Lecture 19

Rigid body dynamics

Definition

N particles subject to holonomic constraints

$|\bar{r}_i - \bar{r}_j| = d_{ij} = \text{constant}$

Independent generalized coordinates

position of any particle in a rigid body determined up to reflection by distance from 3 non co-linear points in the rigid body

coordinates of any point determined by

(1) position of 1 point in the rigid body

(2) orientation of body-fixed coordinate system relative to inertial coordinate system

Formulas

$\bar{M}(\bar{r}_i)$ = coordinate of particle i in the body fixed coordinate system

using

$$|\vec{M}(\bar{r}_i) - \vec{M}(\bar{r}_j)| = |\bar{r}_i - \bar{r}_j|$$

differentiate with respect to $\bar{r}_i$, set $\bar{r}_i = 0$;
differentiate with respect to $\bar{r}_j$, set $\bar{r}_j = 0$.

define

$$O_{mn} \equiv \frac{\partial M^m}{\partial r^n}(\bar{r}=0) \qquad \text{constant matrix}$$

$$\sum_k O_{mk} O_{nk} = \delta_{mn} = \sum_k O_{mk} O^T_{kn}$$

$$\underline{\underline{O}}\,\underline{\underline{O}}^T = \underline{\underline{O}}^T \underline{\underline{O}} = \underline{\underline{I}} \qquad (\underline{\underline{O}}^T = \underline{\underline{O}}^{-1})$$

$$\bar{M}(\bar{r}) = \bar{M}(\bar{0}) + \underline{\underline{O}} \cdot \vec{r}$$

In this case $\bar{M}(\bar{0})$ is the coordinate of the origin of the inertial coordinate system in the body fixed coordinate system

This can be inverted

$$\bar{r} = -\underline{\underline{O}}^T \bar{M}(0) + \underline{\underline{O}}^T \bar{M}(\bar{r})$$

In this case $-\underline{\underline{O}}^T \bar{M}(0)$ is origin of the body fixed coordinate system in the inertial coordinate system

Note both relations have the same form since both $\underline{\underline{Q}}$ and $\underline{\underline{Q}}^T$ are orthogonal transformations

we can write these equations

$$\bar{r}_{Bi} = \bar{r}_{OB} + \underline{\underline{Q}}\,\bar{r}_{Ii}$$

$$\bar{r}_{Ii} = \bar{r}_{OI} + \underline{\underline{Q}}^T \bar{r}_{Bi}$$

where $\bar{r}_{OB}, \bar{r}_{OI}, \underline{\underline{Q}}$ are independent of $\bar{r}_{Bi}$ or $\bar{r}_{Ii}$.

If $\bar{r}_{Ii}(t)$ is the coordinate of particle i as a function of t

$$\bar{r}_{Bi}(t) = \bar{r}_{OB}(t) + \underline{\underline{Q}}(t)\,\bar{r}_{Ii}(t)$$

$$\bar{r}_{Ii}(t) = \bar{r}_{OI}(t) + \underline{\underline{Q}}^T(t)\,\bar{r}_{Bi}(t)$$

comment - since $\underline{\underline{Q}}(t)\,\underline{\underline{Q}}^T(t) = \underline{\underline{I}}$ independent of t it follows that

$$\frac{d\underline{\underline{Q}}}{dt}(t)\,\underline{\underline{Q}}^T(t) + \underline{\underline{Q}}(t)\,\frac{d\underline{\underline{Q}}^T}{dt} = \frac{d\underline{\underline{I}}}{dt} = 0$$

$$\Rightarrow \boxed{\frac{d\underline{\underline{Q}}}{dt}\,\underline{\underline{Q}}^T = -\underline{\underline{Q}}\,\frac{d\underline{\underline{Q}}^T}{dt} = -\left(\frac{d\underline{\underline{Q}}}{dt}\,\underline{\underline{Q}}^T\right)^T}$$

This equation means that

$$\frac{dO}{dt} \cdot O^T$$

is an antisymmetric matrix $\underline{\underline{M}}^T = -\underline{\underline{M}}$

A general 3×3 antisymmetric matrix has the form

$$\underline{\underline{M}} = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} =$$

$$= m_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + m_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + m_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we define

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In this notation a general antisymmetric matrix can be expressed as

$$\underline{\underline{M}} = m_1 \underline{\underline{L}}_1 + m_2 \underline{\underline{L}}_2 + m_3 \underline{\underline{L}}_3 = \vec{m} \cdot \vec{\underline{\underline{L}}}$$

