29:172 Assignment 5 - Due Monday, Feb. 27

1. Let A be a bounded operator with the property that A^2 is compact. Under what conditions does
$$(I - A)^{-1}$$
exist. Hint use the Fredholm alternative.

2. Show that if A is compact then $A^\dagger$ is also compact.

3. Let $\{|\chi_n\rangle\}_{n=1}^{\infty}$ be any orthonormal basis and let K be compact. Consider the finite rank approximations
$$K_N = \sum_{m,n=1}^{N} |\chi_n\rangle\langle\chi_n|K|\chi_m\rangle\langle\chi_m|$$
Show that
$$|||K - K_N|||$$
can be made as small as desired by choosing a large enough N.

4. Show that the integral operator K defined by
$$\langle x|K|f\rangle = \int_{-\infty}^{\infty} \langle x|K|y\rangle dy \langle y|f\rangle$$
with
$$\langle x|K|y\rangle = e^{-x^2-y^2}$$
is compact.

5. The harmonic oscillator Hamiltonian is defined by
$$H := -\frac{d^2}{dx^2} + x^2$$
   a. Show that H is a linear operator.
   b. Show that H is an unbounded operator.
   c. H is known to be Hermitian and has a complete set of orthogonal eigenvectors with eigenvalues $2n+1$, $n = 0, 1, 2, \cdots$. The resolvent of H is the operator
   $$R(z) = (z - H)^{-1}$$
   where z is a complex number. Show that if $z \neq 2n + 1$ that $R(z)$ is compact.

6. It is possible for a unitary operator to be compact? Prove your result.