Electromagnetic current operators for phenomenological relativistic models

W. N. Polyzou
Department of Physics and Astronomy, The University of Iowa, Iowa City, IA 52242

Background: Phenomenological Poincaré invariant quantum mechanical models can provide an efficient description of the dynamics of strongly interacting particles that is consistent with spectral and scattering observables. These models are representation dependent and in order to apply them to reactions with electromagnetic probes it is necessary to have a consistent electromagnetic current operator.

Purpose: The purpose of this work is to use local gauge invariance to construct consistent strong current operators.

Method: Current operators are constructed from a model Hamiltonian by replacing momentum operators in the Weyl representation by gauge covariant derivatives.

Results: The construction provides a systematic method to construct explicit expressions for current operators that are consistent with relativistic models of strong interaction dynamics.

I. INTRODUCTION

Electromagnetic probes are useful tools for studying strong interaction dynamics. This is because they can be accurately treated to lowest non-trivial order in the exchanged photon field. In this approximation scattering matrix elements are linear in matrix elements of the electromagnetic current of the strongly interacting system. Calculations of electromagnetic observables require consistent models of the strong interaction dynamics and the hadronic current operators.

In quantum field theories the current is determined from the dynamics by requiring local gauge invariance. This is achieved by replacing derivatives by gauge covariant derivatives and extracting the coefficient of the term that is linear in the vector potential.

Phenomenological models provide an efficient means to model the structure and dynamics of systems of strongly interacting particles at energy and momentum scales that require a relativistic treatment [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22].

Inverse scattering methods [23] [24](p.245) imply that in any finite energy interval it is in principle possible to formulate phenomenological models involving the experimentally accessible degrees of freedom that give the same S-matrix elements as a more fundamental theory. Cluster properties [25][26] provide a means to build models of complex systems from models of few-body systems that can be constrained by experiment. One challenge is that the relation of phenomenological Poincaré invariant models to an underlying quantum field theory is neither direct nor straightforward.

The dynamics of free particles in relativistic quantum field theories or relativistic quantum models is the same. Free particles of mass m and spin s transform as mass m spin s irreducible representations of the Poincaré group. In relativistic quantum models with interactions there are short-range unitary transformations that preserve the scattering matrix without modifying the representation of the free particles that are used to formulate the scattering asymptotic condition [27]. S-matrix preserving changes of representation of the dynamics require a corresponding change in the representation of the hadronic current. This is easy to understand; changes of representation change the wave functions and hence the probability distribution of the charge carrying constituents. A corresponding change in the current is needed to keep the electromagnetic observables invariant. For short-range unitary transformations the one-body parts of the current remain unchanged, but the many-body parts are representation dependent.

In principle dynamical many-body contributions to the current cannot be ignored because current covariance and current conservation cannot be satisfied without dynamical contributions to the current. While these two conditions lead to representation-dependent dynamical constraints on the hadronic current operator, they do not uniquely determine the hadronic current operator, so additional information that depends on the representation of the dynamics is absent.

It is possible to satisfy current covariance and current conservation by calculating an independent set of current matrix elements in the impulse approximation and use these constraints to generate the remaining matrix elements. This procedure is sensitive to the choice of independent matrix elements and can only be applied to matrix elements. It assumes, without justification, that the required many-body parts of the current do not contribute to the chosen set of independent current matrix elements. It is not clear how to consistently use this method to treat reactions with different initial and final states. These considerations limit the use of electromagnetic probes in relativistic quantum models to reactions were the many-body parts of the hadronic current operator can be ignored. What is needed is a current operator that is consistent with the dynamics, can be used for different initial and final states and satisfies current covariance and current conservation at the operator level.

The requirement of local gauge invariance implies global gauge invariance that results in current conservation. In addition it provides a means to construct the interaction-dependent part of the strong current by replacing derivatives in the Hamiltonian associated with electromagnetically interacting particles by gauge covariant derivatives. In what follows this will be done using the Weyl representation of the dynamics. This leads to a systematic method for constructing a hadronic current operator that is consistent with the model dynamics.

In this work, since Hamiltonians involve degrees of freedom at a fixed time, the gauge invariance of the Hamiltonian is with respect to space-dependent gauge transformations at a common fixed time. The result of this construction is a current operator that is constructed directly from the interaction. Because the result is an operator it has the advantage that it can be used with different initial and final states. Since the Hamiltonian involves quantities at a fixed time, the resulting current is a 3-vector current; however since the current transforms like a four vector density, the dynamical part of the charge density operator can be expressed in terms of the commutator of the dynamical rotationless boost generator with the three-vector current.

While the construction presented in this work is systematic, preserves local gauge invariance and results in an operator that is consistent with the dynamics, the resulting current operator is not unique. One issue is the absence of a unique relativistic position operator. While the momentum operator is $-i\times$ a partial derivative, the derivative can be computed holding different types spin constant. The natural choice is to compute the derivative holding the canonical spin constant, but this is one of an infinite number of possibilities. In addition the construction assumes that the interacting particles are point charges. This is not the case for nucleons, where finite size effects result in one-body currents which depend on invariant form factors. The required form factors could be computed by applying these methods to dynamical models based on sub-nucleon degrees of freedom.

The general structure of relativistic quantum mechanical models of particles is discussed in the next section. The nature of the representation dependence of hadronic current operators is also discussed. The formulation of gauge invariance in phenomenological two-body Hamiltonians is discussed in section three. The Weyl representation of many-body operators is discussed in section four. It expresses operators using the irreducible algebra of one-body canonical coordinate and momentum operators, with the operators ordered so the momentum operators are on the right of the coordinate operators. Gauge invariant extensions can be constructed by replacing the momentum operators in the Weyl representation by gauge covariant derivatives. Section five illustrates the construction of a dynamically generated two-body current using the example of the $\mathbf{L}\cdot\mathbf{S}$, $(\mathbf{L}\cdot\mathbf{S})^2$ and $\mathbf{L}\cdot\mathbf{L}$ parts of the non-relativistic Argonne V18 interaction. The general construction, using the Weyl representation is discussed in section six. Section seven considers the case of current operators generated by translationally invariant non-local two-body interactions. The general construction of section six is applied to a square root kinetic energy operator in section eight. The square root has non-localities that can be treated in the Weyl representation. The resulting current operator is compared to the convection current for a free non-relativistic particle. Both have the form of charge times "average velocity". The method of section six results in a vector current density. The associated charge density is constructed from the vector current density using the dynamical rotationless boost generator in section nine. This ensures that resulting 4-current density transforms like a Lorentz 4-vector with respect to the dynamical representation of the Poincaré group. The result is an operator rather than a matrix element of an operator. The gauge invariant form of the time-dependent Schrödinger equation is discussed in section eleven. The treatment of the spin dependence in the relativistic case is discussed in section twelve. The application to relativistic light-front models is also discussed in this section. A summary of results and conclusions are given in the final section. There is one appendix with a detailed construction of the $(\mathbf{L}\cdot\mathbf{S})^2$ and $\mathbf{L}\cdot\mathbf{L}$ parts of the current in section five.

II. ELECTROMAGNETIC CURRENTS IN PHENOMENOLOGICAL MODELS

A relativistic quantum mechanical model is defined by a unitary ray representation, $U(\Lambda, a)$, of the component of the Poincaré group continuously connected to the identity [28]. (Here Λ labels a Lorentz transformation and a labels the displacement of a spacetime translation.) This ensures that quantum probabilities, expectation values, and ensemble averages are independent of inertial reference frame. This is a weaker condition than microscopic locality, which can only be realized in theories with an infinite number of degrees of freedom.

The relativistic analog of diagonalizing the Hamiltonian is decomposing $U(\Lambda, a)$ into a direct integral of irreducible representations. Once the mass and spin operators are diagonalized, the transformation properties of the states are fixed by group theoretic considerations.

Conserved covariant current operators $J^\mu(x)$ satisfy current conservation

$$[P^\mu, J_\mu(x)] = 0$$

and current covariance

$$U(\Lambda,a)J^\mu(x)U^\dagger(\Lambda,a) = (\Lambda^{-1})^\mu{}_\nu J^\nu(\Lambda x + a)$$

where P^μ is the generator of space-time translations. Both of these constraints involve the dynamics, since the Hamiltonian, P^0, and $U(\Lambda,a)$ are interaction dependent.