We use

$$\frac{dO}{dt} \cdot O^T = \bar{\Omega}(t) \cdot \vec{L}$$

Here

① all of the time dependence appears in $\vec{\Omega}(t)$

② $\underline{\underline{L}}_1$ $\underline{\underline{L}}_2$ $\underline{\underline{L}}_3$ are a basis for 3×3 antisymmetric matrices

③ $\underline{\underline{L}}_1$ $\underline{\underline{L}}_2$ $\underline{\underline{L}}_3$ are constant

Interpretation of $\vec{\Omega}(t)$.

$$\frac{d\underline{\underline{O}}}{dt}\underline{\underline{O}}^T = \vec{\Omega}\cdot\vec{\underline{\underline{L}}} \rightarrow$$

$$\frac{d\underline{\underline{O}}}{dt} = \vec{\Omega}\cdot\vec{\underline{\underline{L}}}\cdot\underline{\underline{O}}(t)$$

This is a first order equation for $\underline{\underline{O}}(t)$. Consider the case where $\vec{\Omega}$ is a constant vector

$$\frac{d^2\underline{\underline{O}}}{dt^2} = \vec{\Omega}\cdot\vec{\underline{\underline{L}}}\,\frac{d\underline{\underline{O}}}{dt} = (\vec{\Omega}\cdot\vec{\underline{\underline{L}}})^2\underline{\underline{O}}$$

by induction

$$\frac{d^n\underline{\underline{O}}}{dt^n} = (\vec{\Omega}\cdot\vec{\underline{\underline{L}}})^n\underline{\underline{O}}$$

We can use this to expand $\underline{\underline{O}}(t)$ in a Taylor series

$$\underline{\underline{Q}}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\bar{\Omega}\cdot\underline{\underline{L}}\, t\right)^n \underline{\underline{Q}}(0) = e^{\bar{\Omega}\cdot\underline{\underline{L}}\, t}\, \underline{\underline{Q}}(0)$$

note for $\bar{\Omega} = \Omega \cdot \hat{e}_3$ $\quad \bar{\Omega}\cdot\underline{\underline{L}} = \begin{pmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$(\bar{\Omega}\cdot\underline{\underline{L}})^2 = \Omega^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\Omega\cdot L)^3 = -\Omega^3\, \hat{e}_3\cdot\underline{\underline{L}}$$

$$(\Omega\cdot L)^4 = \Omega^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

this continues

$$(\bar{\Omega}\cdot\underline{\underline{L}})^0 = \underline{\underline{I}}$$

$$(\Omega\cdot L)^{2n} = \Omega^{2n}(-)^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad n>0$$

$$(\Omega\cdot L)^{2n+1} = \Omega^{2n+1}(-)^n\, \hat{e}_3\cdot\underline{\underline{L}}$$

using this in the series solution

$$\underline{\underline{Q}}(t) = \underline{\underline{I}} + \sum_{n=1}^{\infty} \frac{(\Omega\theta)^{2n}}{(2n)!}(-)^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \hat{e}_3\cdot\underline{\underline{L}} \sum \frac{(\Omega t)^{2n+1}}{(2n+1)!}(-)^n$$

$$= \begin{pmatrix} \cos(\Omega t) & -\sin\Omega(t) & 0 \\ \sin(\Omega t) & \cos\Omega(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a clockwise rotation about the z-axis

The interpretation $\bar{\Omega}$ is the instantaneous angle of rotation

We can use these equations to find the form of Newtons second law is the body fixed coordinate system.

Remark: This allows us to determine the impact of the earth's rotation in an earth fixed coordinate system

We start with

$$\bar{r}_{Bi} = \bar{a}(t) + \underline{\underline{O}}(t)\,\bar{r}_{Ii}$$

$\bar{a}(t)$ = origin of inertial coordinate system in body fixed CS

$$\frac{d\bar{r}_{Bi}}{dt} = \frac{d\bar{a}}{dt} + \frac{d\underline{\underline{O}}}{dt} r_{Ii} + \underline{\underline{O}} \frac{dr_{Ji}}{dt}$$

$$= \frac{d\bar{a}}{dt} + \frac{d\underline{\underline{O}}}{dt}\left(\underline{\underline{O}}^{T}\cdot(\bar{r}_{Bi} - \bar{a}(t))\right) + \underline{\underline{O}} \frac{dr_{Ji}}{dt}$$

