the loop has resistance R

$$\mathcal{E} = IR = \frac{d\Phi}{dt} = B\omega V$$

$$I = \frac{B\omega V}{R}$$

$$F = I\omega B = \frac{B^2\omega^2 V}{R}$$

$$P = I^2 R = \frac{B^2\omega^2 V^2}{R^2} R = \frac{B^2\omega^2 V^2}{R}$$

$$= F \cdot V = \frac{B^2\omega^2 V^2}{R}$$

so the work done moving the loop through the field goes to heating the resistor

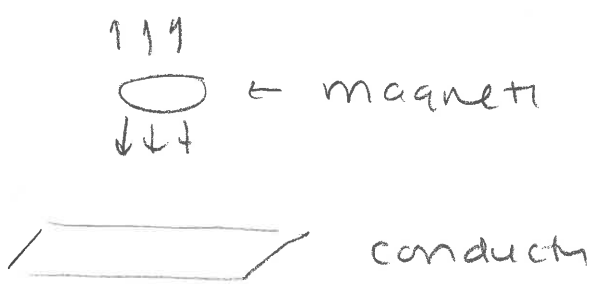
the potential drop around the loop is related to the electric field by

$$-\mathcal{E}MF = -\oint \bar{E}\cdot d\bar{\ell}$$

so amperes law can also be expressed directly in terms of the fields

$$\boxed{\begin{aligned} \frac{d\bar{\Phi}}{dt} &= -\oint \bar{E}\cdot d\bar{\ell} \\ \int_A \frac{\partial \bar{B}}{\partial t}\cdot \hat{n}\, dA &= -\oint \bar{E}\cdot d\bar{\ell} \end{aligned}}$$

where the loop integral is around the boundary of the area



if we drop magnet on the conductor the flux through the conductor increases, resulting in currents that lead to forces that oppose the motion

This is a demonstration of <u>Lenz's law</u>, which states that the change in flux creates an electric field that opposes the change in flux

Differential form of Faraday's law

consider

$$\bar{B}(x\,y\,z\,t)$$

$$\frac{\partial \bar{B}}{\partial t}(x\,y\,z\,t)$$

consider a small square of side $\Delta x\ \Delta y$ centered at x, y, z

$$\frac{\partial \Phi}{\partial t} = \int \frac{\partial \bar{B}}{\partial t} \cdot \hat{z}\, dx\, dy = \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} dx' \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} dy' \frac{\partial B_z}{\partial t}(x'\,y'\,z\,t)$$

for a nice function

$$\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} f(x')\,dx \approx \int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} \left(f(x) + f'(x)(x'-x) + \right.$$

$$= f(x)\left(\frac{\Delta}{2} + \frac{\Delta}{2}\right) + \frac{1}{2} f'(x)\,\frac{\Delta^2}{4} + \ldots$$

$$= f(x)\Delta + \text{term} \sim \Delta^2$$

if we make this approximation

$$\frac{\partial \Phi}{\partial t} = \frac{\partial B_z}{\partial t}(xyzt)\,\Delta x\,\Delta y + O(\Delta^3)$$

$$= -\int_{x-\Delta/2}^{x+\Delta/2} \left(E_x(x', y-\tfrac{\Delta}{2}, z, t) - E_x(x', y+\tfrac{\Delta}{2}, z, t)\right) dx$$

$$- \int_{y-\Delta/2}^{y+\Delta/2} \left(E_y(x+\tfrac{\Delta x}{2}, y', z', t) - E_y(x-\tfrac{\Delta y}{2}, y', z, t)\right)$$

$$= -\Delta x\left(E_x(x\ y-\tfrac{\Delta y}{2} z\ t) - E_x(x\ y+\tfrac{\Delta y}{2} z\ t)\right)$$

$$- \Delta y\left(E_y(x+\tfrac{\Delta x}{2}, y\ z\ t) - E_y(x-\tfrac{\Delta x}{2}, y\ z\ t)\right)$$

$$= \frac{\partial E_x}{\partial y}(xyzt)\,\Delta x\,\Delta y + O(\Delta^3)$$

$$- \frac{\partial E_y}{\partial x}(xyzt)\,\Delta x\,\Delta y + O(\Delta^3)$$

If we divide both sides by $\Delta x \Delta y$ and let them both $\to 0$ we get

$$\frac{\partial B_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z}$$

If we looked at different surfaces we could also get

$$\frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial y}$$

This is called the differential form of Faraday's law. We can write this as

$$\frac{\partial \bar{B}}{\partial t} = -\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \times \bar{E}$$

$$= -\bar{\nabla} \times \bar{E}$$

$$\boxed{\frac{\partial \bar{B}}{\partial t} = -\bar{\nabla} \times \bar{E}}$$

If we integrate

$$\frac{d\Phi}{dt} = -\oint \bar{E} \cdot d\bar{l} = -\int (\nabla \times E) \cdot \hat{n}\, da$$

This leads to the relation

$$\boxed{\oint \bar{E} \cdot d\bar{l} = \int (\bar{\nabla} \times \bar{E}) \cdot \hat{n}\, ds}$$

While we derived it using Faraday law, this is a general result for any vector field