Unitary transformations V satisfying

$$\lim_{t\to\pm\infty} \|V\Pi e^{-iH_0 t}|\psi\rangle\| = 0, \tag{1}$$

where H_0 is the asymptotic Hamiltonian and Π is a channel projector result in a new S-matrix equivalent representation of the dynamics and the associated current:

$$U'(\Lambda,a) = VU(\Lambda,a)V^\dagger$$

$$J^{\mu\prime}(x) = VJ^\mu(x)V^\dagger.$$

The condition (1), with both time limits, is necessary and sufficient for $U(\Lambda,a)$ and $U'(\Lambda,a)$ to have the same S matrix *without* changing the treatment of free particles [27][29].

While this V, which is a small perturbation of the identity in the sense (1), does not change the one-body parts of the current operator, it changes the representation of the many-body parts of the current. What this means for phenomenological models is that the representation of the current depends on the representation of the dynamics.

III. GAUGE INVARIANCE IN PHENOMENOLOGICAL MODELS

While the focus in this work is on the two-body part of the hadronic current, for the purpose of discussing gauge invariance it is useful to express the dynamics in terms of fields. For systems of a fixed numbers of particles the fields are taken as the particle creation or annihilation parts of relativistically covariant free nucleon fields with physical masses. These are not local operators since they do not include the antiparticle operators. They can be normalized to satisfy equal time canonical commutation or anti-commutation relations.

A many-body Hamiltonian with two-body interactions can be expressed in terms of these fields as

$$H = \int \phi_a^\dagger(\mathbf{q}',0)k_{ab}(\mathbf{q}',\mathbf{q})\phi_b(\mathbf{q},0)d\mathbf{q}'d\mathbf{q}+$$

$$\int \phi_b^\dagger(\mathbf{q}_b,0)\phi_a^\dagger(\mathbf{q}_a,0)v_{ab:cd}(\mathbf{q}_a,\mathbf{q}_b;\mathbf{q}_c,\mathbf{q}_d)\phi_c(\mathbf{q}_c,0)\phi_d(\mathbf{q}_d',0)d\mathbf{q}_a d\mathbf{q}_b d\mathbf{q}_c d\mathbf{q}_d \tag{2}$$

where the coefficients $k_{ab}(\mathbf{q}',\mathbf{q})$ and $v_{ab:cd}(\mathbf{q}_a,\mathbf{q}_b;\mathbf{q}_c,\mathbf{q}_d)$ are matrix elements of the kinetic energy and interaction in the single-particle basis used in the field and creation operators.

The spin degrees of freedom have been suppressed in these expressions. This Hamiltonian is not necessarily local (in the sense of local potentials) and is not invariant under local gauge transformations. It is constructed to model the strong interaction dynamics for some range of energies.

The Hamiltonian commutes with itself so it is independent of time. This means that all of the fields in the Hamiltonian can be evaluated at a fixed common time.

Under a local gauge transformation all of the field operators are multiplied by position dependent phases, $e^{i\chi(\mathbf{q},t)}$, at a fixed common time. The Hamiltonian (2) will be gauge invariant if the phases intertwine with the kernels in (2) in the following sense

$$k_{ab}(\mathbf{q}',\mathbf{q})e^{i\chi(\mathbf{q},t)} = e^{i\chi(\mathbf{q}',t)}k_{ab}(\mathbf{q}',\mathbf{q}) \tag{3}$$

$$v_{ab:cd}(\mathbf{q}_a,\mathbf{q}_b;\mathbf{q}_c,\mathbf{q}_d)e^{i\chi(\mathbf{q}_c,t)+i\chi(\mathbf{q}_d,t)} = e^{i\chi(\mathbf{q}_a,t)+i\chi(\mathbf{q}_b,t)}v_{ab:cd}(\mathbf{q}_a,\mathbf{q}_b;\mathbf{q}_c,\mathbf{q}_d). \tag{4}$$

Satisfying this condition requires a modification of the original Hamiltonian. The modified Hamiltonian should become the original Hamiltonian in the limit of 0 electric charge.

Since the interactions are non-local it is useful to consider a geometric treatment of gauge transformations following [30]. Under a local gauge transformation all of the charged fields acquire spacetime dependent phases

$$\phi_a(\mathbf{q}_i,t) \to \phi'_a(\mathbf{q}_i,t) = e^{i\chi(\mathbf{q},t)_i}\phi_a(\mathbf{q}_i,t) =: U(\mathbf{q}_i,t)\phi_a(\mathbf{q}_i,t) \tag{5}$$

$$\phi^\dagger_a(\mathbf{q}_i,t) \to \phi^{\dagger\prime}_a(\mathbf{q}_i,t) = \phi^\dagger_a(\mathbf{q}_i,t)U^\dagger(\mathbf{q}_i,t). \tag{6}$$

The derivatives of the gauge transformed fields transform like

$$-i\partial_{i\mu}\phi(\mathbf{q}_i,t) \to -i\partial_{i\mu}\phi'(\mathbf{q}_i,t) = e^{i\chi(\mathbf{q}_i,t)}(-i\partial_{i\mu} + \partial_{i\mu}\chi_i(\mathbf{q}_i,t))\phi(\mathbf{q}_i,t). \tag{7}$$

This means that the derivative of the gauge transformed field is not the gauge transformation of the derivative field.

While the Hamiltonian involves degrees of freedom at a common time, the interactions are generally non-local, which implicitly involves spatial derivatives. The Hamiltonian can be transformed to a gauge invariant operator by replacing the spatial derivatives by gauge covariant derivatives.

For the non-local case it is useful to introduce a Wilson line operator $W(\mathbf{q}',t';\mathbf{q},t)$ that intertwines the gauge transformations at two different spacetime points

$$W(\mathbf{q}',t',\mathbf{q},t)U(\mathbf{q},t) = U(\mathbf{q}',t')W(\mathbf{q}',t';\mathbf{q},t). \tag{8}$$

For $\mathbf{q}' = \mathbf{q}+\delta\mathbf{q}$ close to $\mathbf{q}$ and fixed t (8) becomes

$$W(\mathbf{q}+\delta\mathbf{q},t;\mathbf{q},t)U(\mathbf{q},t) = U(\mathbf{q}+\delta\mathbf{q},t)W(\mathbf{q}+\delta\mathbf{q},t;\mathbf{q},t). \tag{9}$$

This operator can be used to construct a gauge covariant derivative. To do this consider the difference

$$\phi_a(\mathbf{q}+\delta\mathbf{q},t) - W(\mathbf{q}+\delta\mathbf{q},t;\mathbf{q},t)\phi_a(\mathbf{q},t). \tag{10}$$

Both terms in this equation undergo the same gauge transformation when $\phi_a(\mathbf{q}+\delta\mathbf{q},t)$ and $\phi_a(\mathbf{q},t)$ are gauge transformed. Dividing by $\delta\mathbf{q}$ and taking the limit as $\delta\mathbf{q}\to 0$ gives a gauge covariant derivative:

$$\lim_{\delta\mathbf{q}\to 0}\frac{\phi_a(\mathbf{q}+\delta\mathbf{q},t) - (W(\mathbf{q}+\delta\mathbf{q},t;\mathbf{q},t) - W(\mathbf{q},t;\mathbf{q},t) + W(\mathbf{q},t;\mathbf{q},t))\phi_a(\mathbf{q}),t)}{\delta\mathbf{q}} =$$

$$\lim_{\delta\mathbf{q}\to 0}[\frac{\phi_a(\mathbf{q}+\delta\mathbf{q},t) - \phi_a(\mathbf{q},t)}{\delta\mathbf{q}} - \frac{(W(\mathbf{q}+\delta\mathbf{q},t;\mathbf{q},t) - W(\mathbf{q},t;\mathbf{q},t)}{\delta\mathbf{q}}]\phi_a(\mathbf{q},t) =$$

$$(\boldsymbol{\partial} - \boldsymbol{\partial}W(\mathbf{q},t;\mathbf{q}),t)\phi_a(\mathbf{q},t) \tag{11}$$

where the derivative in the second argument acts on the first $\mathbf{q}$ in $W(\mathbf{q},t;\mathbf{q}',t)$. The vector potential is, up to a multiplicative constant, the term linear in expanding $W(\mathbf{q}+\delta\mathbf{q},t;\mathbf{q},t)$ in powers of $\delta\mathbf{q}$:

$$W(\mathbf{q}+\delta\mathbf{q},t;\mathbf{q},t) = 1 + \partial_{\mathbf{q}'}W(\mathbf{q}',t;\mathbf{q},t)_{\mathbf{q}'=\mathbf{q}}\cdot\delta\mathbf{q} + \cdots =$$

$$1 - ie\mathbf{A}(\mathbf{q},t)\cdot\delta\mathbf{q} + o(\mathbf{A}(\mathbf{q},t)^2). \tag{12}$$

