29:171 - Homework Assignment #9

1. The commutator and anti-commutator of two linear operators are defined by
$$[A,B] := AB - BA \qquad \{A,B\} := AB - BA$$
Prove the following identities
$$[A[B,C]] + [B[C,A]] + [C[A,B]] = 0$$
$$[A,BC] = [A,B]C + B[A,C]$$
$$[A,BC] = \{A,B\}C - B\{A,C\}$$

2. Let K be a linear Hermitian operator. Define
$$W := (I + iK)(I - iK)^{-1}$$
Show that W is a unitary operator.

   Express K in terms of W.(K is called the Cayley transform of W)

3. Let P be an orthogonal projection operator. Let $Q := I - P$.

   Show that Q is an orthogonal projection operator.

   Evaluate QP.

4. A linear operator N is Nilpotent if for some finite n, $N^n = 0$. Show that e^N is a finite degree polynomial in N if N is nilpotent. Show that $e^{\alpha N} e^{\beta N} = e^{(\alpha+\beta)N}$ still holds when N is nilpotent.

5. Let A be a bounded linear operator on a normed linear space. Define the partial sums
$$F_n(A) = I + \sum_{m=1}^{n} \frac{1}{m!} A^m$$
Show that this is a Cauchy sequence of operators.

6. Show that if $[A,B] = 0$ that
$$exp(A + B) = exp(A)exp(B) = exp(B)exp(A)$$
What happens to these relations if $[A,B] = \alpha I$ where α is complex and I is the identity operator?

7. Let P be a positive operator. Prove the generalized Cauchy Schwartz inequality:
$$|\langle a|P|b\rangle|^2 \leq \langle a|P|a\rangle\langle b|P|b\rangle$$