29:171 - Homework Assignment #10

1. In classical mechanics we can consider the space of functions $f(x,p)$ of coordinates and linear momentum as vectors in a vector space. The energy $E = E(x,p)$ of a system can be expressed the sum of a kinetic and potential energy, which is a function of coordinates and linear momentum.

   Define the operator A by

$$(Af)(x,p) = \frac{\partial E}{\partial x}\frac{\partial f}{\partial p} - \frac{\partial E}{\partial p}\frac{\partial f}{\partial x}$$

   Show that A is a linear operator. We sometimes write

$$Af = \{E, f\}_{p.b}$$

   which is called the Poisson Bracket of E and f

   Let $E = \frac{1}{2}(p^2 + x^2)$ be the energy for a one dimensional simple harmonic oscillator with mass 1 and spring constant 1. Calculate

$$Ax = \{E, x\}_{p.b}$$

$$Ap = \{E, p\}_{p.b}$$

   Use these results to calculate

$$x(t) = e^{-tA}x$$

   Show that this solution describes simple harmonic motion corresponding to an initial coordinate and momentum given by x and p

2. Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   These matrices are called the Pauli spin matrices.

a. Let $\hat{n}$ be a real unit vector. Show that

$$\hat{n} \cdot \vec{\sigma} = \hat{n}_x \sigma_x + \hat{n}_y \sigma_y + \hat{n}_z \sigma_z$$

   is a Hermitian matrix.

b. Show that $e^{\lambda \hat{n} \cdot \vec{\sigma}}$ is a positive operator for every real λ

3. Let N be a nilpotent operator satisfying $N^3 = 0$. Let $|v\rangle$ be an initial vector and define the polynomial in t

$$|v(t)\rangle = |v\rangle + tN|v\rangle + \frac{t^2}{2!}N^2|v\rangle$$

Show that $|v(t)\rangle$ satisfies the differential equation

$$\frac{d|v(t)\rangle}{dt} := \lim_{\epsilon\to 0} \frac{|v(t+\epsilon)\rangle - |v(t)\rangle}{\epsilon} = N|v(t)\rangle$$

4. Show that three vectors, $\vec{a}, \vec{b}, \vec{c}$, in a three dimensional space are linearly dependent if and only if the component vectors satisfy

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

5. Show that $\det(A) = \det(A^T)$ where $A^T_{mn} = A_{nm}$

6. Complex numbers are vectors in a one-dimensional vector space. Positive operators on this space are given by multiplying by a real positive number. Let $A = 1/3$. Approximate the positive square root of this operator using the method used in class:

$$C = 1 - A = 2/3 \qquad X = 1 - \sqrt{A}$$

$$X_n = \frac{1}{2}(C + X_{n-1}^2) \qquad X_0 = C/2 \qquad X = \lim_{n\to\infty} X_n$$

Compare your approximation to what you get using a calculator.