This result in the familiar expression for the covariant derivative

$$\mathbf{D}\phi_a(\mathbf{q},t) := \boldsymbol{\partial}\phi_a(\mathbf{q},t) - ie\mathbf{A}(\mathbf{q},t)\phi_a(\mathbf{q},t). \tag{13}$$

IV. INTERACTIONS - THE WEYL REPRESENTATION

The starting point for a general discussion of the dynamics is to note that for a system of a finite number of degrees of freedom any linear operator O can be represented in the Weyl representation [31] as

$$\hat{O} = \int d\mathbf{a}d\mathbf{b}o(\mathbf{a},\mathbf{b})e^{i\mathbf{a}\cdot\hat{\mathbf{q}}}e^{i\mathbf{b}\cdot\hat{\mathbf{p}}} \tag{14}$$

where in this section $\hat{\mathbf{p}} = (\hat{p}_1, \cdots, \hat{p}_N)$ and $\hat{\mathbf{q}} = (\hat{q}_1, \cdots, \hat{q}_N)$ are conjugate coordinate and momentum operators:

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij}. \tag{15}$$

In this expression the hats distinguish operators from variables. This follows from the Stone-Von Neumann theorem [32][33][34] [35] which demonstrated the irreducibility of the Weyl algebra for systems with finite number of degrees of freedom. See [36] for an elementary treatment based on limits of finite systems. For the purpose of considering local gauge transformations the coordinates and conjugate momenta should be organized into three-vectors, but in this section is it more efficient to use a notation that groups the coordinates and momenta into a set of N canonically conjugate pairs of operators. In a relativistic theory the single-particle "coordinates" are functions of the single-particle Poincaré generators. They can be taken as the one-body Newton-Wigner position operators, [37] which in the single-particle momentum-canonical spin representation are i times the partial derivative with respect to the single-particle momenta holding the single-particle canonical spin constant (recall relativistic spins undergo momentum-dependent Wigner rotations)[38]. These operators are canonically conjugate to the momentum generators and commute with the canonical spin.

The complex coefficients $o(\mathbf{a}, \mathbf{b})$ can be expressed in terms of matrix elements of the operator $\hat{O}$. It is simplest to start with a mixed (coordinate-momentum) representation where, due to the ordering of the operators in (14), the operators are replaced by numbers

$$\langle \mathbf{q}|\hat{O}|\mathbf{p}\rangle = \int d\mathbf{a}d\mathbf{b}o(\mathbf{a}, \mathbf{b})e^{i\mathbf{a}\cdot\mathbf{q}}\langle \mathbf{q}|\mathbf{p}\rangle e^{i\mathbf{b}\cdot\mathbf{p}}$$

$$= \frac{1}{(2\pi)^{N/2}} \int d\mathbf{a}d\mathbf{b}o(\mathbf{a}, \mathbf{b})e^{i\mathbf{a}\cdot\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{p}}e^{i\mathbf{b}\cdot\mathbf{p}}. \tag{16}$$

This can be re-expressed in a form where the kernel $o(\mathbf{a}, \mathbf{b})$ can be computed by Fourier transform:

$$(2\pi)^{N/2}\langle \mathbf{q}|\hat{O}|\mathbf{p}\rangle e^{-i\mathbf{q}\cdot\mathbf{p}} = \int d\mathbf{a}d\mathbf{b}o(\mathbf{a}, \mathbf{b})e^{i\mathbf{a}\cdot\mathbf{q}}e^{i\mathbf{b}\cdot\mathbf{p}}. \tag{17}$$

To extract the kernel multiply both sides by

$$e^{-i\mathbf{a}'\cdot\mathbf{q}}e^{-i\mathbf{b}'\cdot\mathbf{p}}d\mathbf{p}d\mathbf{q} \tag{18}$$

and integrate over $\mathbf{p}$ and $\mathbf{q}$ to get

$$(2\pi)^{2N}o(\mathbf{a}', \mathbf{b}') = \int (2\pi)^{N/2}\langle \mathbf{q}|\hat{O}|\mathbf{p}\rangle e^{-i(\mathbf{q}\cdot\mathbf{p}+\mathbf{a}'\cdot\mathbf{q}+\mathbf{b}'\cdot\mathbf{p})}d\mathbf{p}d\mathbf{q}. \tag{19}$$

This expresses $o(\mathbf{a}', \mathbf{b}')$ in terms of the coordinate-momentum matrix elements of O:

$$o(\mathbf{a}, \mathbf{b}) = \int (2\pi)^{N/2-2N}\langle \mathbf{q}|\hat{O}|\mathbf{p}\rangle e^{-i(\mathbf{q}\cdot\mathbf{p}+\mathbf{a}\cdot\mathbf{q}+\mathbf{b}\cdot\mathbf{p})}d\mathbf{p}d\mathbf{q}. \tag{20}$$

Replacing the matrix element in the mixed representation by one in the momentum representation gives

$$o(\mathbf{a}, \mathbf{b}) =$$

$$\int (2\pi)^{N/2-2N}\langle \mathbf{q}|\mathbf{p}'\rangle d\mathbf{p}'\langle \mathbf{p}'|\hat{O}|\mathbf{p}\rangle e^{-i(\mathbf{q}\cdot\mathbf{p}+\mathbf{a}\cdot\mathbf{q}+\mathbf{b}\cdot\mathbf{p})}d\mathbf{p}d\mathbf{q} =$$

$$\int (2\pi)^{-2N}\langle \mathbf{p}'|\hat{O}|\mathbf{p}\rangle e^{i(\mathbf{q}\cdot\mathbf{p}'-\mathbf{a}\cdot\mathbf{q}-\mathbf{b}\cdot\mathbf{p}-\mathbf{q}\cdot\mathbf{p})}d\mathbf{p}d\mathbf{p}'d\mathbf{q}. \tag{21}$$

Integrating over $\mathbf{q}$ gives a delta function

$$\int (2\pi)^{-N}\langle \mathbf{p}'|\hat{O}|\mathbf{p}\rangle e^{-i\mathbf{b}\cdot\mathbf{p}}\delta(\mathbf{p}'-\mathbf{a}-\mathbf{p})d\mathbf{p}d\mathbf{p}'. \tag{22}$$

Integrating over $\mathbf{p}'$ gives

$$\int (2\pi)^{-N} \langle \mathbf{p}+\mathbf{a}|\hat{O}|\mathbf{p}\rangle e^{-i\mathbf{b}\cdot\mathbf{p}} d\mathbf{p}. \tag{23}$$

This can be put in a more symmetric form by defining $\mathbf{p} = \mathbf{p}' - \mathbf{a}/2$

$$o(\mathbf{a},\mathbf{b}) =$$

$$\int (2\pi)^{-N} \langle \mathbf{p}'+\mathbf{a}/2|\hat{O}|\mathbf{p}'-\mathbf{a}/2\rangle e^{-i\mathbf{b}\cdot(\mathbf{p}'-\mathbf{a}/2)} d\mathbf{p}'. \tag{24}$$

Removing the primes gives an expression for the coefficient $o(\mathbf{a},\mathbf{b})$ in terms of momentum-space matrix elements of $\hat{O}$:

$$o(\mathbf{a},\mathbf{b}) = \int (2\pi)^{-N} \langle \mathbf{p}+\mathbf{a}/2|\hat{O}|\mathbf{p}-\mathbf{a}/2\rangle e^{-i\mathbf{b}\cdot(\mathbf{p}-\mathbf{a}/2)} d\mathbf{p}. \tag{25}$$

These relations are for a system of N degrees of freedom. They relate momentum-space matrix elements of operators to the expansion coefficients of the operators in the Weyl representation.