$$\boxed{\frac{d\bar{r}_{Bi}}{dt} = \frac{d\bar{a}}{dt} + \bar{\Omega}(t)\cdot\underline{\underline{L}}\,(\bar{r}_{Bi} - \bar{a}(t)) + \underline{\underline{O}} \frac{dr_{Ji}}{dt}}$$

note that

$$(\bar{\Omega}\cdot\underline{\underline{L}})\cdot\bar{V} = \begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \Omega_y V_z - \Omega_z V_y \\ \Omega_z V_x - \Omega_x V_z \\ \Omega_x V_y - \Omega_y V_x \end{pmatrix}$$

$$= \vec{\Omega}\times\vec{V}$$

using this gives

$$\boxed{\frac{d\bar{r}_{Bi}}{dt} = \frac{d\bar{a}}{dt} + \bar{\Omega}\times(\bar{r}_{Bi} - \bar{a}(t)) + \underline{\underline{Q}}\frac{d\bar{r}_I}{dt}}$$

differentiate a second time to get

$$\frac{d^2\bar{r}_{Bi}}{dt^2} = \frac{d^2\bar{a}}{dt^2} + \frac{d\bar{\Omega}}{dt}\times(\bar{r}_{Bi} - \bar{a}) + \bar{\Omega}\times\left(\frac{dr_{Bi}}{dt} - \frac{d\bar{a}}{dt}\right)$$

$$+ \frac{d\underline{\underline{Q}}}{dt}\underline{\underline{Q}}^T\underline{\underline{Q}}\frac{dr_I}{dt} + \underline{\underline{Q}}\frac{d^2r_I}{dt^2}$$

we can write

$$\frac{d\underline{\underline{Q}}}{dt}\underline{\underline{Q}}^T\underline{\underline{Q}}\frac{dr_I}{dt} = \bar{\Omega}\times\left(\underline{\underline{Q}}\frac{dr_I}{dt}\right) =$$

$$\bar{\Omega}\times\frac{dr_{Bi}}{dt} - \bar{\Omega}\times\frac{da}{dt} - \bar{\Omega}\times(\bar{\Omega}\times(\bar{r}_{Bi} - a(t)))$$

using this in the above gives

$$\boxed{\frac{d^2\bar{r}_{Bi}}{dt^2} = \frac{d^2a}{dt^2} + 2\bar{\Omega}\times\left(\frac{dr_{Bi}}{dt} - \frac{da}{dt}\right) - \bar{\Omega}\times\bar{\Omega}\times(\bar{r}_{Bi} - a(t)) + \frac{d\bar{\Omega}}{dt}\times(\bar{r}_{Bi} - \bar{a}) + \underline{\underline{Q}}\frac{d^2r_I}{dt^2}}$$

This gives the form of Newton's second Law in a body fixed coordinate system

$$m\frac{d^2\bar{r}_{Bi}}{dt^2} = \sum \vec{F}_i + 2m\bar{\Omega}\times\left(\frac{d\bar{r}_{Bi}}{dt} - \frac{d\bar{a}}{dt}\right)$$
$$- m\,\bar{\Omega}\times(\bar{\Omega}\times(\bar{r}_{Bi} - \bar{a}(t)))$$
$$+ m\frac{d\bar{\Omega}}{dt}\times(\bar{r}_{Bi} - a) + m\frac{d^2a}{dt^2}$$

The first term comes from expressing the force in the body fixed system — the rest of the terms are inertial forces

$2m\bar{\Omega}\times(\bar{v}_B - \dot{\bar{a}})$ vanishes when $\bar{v}_B - \bar{a}$ is at rest in the body fixed system

$m\frac{d\bar{\Omega}}{dt}\times(\bar{r}_{Bi} - \bar{a})$ vanishes when $\bar{\Omega}$ = constant

$-m\bar{\Omega}\times(\bar{\Omega}\times(\bar{r}_{Bi} - \bar{a}))$ is a centripetal force

Application - effect of earths rotation on dropping a ball from a height h above the surface of the earth.

step 1 choose coordinates

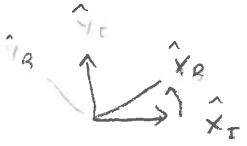
* body fixed and space fixed origins are the same and at the center of the earth