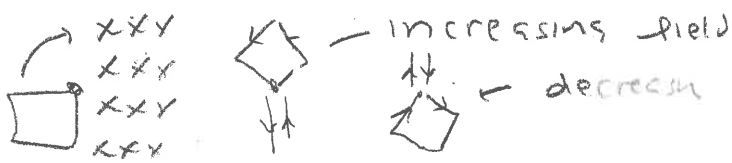
$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I = \mu_0 \int \bar{J} \cdot \hat{n}\, dA = \int (\bar{\nabla} \times \bar{B}) \cdot \hat{n}\, dA$$

comparing the 2 area integrals we get (for arbitrary area)

$$\boxed{\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}}$$

which is the differential form of Gauss law.

generating electricity



by rotating the current loop through the field the change in flux generates an oscillating current

Inductors

consider a solenoid

Amperes law gives

$$B\ell = \mu_0 I n \ell$$

$$\bar{B} = \mu_0 I n \hat{z}$$

The flux per unit length is

$$\Phi = n\ell B \cdot A = n\ell(\mu_0 I n \hat{z}) \cdot A\hat{z}$$

$$= n^2 \ell \mu_0 I A$$

The ratio of the magnetic flux to the current is called the inductance.

$$L = \frac{\Phi}{I}$$

The unit of inductance in the Henry

$$1 \text{ Henry} = \frac{1 \text{ Tesla} \cdot m^2}{C} \cdot s$$

For the solenoid

$$\boxed{\frac{L}{l} = \text{inductance / length} = \mu_0 n^2 A}$$

Note that

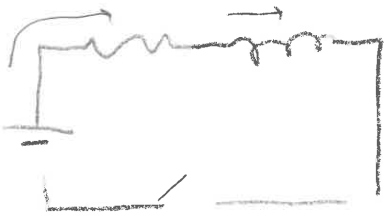
$$\Phi = LI \Rightarrow$$

$$\boxed{\frac{d\Phi}{dt} = -\mathcal{E} = L\frac{dI}{dt}}$$

This relates the drop in EMF across the inductor to the rate of change of current

By lenz's law the induced EMF tries to prevent the current from changing

RL circuits



a helical curve — is used to represent an inductor

Assume no initial current – turn on switch – use loop rule

$$\mathcal{E} - IR - L\frac{dI}{dt}$$ ← the potential across the inductor opposes the increase in current

$$\frac{dI}{dt} = \frac{1}{L}(\mathcal{E} - IR) = \frac{R}{L}\left(\frac{\mathcal{E}}{R} - I\right)$$

this is a differential equation

$$\frac{dI}{\frac{\mathcal{E}}{R} - I} = \frac{R}{L} \qquad u = \frac{\mathcal{E}}{R} - I \qquad du = -dI$$

$$-\frac{du}{u} = \frac{R}{L}dt$$

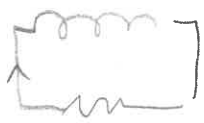
$$\ln u = -\frac{R}{L}t$$

$$\left(\frac{\frac{\mathcal{E}}{R} - I}{\frac{\mathcal{E}}{R}}\right) = e^{-\frac{R}{L}t} \qquad \frac{\mathcal{E}}{R} - I = \frac{\mathcal{E}}{R}e^{-\frac{R}{L}t}$$

$$\boxed{I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

In this case we see that we get a steady state current of $\frac{\mathcal{E}}{R}$ after a long time

If we remove the emf and start with a current



as the current decrease, the inductor creates a field that opposes the decrease in current

$$-IR - L\frac{dI}{dt} = 0 \qquad (\text{setting } \mathcal{E} = 0)$$

$$\frac{dI}{dt} = -\frac{R}{L} I$$

$$\frac{dI}{I} = -\frac{R}{L} dt$$

$$I(t) = I(0)\, e^{-\frac{R}{L}t}$$

note

$$P = I^2 R = I^2(0) R\, e^{-\frac{2R}{L}t}$$

Integrating over time

$$\int_0^\infty = I^2(0) R \int_0^\infty e^{-\frac{2Rt}{L}}\, dt \qquad u = \frac{2Rt}{L} \quad du = \frac{2R}{L} dt$$

$$= I_0^2 R \frac{L}{2R} \cdot 1 = I_0^2 \frac{L}{2}$$

This represents the energy stored in an inductor with current I

$$\text{energy} = \frac{1}{2} I^2 L$$

for a solenoid

$$L = \mu_0 n^2 A l$$

$$\Phi = I L$$

$$I = \Phi / L$$

$$\text{energy} = \frac{1}{2} \Phi^2 \frac{1}{L} = \frac{1}{2} \Phi^2 \frac{1}{\mu_0 n^2 A l}$$

$$\frac{\text{energy}}{A l} = \frac{1}{2\mu_0} \Phi^2 \frac{\Phi^2}{n^2 A^2 l^2}$$

$$\Phi = NBA \qquad nlBA$$

$$\frac{\text{energy}}{\text{volume}} = \frac{1}{2\mu_0} \frac{n^2 l^2 B^2 A^2}{n^2 l^2 A^2} = \frac{1}{2\mu_0} B^2$$

This suggests that magnetic fields have a volume energy density

$$\frac{\mathcal{E}}{\text{Vol}} = \frac{1}{2\mu_0} B^2$$