For applications of interest $\hat{O}$ is a translationally invariant two-body interaction. For two particles equation (25) becomes

$$o(\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2) =$$

$$\int (2\pi)^{-6} \langle \mathbf{p}_1+\mathbf{a}_1/2, \mathbf{p}_2+\mathbf{a}_2/2|\hat{O}|\mathbf{p}_1-\mathbf{a}_1/2, \mathbf{p}_2-\mathbf{a}_2/2\rangle \times$$

$$e^{-i\mathbf{b}_1\cdot(\mathbf{p}_1-\mathbf{a}_1/2)-i\mathbf{b}_2\cdot(\mathbf{p}_2-\mathbf{a}_2/2)} d\mathbf{p}_1 d\mathbf{p}_2. \tag{26}$$

Translational invariance implies the operator $\hat{O}$ commutes with the total momentum so the matrix element in (26) can be expressed as

$$\delta(\mathbf{p}_1+\frac{\mathbf{a}_1}{2}+\mathbf{p}_2+\frac{\mathbf{a}_2}{2}-\mathbf{p}_1+\frac{\mathbf{a}_1}{2}-\mathbf{p}_2+\frac{\mathbf{a}_2}{2})\langle \mathbf{p}_1+\mathbf{a}_1 2, \mathbf{p}_2+\frac{\mathbf{a}_2}{2}\|\hat{O}\|\mathbf{p}_1-\frac{\mathbf{a}_1}{2}, \mathbf{p}_2-\frac{\mathbf{a}_2}{2}\rangle =$$

$$\delta(\mathbf{a}_1+\mathbf{a}_2)\langle \mathbf{p}_1+\frac{\mathbf{a}_1}{2}, \mathbf{p}_2-\frac{\mathbf{a}_1}{2}\|\hat{O}\|\mathbf{p}_1-\frac{\mathbf{a}_1}{2}, \mathbf{p}_2+\frac{\mathbf{a}_1}{2}\rangle. \tag{27}$$

Since momentum is conserved it is useful to make the variable change $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$, $d\mathbf{p}_1 d\mathbf{p}_2 = d\mathbf{P} d\mathbf{p}$. In terms of these variables the previous equation is replaced by

$$\delta(\mathbf{a}_1+\mathbf{a}_2)\langle \frac{1}{2}\mathbf{P}+\mathbf{p}+\mathbf{a}_1/2, \frac{1}{2}\mathbf{P}-\mathbf{p}-\mathbf{a}_1/2\|\hat{O}\|\frac{1}{2}\mathbf{P}+\mathbf{p}-\mathbf{a}_1/2, \frac{1}{2}\mathbf{P}-\mathbf{p}+\mathbf{a}_1/2\rangle. \tag{28}$$

If $\hat{O}$ is a self-adjoint operator it follows that the Weyl coefficients satisfy

$$o(\mathbf{a},\mathbf{b}) = o^*(-\mathbf{a},-\mathbf{b}) e^{i\mathbf{a}\cdot\mathbf{b}}. \tag{29}$$

V. EXAMPLE

It is useful to consider an example that illustrates the strategy for the general method. For simplicity this method will be applied to a non-relativistic model. For local potentials the potential is locally gauge invariant. For a realistic interaction like the Argonne V18 interaction [39] the operators that involve the orbital angular momentum like spin orbit terms, $\mathbf{S}\cdot\mathbf{L}$, $(\mathbf{S}\cdot\mathbf{L})^2$ and L^2 will lead to interaction dependent currents, since the orbital angular momentum involves derivatives.

The Weyl representation is useful because the operators are ordered so the momentum operators are to the right of the position operators. For the Argonne V18 interaction the ordering can be more simply realized by using canonical commutation relations, which necessarily gives the same result as the Weyl representation.

The two-body current associated with a non-relativistic spin-orbit interaction is derived first. In this case the derivative term appears linearly. The spin-orbit contribution to the current is derived by replacing the momentum in the orbital angular momentum operator by a gauge covariant derivative and extracting the coefficient of the vector potential.

In what follows it is useful to define the following conjugate pairs of momenta and coordinates

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \qquad \mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$$

$$\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2 \qquad \mathbf{Q} = \frac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2). \tag{30}$$

Assume a spin-orbit interaction of the form

$$v_{so}(|\mathbf{q}|)\mathbf{L} \cdot \mathbf{S} = v_{so}(|\mathbf{q}|)(\mathbf{q} \times \mathbf{p}) \cdot \mathbf{S} = v_{so}(|\mathbf{q}|)(\mathbf{S} \times \mathbf{q}) \cdot \mathbf{p} \tag{31}$$

where $\mathbf{L} = \mathbf{q} \times \mathbf{p}$. For this interaction $[\mathbf{L}, v_{so}(|\mathbf{q}|)] = 0$ so $\mathbf{p}$ can be placed on the right of v_{so} as in the Weyl representation. Replacing the single particle momentum operators by covariant derivatives gives

$$v_{so}(|\mathbf{q}|)(\mathbf{S} \times \mathbf{q}) \cdot (\mathbf{p} - \frac{e_1}{2}\mathbf{A}(\mathbf{q}_1) + \frac{e_2}{2}\mathbf{A}(\mathbf{q}_2)). \tag{32}$$

Note that this replacement assumes that the nucleons are structureless point charges with charge e_i. Real nucleons have a finite size with non-trivial charge and magnetic moment distributions. The electromagnetic structure of individual nucleons does not directly impact the structure of the nucleon-nucleon interaction, but it contributes to the nuclear currents. This will be ignored in this section, since the focus is on the role of local gauge invariance in constructing consistent two-body currents.

In this case it is not necessary to use the Weyl representation because (31) can be expressed directly with the momentum operators to the right for the coordinate operators.

The part of (32) that is linear in the vector potential can be expressed as

$$\int \mathbf{J}(\mathbf{x}, 0) \cdot \mathbf{A}(\mathbf{x}, 0) d\mathbf{x} \tag{33}$$

where

$$\mathbf{J}(\mathbf{x}, 0) = -\frac{1}{2} v_{so}(|\mathbf{q}|)(\mathbf{S} \times \mathbf{q})(e_1 \delta(\mathbf{x} - \mathbf{q}_1) - e_2 \delta(\mathbf{x} - \mathbf{q}_2)). \tag{34}$$

In this expression it is understood that the $\mathbf{q}_i$ are operators. They become numbers after taking mixed basis matrix elements. Taking momentum-space matrix elements of (34) gives

$$\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{J}(\mathbf{x}, 0) | \mathbf{p}_1, \mathbf{p}_2 \rangle =$$

$$-\frac{1}{2} \int \langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{Q}', \mathbf{q}' \rangle v_{so}(|\mathbf{q}|)(\mathbf{S} \times \mathbf{q}) \delta(\mathbf{Q}' - \mathbf{Q}) \times$$

$$\delta(\mathbf{q}' - \mathbf{q})(e_1 \delta(\mathbf{x} - \mathbf{q}_1) - e_2 \delta(\mathbf{x} - \mathbf{q}_2)) d\mathbf{Q} d\mathbf{Q}' d\mathbf{q} d\mathbf{q}' \langle \mathbf{Q}, \mathbf{q} | \mathbf{p}_1, \mathbf{p}_2 \rangle =$$

$$-\frac{1}{2(2\pi)^6} \int e^{-i(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{Q}} e^{-i\mathbf{q} \cdot (\mathbf{p}' - \mathbf{p})} v_{so}(|\mathbf{q}|)(\mathbf{S} \times \mathbf{q}) \times$$