$\bar{a} = 0$

* $\bar{r}_I = \underline{\underline{Q}}^T(t)\,\bar{r}_B(t)$ $\qquad \underline{\underline{Q}}^T = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Omega = \frac{2\pi}{24\times 60\times 60}\,\text{sec}^{-1} = 7.2\times 10^{-5}$

body fixed axes move counter clock wise about z axis in inertial cs



$\bar{\Omega} = (0\ 0\ \Omega)$ $\qquad \Omega = 7.2\times 10^{-5}\,r/s$ = constant

* in this case

$$m\ddot{\bar{r}}_B = \underline{\underline{Q}}\bar{F}_i + 2m\,\bar{\Omega}\times\frac{d\bar{r}_B}{dt} - m\,\bar{\Omega}\times(\bar{\Omega}\times\bar{r}_{Bi})$$

$$V_1 = \frac{GmM_e}{r_B^2} \qquad \bar{F} = -\frac{GMm_e}{r^3}\bar{r}$$

initial conditions (z x x plane

$$\bar{r}_B = ((R+h)\sin\theta, 0, (R+h)\cos\theta)$$

$$\dot{\bar{r}}_B = (0,0,0) \quad \text{initial velocity} = 0$$

In general

$$\bar{r}_B(t) = \bar{r}_B(0) + \dot{\bar{r}}_B(0)\,t + \frac{1}{2}\ddot{\bar{r}}_B t^2 + \cdots$$

$$\underline{\underline{Q}}\,\bar{F} = -\frac{GMm}{1}\underline{\underline{Q}}\left(\frac{\bar{r}_1}{r_1^3}\right) = -\frac{GMm}{1}\underline{\underline{Q}}\,\underline{\underline{Q}}^T\frac{\bar{r}_B}{r_B^3} =$$

$$= -\frac{GMm}{r_B^3}\bar{r}_B$$

$$2m\,\bar{\Omega}\times\frac{d\bar{r}_B}{dt} = 2m(0\,0\,\Omega)\times(V_x\,V_y\,V_z) =$$

$$= 2m\,\Omega(-V_{yB}, V_{xB}, 0)$$

$$-m\,\bar{\Omega}\times(\bar{\Omega}\times\bar{r}_{Bi}) = -m[\bar{\Omega}(\bar{\Omega}\cdot\bar{r}_{Bi}) - \Omega^2\bar{r}_{Bi}]$$

$$= m\Omega^2(r_{xB}\; r_{yB}\; 0)$$

The differential equations become

$$m\ddot{x} = -\frac{GMm}{r^3}x - 2m\Omega\dot{y} + m\Omega^2 x$$

$$m\ddot{y} = -\frac{GMm}{r^3}y + 2m\Omega\dot{x} + m\Omega^2 y$$

$$m\ddot{z} = -\frac{GMm}{r^3}z$$

we can cancel m

$$\ddot{x} = -\frac{GM}{r^3}x + \Omega^2 x - 2m\Omega\dot{y}$$

$$\ddot{y} = -\frac{GM}{r^3}y + \Omega^2 y + 2m\Omega\dot{x}$$

$$\ddot{z} = -\frac{GM}{r^3}z$$

for small time we make an equivalent first order system

$$\dot{x} = v_x \qquad \dot{v}_x = -\frac{GM}{r^3}x + \Omega^2 x - 2\Omega v_y$$

$$\dot{y} = v_y \qquad \dot{v}_y = -\frac{GM}{r^3}y + \Omega^2 y + 2\Omega v_x$$

$$\dot{z} = v_z \qquad \dot{v}_z = -\frac{GM}{r^3}z$$

next we write this as a system of integral equations

$$x(t) = x(0) + \int_0^t v_x(t')\,dt'$$

$$v_x(t) = v_x(0) + \int_0^t \left(-\frac{GM}{r^3}x(t') + \Omega^2 x(t') - 2\Omega v_y(t')\right)dt'$$

$$y(t) = y(0) + \int_0^t v_y(t')\,dt'$$

$$v_y(t) = v_y(0) + \int_0^t \left(-\frac{GM}{r^3}y'(t') + \Omega^2 y(t') + 2\Omega v_x(t')\right)dt'$$

$$z(t) = z(0) + \int_0^t v_z(t')\,dt'$$

$$v_z(t) = v_z(0) + \int_0^t \left(-\frac{GM}{r^3}z'\,dt\right)$$

We solve this by iteration

$$x_1(t) = (R+h)\sin\theta$$

$$v_{1x}(t) = \int_0^t \left(-\frac{GM}{(R+h)^3}(R+h)\sin\theta + \Omega^2(R+h)\sin\theta\right) dt$$