$$(e_1 \delta(\mathbf{x} - \mathbf{Q} - \frac{1}{2}\mathbf{q}) - e_2 \delta(\mathbf{x} - \mathbf{Q} + \frac{1}{2}\mathbf{q})) d\mathbf{Q} d\mathbf{q} =$$

$$-\frac{e_1}{2(2\pi)^6} \int e^{-i(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{x} - \frac{1}{2}\mathbf{q})} e^{-i\mathbf{q} \cdot (\mathbf{p}' - \mathbf{p})} v_{so}(|\mathbf{q}|)(\mathbf{S} \times \mathbf{q}) d\mathbf{q}$$

$$+\frac{e_2}{2(2\pi)^6}\int e^{-i(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)\cdot(\mathbf{x}+\frac{1}{2}\mathbf{q})}e^{-i\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p})}v_{so}(|\mathbf{q}|)(\mathbf{S}\times\mathbf{q})d\mathbf{q} =$$

$$\frac{1}{(2\pi)^6}e^{-i(\mathbf{P}'-\mathbf{P})\cdot\mathbf{x}}\int[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P})}{2}\right)-i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P})}{2}\right)]\times$$

$$e^{-i\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p})}v_{so}(|\mathbf{q}|)(\mathbf{S}\times\mathbf{q})d\mathbf{q}. \tag{35}$$

Setting $\mathbf{x}=0$ this becomes

$$\langle\mathbf{p}_1',\mathbf{p}_2'|\mathbf{J}(\mathbf{0},0)|\mathbf{p}_1,\mathbf{p}_2\rangle =$$

$$\frac{1}{(2\pi)^6}\int[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P})}{2}\right)-i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P})}{2}\right)]\times$$

$$e^{-i\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p})}v_{so}(|\mathbf{q}|)(\mathbf{S}\times\mathbf{q})d\mathbf{q}. \tag{36}$$

This operator has a non-trivial dependence on the interaction $v_{so}(|\mathbf{q}|)(\mathbf{S}\times\mathbf{q})$.

Next consider operators of the form $(\mathbf{L}\cdot\mathbf{S})^2$ and $(\mathbf{L}\cdot\mathbf{L})$. These operators are quadratic in the momentum.

In order to carry out the above analysis the first step is to order the operators so the coordinate operators are on the left of the momentum operators. This can be done using the Weyl representation, but for these interactions a direct approach using the canonical commutation relations can be used to keep the vector potential to the left of the momentum operators. The end result must be the same. The steps are similar to the steps used with the spin-orbit interaction. The resulting contributions to the current are:

$$\langle\mathbf{P}',\mathbf{p}'|\mathbf{J}(\mathbf{x},0)_{(\mathbf{L}\cdot\mathbf{S})^2}|\mathbf{P},\mathbf{p}\rangle =$$

$$\frac{1}{(2\pi)^6}e^{i(\mathbf{P}-\mathbf{P}')\cdot\mathbf{x}}\int[(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P})}{2}\right)+i(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P})}{2}\right)]\times$$

$$e^{i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)\left[-\frac{1}{2}(\mathbf{S}\times\mathbf{q})\cdot(\mathbf{p}+\mathbf{p}')(\mathbf{S}\times\mathbf{q}))-\frac{i}{2}(\mathbf{S}^2\mathbf{q}-(\mathbf{S}\cdot\mathbf{q})\mathbf{S})\right]d\mathbf{q} \tag{37}$$

and

$$\langle\mathbf{P}',\mathbf{p}'|\mathbf{J}(\mathbf{x},0)_{\mathbf{L}\cdot\mathbf{L}}|\mathbf{P},\mathbf{p}\rangle =$$

$$\int d\mathbf{q}v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\frac{1}{(2\pi)^6}e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\times$$

$$[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}-\mathbf{P}'}{2}\right)+i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}-\mathbf{P}'}{2}\right)]\times$$

$$\frac{1}{2}\mathbf{q}\times(\mathbf{q}\times(\mathbf{p}+\mathbf{p}')) \tag{38}$$

The derivations are given in the appendix.

VI. GAUGE COVARIANT NON-LOCAL OPERATORS

A general two-body Hamiltonian of the form (2) will not be locally gauge invariant in the sense (3-4). A gauge invariant extension of (2) can be constructed by replacing the momentum operators in the Weyl representation by gauge covariant derivatives:

$$\langle \mathbf{q}', t| e^{i\hat{\mathbf{p}}\cdot \mathbf{a}} |\mathbf{q}'', t\rangle \to \langle \mathbf{q}', t| e^{i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}},t))\cdot \mathbf{a}} |\mathbf{q}'', t\rangle. \tag{39}$$

To show this has the desired property use the Trotter product representation [31] of the exponent of the sum of non-commuting operators:

$$\lim_{N\to\infty} \langle \mathbf{q}'|[e^{i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}}))\cdot \frac{\mathbf{a}}{N}}]^N |\mathbf{q}''\rangle. \tag{40}$$

The important thing about this expression is that in the $N \to \infty$ limit only the first order terms in the expansion of the exponential contribute. It follows that (40) is equal to

$$= \lim_{N\to\infty} \langle \mathbf{q}', t|[1 + i(\hat{\mathbf{p}} - eA(\hat{\mathbf{q}}, t)) \cdot \frac{\mathbf{a}}{N}]^N |\mathbf{q}'', t\rangle =$$

$$\lim_{N\to\infty} \langle \mathbf{q}', t|[(1 + i\hat{\mathbf{p}}\frac{\mathbf{a}}{N})(1 - ieA(\hat{\mathbf{q}}, t)) \cdot \frac{\mathbf{a}}{N})]^N |\mathbf{q}''\rangle =$$

$$\lim_{N\to\infty} \langle \mathbf{q}', t|[e^{i\hat{\mathbf{p}}\frac{\mathbf{a}}{N}} e^{-ieA(\hat{\mathbf{q}},t)\cdot \frac{\mathbf{a}}{N}}]^N |\mathbf{q}'', t\rangle \tag{41}$$

where

$$e^{i\hat{\mathbf{p}}\frac{\mathbf{a}}{N}} = \hat{T}(\frac{\mathbf{a}}{N}) \tag{42}$$

is the operator that translates the coordinates by $\frac{\mathbf{a}}{N}$ and

$$e^{-ieA(\hat{\mathbf{q}},t)\cdot \frac{\mathbf{a}}{N}} = \hat{W}(\mathbf{q} + \frac{\mathbf{a}}{N}, \mathbf{q}) \tag{43}$$

is the operator that transforms the phase at $\mathbf{q}$ to the phase at $\mathbf{q} + \mathbf{a}/N$ for small $\mathbf{a}/N$ (see 12)).. With these identifications, inserting complete sets of intermediate states expression (41) becomes

$$\lim_{N\to\infty} \langle \mathbf{q}', t|[e^{i\hat{\mathbf{p}}\frac{\mathbf{a}}{N}} e^{-ieA(\hat{\mathbf{q}},t)\cdot \frac{\mathbf{a}}{N}}]^N |\mathbf{q}'', t\rangle =$$

$$\int \langle \mathbf{q} + \mathbf{a}, t|\hat{T}(\frac{\mathbf{a}}{N})\hat{W}(\mathbf{q} + \mathbf{a}, t, \mathbf{q} + \mathbf{a}(\frac{N-1}{N}), t)|\mathbf{q}_{N-1}\rangle d\mathbf{q}_{N-1}\times$$

$$\langle \mathbf{q}_{N-1}, t|\hat{T}(\frac{\mathbf{a}}{N})\hat{W}(\mathbf{q} + \mathbf{a}\frac{N-1}{N}, t; \mathbf{q} + \mathbf{a}\frac{N-2}{N})t|\mathbf{q}_{N-2}, t\rangle d\mathbf{q}_{N-2}\langle d\mathbf{q}_{N-2}$$

$$\cdots |\mathbf{q}_2\rangle d\mathbf{q}_2\langle \mathbf{q}_2|\hat{T}(\frac{\mathbf{a}}{N})\hat{W}(\mathbf{q} + 2\frac{\mathbf{a}}{N}, t; \mathbf{q} + \frac{\mathbf{a}}{N}, t)|\mathbf{q}_1\rangle\times$$

$$d\mathbf{q}_1\langle \mathbf{q}_1|\hat{T}(\frac{\mathbf{a}}{N})\hat{W}(\mathbf{q} + \frac{\mathbf{a}}{N}, t; \mathbf{q}, t)|\mathbf{q}'', t\rangle. \tag{44}$$