$$y_1(t) = 0$$

$$v_{1y}(t) = 0$$

$$z_1(t) = (R+h)\cos\theta$$

$$v_{z1}(t) = \int_0^t \left(-\frac{GM}{(R+h)^3}(R+h)\cos\theta\right.$$

integrating

$$x_1(t) = (R+h)\sin\theta$$

$$v_{1x}(t) = (R+h)\left[-\frac{GM}{(R+h)^3} + \Omega^2\sin\theta\right]t$$

$$y_1(t) = 0$$

$$v_{1y}(t) = 0$$

$$z_1(t) = (R+h)\cos\theta$$

$$v_{1z}(t) = -\frac{GM}{(R+h)^3}(R+h)\cos\theta\, t$$

We insert these back in the equation to get the $O(t^1)$ solutions

$$x_2(t) = (R+h)\sin\theta + \int_0^t (R+h)\left[-\frac{GM}{(R+h)^3} + \Omega^2\sin\theta\right]t'\,dt'$$

$$= (R+h)\sin\theta\left[1 + \frac{t^2}{2}\left[-\frac{GM}{(R+h)^3} + \Omega^2\sin\theta\right]\right]$$

$$v_{2x}(t) = \int_0^t \left[-\frac{GM}{(R+h)^3}(R+h)\sin\theta + \Omega^2(R+h)\sin\theta\right]dt'$$

$$= -\frac{GM}{(R+h)^3}(R+h)\sin\theta\, t + \Omega^2(R+h)\sin\theta\, t$$

we solve by iteration assuming small Ω.

$$x_K(t) = x(0) + \int_0^t v_{xK-1}(t')\,dt'$$

$$v_{xK}(t) = \int_0^t \left(-\frac{GM}{r_{K-1}^3} x_{K-1}(t') + \Omega^2 x_{K-1}(t) - 2\Omega v_{K-1\,y}(t') \right) dt'$$

$$v_y(t) = \int_0^t \left(-\frac{GM}{r_{K-1}^3} y_{K-1}(t') + \Omega^2 y_{K-1}(t) + 2\Omega v_{K-1\,x}(t') \right) dt'$$

$$z_K(t) = z(0) + \int_0^t v_{zK-1}(t')\,dt \qquad y_K(t) = y(0) + \int_0^t v_{yK-1}(t')\,dt'$$

$$v_{Kz}(t) = 0 + \int_0^t -\frac{GM}{r_{K-1}^3} z_{K-1}(t')\,dt'$$

where $x(0) = (R+h)\sin\theta \qquad y(0) = (R+h)\cos\theta$

$$x_{K1}(t) = (R+h)\sin\theta$$

$$v_{K1}(t) = \left(-\frac{GM}{(R+h)^3} + \Omega^2 \right)(R+h)\sin\theta\, t$$

$$y_{K1}(t) = 0$$

$$v_{y1}(t) = 0$$

$$z_{K1}(t) = (R+h)\cos\theta\, t$$

$$v_K(t) = \int_0^t -\frac{GM}{(R+h)^3}(R+h)\cos\theta = -\frac{GM\cos\theta\, t}{(R+h)^2}$$

we iterate again to get terms to second order in t

$$x_2(t) = (R+h)\sin\theta + \left(-\frac{GM}{(R+h)^3} + \Omega^2\right)(R+h)\frac{t^2}{2}$$

$$v_{2x}(t) = -\frac{GM}{(R+h)}(R+h)\sin\theta\,\frac{t}{2} + \Omega^2(R+h)\sin\theta\,\frac{t}{2}$$

$$y_2(t) = 0$$

$$v_{2y}(t) = 2\Omega\left(-\frac{GM}{(R+h)^3} + \Omega^2\right)(R+h)\sin\theta\,\frac{t}{2}$$

$$z_2(t) = (R+h)\sin\theta - \frac{GM\cos\theta}{(R+h)}\frac{t}{2}$$

$$v_{z_2}(t) = -\frac{GM}{R+h)^3}(R+h)\cos\theta\,\frac{t}{2}$$

m1 shows that to second order in there is an additional displacement in x, in addition there are $O(\Omega^2)$ corrections in v_x and order $\Omega 3$ in v_y