After a local gauge transform on the initial coordinate, each step translates the coordinate by $\mathbf{a}/N$ and transports the phase to the phase associated with the translated coordinate. This implies that in the limit $N \to \infty$ (44) has the property [40]

$$\langle \mathbf{q}' + \mathbf{a}, t| e^{i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}},t))\cdot \mathbf{a}} |\mathbf{q}'\rangle e^{i\chi(\mathbf{q}',t)} =$$

$$e^{i\chi(\mathbf{q}'+\mathbf{a},t)}\langle \mathbf{q}'+\mathbf{a}|e^{i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}}))\cdot\mathbf{a}}|\mathbf{q}'\rangle. \tag{45}$$

The operator $e^{i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}}))\cdot\mathbf{a}} = e^{i\mathbf{D}\cdot\mathbf{a}}$ in (45) is the exponential of the gauge covariant operator $\mathbf{D}\cdot\mathbf{a}$.

This shows that

$$\hat{O} = \int d\mathbf{a}d\mathbf{b}o(\mathbf{a},\mathbf{b})e^{i\mathbf{a}\cdot\hat{\mathbf{q}}}e^{i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}}))\cdot\mathbf{b}} \tag{46}$$

is a gauge covariant operator (in the relativistic case $\hat{\mathbf{q}}$ represents a single-particle Newton-Wigner position operator).

Note that if the original $\hat{O}$ is self-adjoint then so is the corresponding gauge covariant operator. This follows because

$$[e^{i\mathbf{a}\cdot\hat{\mathbf{q}}}e^{i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}}))\cdot\mathbf{b}}]^\dagger = e^{-i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}}))\cdot\mathbf{b}}e^{-i\mathbf{a}\cdot\hat{\mathbf{q}}} =$$

$$e^{-i\mathbf{a}\cdot\hat{\mathbf{q}}}e^{-i(\hat{\mathbf{p}}-\mathbf{a}-eA(\hat{\mathbf{q}}))\cdot\mathbf{b}} = e^{-i\mathbf{a}\cdot\hat{\mathbf{q}}}e^{-i(\hat{\mathbf{p}}-eA(\hat{\mathbf{q}}))\cdot\mathbf{b}}e^{i\mathbf{a}\cdot\mathbf{b}} \tag{47}$$

which shows that the gauge covariant operator is self adjoint if the coefficients $o(\mathbf{a},\mathbf{b})$ satisfy condition (29).

VII. TWO-BODY VECTOR CURRENTS

The interesting quantity in the one-photon exchange approximation is the hadronic current which is the coefficient of the part of this gauge-invariant Hamiltonian that is linear in the gauge field. The structure of the interaction-dependent two-body contribution to this current is discussed in this section.

A two-body interaction in the Weyl representation has the form

$$\hat{v} = \int d\mathbf{a}d\mathbf{b}v(\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2)e^{i(\mathbf{a}_1\cdot\hat{\mathbf{q}}_1+\mathbf{a}_2\cdot\hat{\mathbf{q}}_2)}e^{i(\mathbf{b}_1\cdot\hat{\mathbf{p}}_1+\mathbf{b}_2\cdot\hat{\mathbf{p}}_2)}. \tag{48}$$

Replacing the momentum operators by covariant derivatives in the Weyl representation results in the following gauge covariant kernel for the two-body interaction:

$$\hat{v}_g = \int d\mathbf{a}_1d\mathbf{a}_2d\mathbf{b}_1d\mathbf{b}_2v(\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2)e^{i(\mathbf{a}_1\cdot\hat{\mathbf{q}}_1+\mathbf{a}_2\cdot\hat{\mathbf{q}}_2)}\times$$

$$e^{i((\mathbf{b}_1\cdot(\hat{\mathbf{p}}_1-e_1\mathbf{A}(\hat{\mathbf{q}}_1,t))+(\mathbf{b}_2\cdot(\hat{\mathbf{p}}_2-e_2\mathbf{A}(\hat{\mathbf{q}}_2,t)))}. \tag{49}$$

The two-body part of the current operator is the coefficient of the part of $\hat{v}_g$ that is linear in the charges e_i. Since the momenta and vector potential do not commute, in order to find the term linear in the e_i use for B and A non-commuting operators

$$\frac{d}{dx}e^{\hat{B}+\hat{A}x}_{|x=0} = \int_0^1 d\lambda e^{\lambda\hat{B}}\hat{A}e^{(1-\lambda)\hat{B}}. \tag{50}$$

This formula follows using the Trotter product formula

$$e^{\hat{A}+\hat{B}}|\Psi\rangle = \lim_{N\to\infty}(e^{\hat{A}/N}e^{\hat{B}/N})^N|\Psi\rangle \tag{51}$$

and the chain rule [41]. The part of the interaction that is linear in the vector potential is

$$\langle\mathbf{q}_2,\mathbf{q}_1|\sum_i e_i\frac{\partial\hat{v}_g}{\partial e_i}|\mathbf{p}_1,\mathbf{p}_2\rangle_{e_i=0} =$$

$$\int d\mathbf{a}_1d\mathbf{a}_2d\mathbf{b}_1d\mathbf{b}_2\int_0^1 d\lambda v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)e^{i(\mathbf{a}_1\cdot\hat{\mathbf{q}}_1+\mathbf{a}_2\cdot\hat{\mathbf{q}}_2)}$$

$$e^{i\lambda(\mathbf{b}_1\cdot\hat{\mathbf{p}}_1+\mathbf{b}_2\cdot\hat{\mathbf{p}}_2)}(-ie_1\mathbf{A}(\hat{\mathbf{q}}_1,t)\cdot\mathbf{b}_1-ie_2\mathbf{A}(\hat{\mathbf{q}}_2,t)\cdot\mathbf{b}_2)e^{i(1-\lambda)(\mathbf{b}_1\cdot\hat{\mathbf{p}}_1+\mathbf{b}_2\cdot\hat{\mathbf{p}}_2)} = \tag{52}$$

$$\int d\mathbf{a}_1 d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2) e^{i(\mathbf{a}_1\cdot\hat{\mathbf{q}}_1+\mathbf{a}_2\cdot\hat{\mathbf{q}}_2)}$$

$$(-ie_1\mathbf{A}(\hat{\mathbf{q}}_1+\lambda\mathbf{b}_1,t)\cdot\mathbf{b}_1 - ie_2\mathbf{A}(\hat{\mathbf{q}}_2+\lambda\mathbf{b}_2,t)\cdot\mathbf{b}_2 e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)} + o(e^2). \tag{53}$$

In this expression the $\hat{\mathbf{q}}_i$ are operators and all of the $\hat{\mathbf{q}}_i$'s are to the left of the $\hat{\mathbf{p}}_i$'s. By taking mixed matrix elements the operators become numbers

$$\sum_i e_i \langle \mathbf{q}_2,\mathbf{q}_1| \frac{\partial \hat{v}_g}{\partial e_i} |\mathbf{p}_1,\mathbf{p}_2\rangle =$$

$$\int d\mathbf{a}_1 d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2) e^{i(\mathbf{a}_1\cdot\mathbf{q}_1+\mathbf{a}_2\cdot\mathbf{q}_2)}\times$$

$$(-ie_1\mathbf{A}(\mathbf{q}_1+\lambda\mathbf{b}_1,t)\cdot\mathbf{b}_1 - ie_2\mathbf{A}(\mathbf{q}_2+\lambda\mathbf{b}_2,t)\cdot\mathbf{b}_2)\times$$

$$\langle \mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2\rangle e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)} + o(e^2). \tag{54}$$

To factor out the vector potential insert delta functions

$$\int d\mathbf{x}(-i\mathbf{A}(\mathbf{x}))\cdot \int d\mathbf{a}_1 d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)\times$$

$$e^{i(\mathbf{a}_1\cdot\mathbf{q}_1+\mathbf{a}_2\cdot\mathbf{q}_2)}(e_1\delta(\mathbf{x}-\mathbf{q}_1-\lambda\mathbf{b}_1)\mathbf{b}_1 + e_2\delta(\mathbf{x}-\mathbf{q}_2-\lambda\mathbf{b}_2)\mathbf{b}_2)\times$$

$$\langle \mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2\rangle e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)} + o(e^2) = \tag{55}$$

$$\frac{1}{(2\pi)^3}\int d\mathbf{x}(-i\mathbf{A}(\mathbf{x}))\cdot \int d\mathbf{a}_1 d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)\times$$

$$e^{i(\mathbf{a}_1\cdot\mathbf{q}_1+\mathbf{a}_2\cdot\mathbf{q}_2)}\times(e_1\delta(\mathbf{x}-\mathbf{q}_1-\lambda\mathbf{b}_1)\mathbf{b}_1 + e_2\delta(\mathbf{x}-\mathbf{q}_2-\lambda\mathbf{b}_2)\mathbf{b}_2)\times$$

$$e^{i(\mathbf{q}_1\cdot\mathbf{p}_1+\mathbf{q}_2\cdot\mathbf{p}_2)} e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)} + o(e^2). \tag{56}$$

From this expression matrix elements of the current density in the mixed representation can be read off:

$$\langle \mathbf{q}_1,\mathbf{q}_2|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1,\mathbf{p}_2\rangle :=$$

$$-\frac{i}{(2\pi)^3}\int d\mathbf{a}_1 d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2) e^{i(\mathbf{a}_1\cdot\mathbf{q}_1+\mathbf{a}_2\cdot\mathbf{q}_2)}\times$$

$$(e_1\delta(\mathbf{x}-\mathbf{q}_1-\lambda\mathbf{b}_1)\mathbf{b}_1 + e_2\delta(\mathbf{x}-\mathbf{q}_2-\lambda\mathbf{b}_2)\mathbf{b}_2) e^{i(\mathbf{q}_1\cdot\mathbf{p}_1+\mathbf{q}_2\cdot\mathbf{p}_2)} e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)}. \tag{57}$$

This can be Fourier transformed to give a momentum space kernel

$$\langle \mathbf{p}_2',\mathbf{p}_1'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1,\mathbf{p}_2\rangle :=$$

$$-\frac{i}{(2\pi)^6}\int d\mathbf{a}_1 d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 d\mathbf{q}_1 d\mathbf{q}_2 \int_0^1 d\lambda v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2) e^{i(\mathbf{a}_1\cdot\mathbf{q}_1+\mathbf{a}_2\cdot\mathbf{q}_2)}\times$$

$$(e_1\delta(\mathbf{x}-\mathbf{q}_1-\lambda\mathbf{b}_1)\mathbf{b}_1+e_2\delta(\mathbf{x}-\mathbf{q}_2-\lambda\mathbf{b}_2)\mathbf{b}_2)\times$$

$$e^{i(\mathbf{q}_1\cdot\mathbf{p}_1+\mathbf{q}_2\cdot\mathbf{p}_2)}e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)}e^{-i(\mathbf{q}_1\cdot\mathbf{p}_1'+\mathbf{q}_2\cdot\mathbf{p}_2')}. \tag{58}$$

One of the two $\mathbf{q}$ integrals can be performed using the delta functions giving

$$\langle\mathbf{p}_2',\mathbf{p}_1'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1,\mathbf{p}_2\rangle=-\frac{ie_1}{(2\pi)^6}\int d\mathbf{a}_1d\mathbf{a}_2d\mathbf{b}_1d\mathbf{b}_2d\mathbf{q}_2\int_0^1 d\lambda\times$$

$$v(\mathbf{a}_2,\mathbf{a}_1;\mathbf{b}_1,\mathbf{b}_2)e^{i(\mathbf{a}_1\cdot(\mathbf{x}-\lambda\mathbf{b}_1)+\mathbf{a}_2\cdot\mathbf{q}_2)}\times$$

$$\mathbf{b}_1e^{i((\mathbf{x}-\lambda\mathbf{b}_1)\cdot\mathbf{p}_1+\mathbf{q}_2\cdot\mathbf{p}_2)}e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)}e^{-i((\mathbf{x}-\lambda\mathbf{b}_1)\cdot\mathbf{p}_1'+\mathbf{q}_2\cdot\mathbf{p}_2')}$$

$$-\frac{ie_2}{(2\pi)^6}\int d\mathbf{a}_1d\mathbf{a}_2d\mathbf{b}_1d\mathbf{b}_2d\mathbf{q}_1\int_0^1 d\lambda\times$$

$$v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)e^{i(\mathbf{a}_1\cdot\mathbf{q}_1+\mathbf{a}_2\cdot(\mathbf{x}-\lambda\mathbf{b}_2)}\times$$

$$\mathbf{b}_2e^{i(\mathbf{q}_1\cdot\mathbf{p}_1+(\mathbf{x}-\lambda\mathbf{b}_2)\cdot\mathbf{p}_2)}e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)}e^{-i(\mathbf{q}_1\cdot\mathbf{p}_1'+(\mathbf{x}-\lambda\mathbf{b}_2)\cdot\mathbf{p}_2')}. \tag{59}$$

Integrating over the remaining $\mathbf{q}_2$ (resp $\mathbf{q}_1$) gives $(2\pi)^3\times$ delta functions:

$$\langle\mathbf{p}_2',\mathbf{p}_1'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1,\mathbf{p}_2\rangle=-\frac{ie_1}{(2\pi)^3}\int d\mathbf{a}_1d\mathbf{a}_2d\mathbf{b}_1d\mathbf{b}_2\int_0^1 d\lambda\times$$

$$v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)\delta(\mathbf{a}_2+\mathbf{p}_2-\mathbf{p}_2')e^{i\mathbf{a}_1\cdot(\mathbf{x}-\lambda\mathbf{b}_1)}\mathbf{b}_1e^{i(\mathbf{x}-\lambda\mathbf{b}_1)\cdot\mathbf{p}_1}e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)}e^{-i(\mathbf{x}-\lambda\mathbf{b}_1)\cdot\mathbf{p}_1'}+$$

$$-\frac{ie_2}{(2\pi)^3}\int d\mathbf{a}_1d\mathbf{a}_2d\mathbf{b}_1d\mathbf{b}_2\int_0^1 d\lambda\times$$

$$v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)\delta(\mathbf{a}_1+\mathbf{p}_1-\mathbf{p}_1')e^{i\mathbf{a}_2\cdot(\mathbf{x}-\lambda\mathbf{b}_2)}\mathbf{b}_2e^{i(\mathbf{x}-\lambda\mathbf{b}_2)\cdot\mathbf{p}_2}e^{i(\mathbf{b}_1\cdot\mathbf{p}_1+\mathbf{b}_2\cdot\mathbf{p}_2)}e^{-i(\mathbf{x}-\lambda\mathbf{b}_2)\cdot\mathbf{p}_2'}. \tag{60}$$

Integrating $\mathbf{a}_2=\mathbf{p}_2'-\mathbf{p}_2$ resp $\mathbf{a}_1=\mathbf{p}_1'-\mathbf{p}_1$ using the delta functions gives:

$$\langle\mathbf{p}_2',\mathbf{p}_1'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1,\mathbf{p}_2\rangle=-\frac{ie_1}{(2\pi)^3}\int d\mathbf{a}_1d\mathbf{b}_1d\mathbf{b}_2\int_0^1 d\lambda\times$$

$$v(\mathbf{a}_1,(\mathbf{p}_2'-\mathbf{p}_2);\mathbf{b}_1,\mathbf{b}_2)\mathbf{b}_1e^{i\mathbf{x}\cdot(\mathbf{a}_1+\mathbf{p}_1-\mathbf{p}_1')}e^{i\mathbf{b}_2\cdot\mathbf{p}_2}e^{i\mathbf{b}_1\cdot(-\lambda\mathbf{a}_1-\lambda\mathbf{p}_1+\lambda\mathbf{p}_1'+\mathbf{p}_1)}$$

$$-\frac{ie_2}{(2\pi)^3}\int d\mathbf{a}_2d\mathbf{b}_1d\mathbf{b}_2\int_0^1 d\lambda\times$$

$$v((\mathbf{p}_1'-\mathbf{p}_1),\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)\mathbf{b}_2e^{i\mathbf{x}\cdot(\mathbf{a}_2+\mathbf{p}_2-\mathbf{p}_2')}e^{i\mathbf{b}_1\cdot\mathbf{p}_1}e^{i\mathbf{b}_2\cdot(-\lambda\mathbf{a}_2-\lambda\mathbf{p}_2+\lambda\mathbf{p}_2'+\mathbf{p}_2)}. \tag{61}$$

The next step is to replace the kernel of the interaction in the Weyl representation by its expression in terms of potential matrix elements using

$$v(\mathbf{a}_1,\mathbf{a}_2;\mathbf{b}_1,\mathbf{b}_2)=$$

$$\int (2\pi)^{-6} d\mathbf{k}_1 d\mathbf{k}_2 \langle \mathbf{k}_1 + \mathbf{a}_1/2, \mathbf{k}_2 + \mathbf{a}_2/2 | \hat{v} | \mathbf{k}_1 - \mathbf{a}_1/2, \mathbf{k}_2 - \mathbf{a}_2/2 \rangle \times$$

$$e^{-i(\mathbf{b}_1 \cdot (\mathbf{k}_1 - \mathbf{a}_1/2) + \mathbf{b}_2 \cdot (\mathbf{k}_2 - \mathbf{a}_2/2)}.$$

The Weyl kernels in (61) become

$$v(\mathbf{a}_1, (\mathbf{p}_2' - \mathbf{p}_2); \mathbf{b}_1, \mathbf{b}_2) =$$

$$\int (2\pi)^{-6} d\mathbf{k}_1 d\mathbf{k}_2 \langle \mathbf{k}_1 + \mathbf{a}_1/2, \mathbf{k}_2 + (\mathbf{p}_2' - \mathbf{p}_2)/2 | \hat{v} | \mathbf{k}_1 - \mathbf{a}_1/2, \mathbf{k}_2 - (\mathbf{p}_2' - \mathbf{p}_2)/2 \rangle \times$$

$$e^{-i(\mathbf{b}_1 \cdot (\mathbf{k}_1 - \mathbf{a}_1/2) + \mathbf{b}_2 \cdot (\mathbf{k}_2 - (\mathbf{p}_2' - \mathbf{p}_2)/2)}$$

$$v((\mathbf{p}_1' - \mathbf{p}_1), \mathbf{a}_2, ; \mathbf{b}_1, \mathbf{b}_2) =$$

$$\int (2\pi)^{-6} d\mathbf{k}_1 d\mathbf{k}_2 \langle \mathbf{k}_1 + \mathbf{a}_1/2, \mathbf{k}_2 + (\mathbf{p}_1' - \mathbf{p}_1)/2 | \hat{v} | \mathbf{k}_1 - (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 - \mathbf{a}_2/2 \rangle \times$$

$$e^{-i(\mathbf{b}_1 \cdot (\mathbf{k}_1 - (\mathbf{p}_1' - \mathbf{p}_1)/2) + \mathbf{b}_2 \cdot (\mathbf{k}_2 - \mathbf{a}_2/2)}. \tag{62}$$

Inserting these expressions in the expression for the current matrix elements gives

$$\langle \mathbf{p}_2', \mathbf{p}_1' | \mathbf{J}(\mathbf{x}, 0) | \mathbf{p}_1, \mathbf{p}_2 \rangle =$$

$$-\frac{ie_1}{(2\pi)^9} \int d\mathbf{a}_1 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda \times$$

$$\int d\mathbf{k}_1 d\mathbf{k}_2 \langle \mathbf{k}_1 + \mathbf{a}_1/2, \mathbf{k}_2 + (\mathbf{p}_2' - \mathbf{p}_2)/2 | \hat{v} | \mathbf{k}_1 - \mathbf{a}_1/2, \mathbf{k}_2 - (\mathbf{p}_2' - \mathbf{p}_2)/2 \rangle \times$$

$$e^{-i(\mathbf{b}_1 \cdot (\mathbf{k}_1 - \mathbf{a}_1/2) + \mathbf{b}_2 \cdot (\mathbf{k}_2 - (\mathbf{p}_2' - \mathbf{p}_2)/2)} \times$$

$$\mathbf{b}_1 e^{i\mathbf{x} \cdot (\mathbf{a}_1 + \mathbf{p}_1 - \mathbf{p}_1')} e^{i\mathbf{b}_2 \cdot \mathbf{p}_2} e^{i\mathbf{b}_1 \cdot (-\lambda \mathbf{a}_1 - \lambda \mathbf{p}_1 + \lambda \mathbf{p}_1' + \mathbf{p}_1)}$$

$$-\frac{ie_2}{(2\pi)^9} \int d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda \times$$

$$\int d\mathbf{k}_1 d\mathbf{k}_2 \langle \mathbf{k}_1 + (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 + \mathbf{a}_2/2 | \hat{v} | \mathbf{k}_1 - (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 - \mathbf{a}_2/2 \rangle \times$$

$$e^{-i(\mathbf{b}_1 \cdot (\mathbf{k}_1 - (\mathbf{p}_1' - \mathbf{p}_1)/2) + \mathbf{b}_2 \cdot (\mathbf{k}_2 - \mathbf{a}_2/2)} \times$$

$$\mathbf{b}_2 e^{i\mathbf{x} \cdot (\mathbf{a}_2 + \mathbf{p}_2 - \mathbf{p}_2')} e^{i\mathbf{b}_1 \cdot \mathbf{p}_1} e^{i\mathbf{b}_2 \cdot (-\lambda \mathbf{a}_2 - \lambda \mathbf{p}_2 + \lambda \mathbf{p}_2' + \mathbf{p}_2)}. \tag{63}$$

Collecting terms in the exponents gives

$$\langle \mathbf{p}_2', \mathbf{p}_1' | \mathbf{J}(\mathbf{x}, 0) | \mathbf{p}_1, \mathbf{p}_2 \rangle = -\frac{ie_1}{(2\pi)^9} \int d\mathbf{a}_1 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda \times$$

$$\int d\mathbf{k}_1 d\mathbf{k}_2 \langle \mathbf{k}_1 + \mathbf{a}_1/2, \mathbf{k}_2 + (\mathbf{p}_2' - \mathbf{p}_2)/2, |\hat{v}|\mathbf{k}_1 - \mathbf{a}_1/2, \mathbf{k}_2 - (\mathbf{p}_2' - \mathbf{p}_2)/2\rangle \times$$

$$\mathbf{b}_1 e^{i\mathbf{b}_1\cdot(-\mathbf{k}_1+\mathbf{a}_1/2-\lambda\mathbf{a}_1-\lambda\mathbf{p}_1+\lambda\mathbf{p}_1'+\mathbf{p}_1)} e^{i\mathbf{b}_2\cdot(-\mathbf{k}_2+(1/2)(\mathbf{p}_2'-\mathbf{p}_2)+\mathbf{p}_2)} e^{i\mathbf{x}\cdot(\mathbf{a}_1+\mathbf{p}_1-\mathbf{p}_1')}$$

$$-\frac{ie_2}{(2\pi)^9}\int d\mathbf{a}_2 d\mathbf{b}_1 d\mathbf{b}_2 \int_0^1 d\lambda \times$$

$$\int d\mathbf{k}_1 d\mathbf{k}_2 \langle \mathbf{k}_1 + (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 + \mathbf{a}_2/2|\hat{v}|\mathbf{k}_1 - (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 - \mathbf{a}_2/2\rangle \times$$

$$\mathbf{b}_2 e^{i\mathbf{b}_1\cdot(-\mathbf{k}_1+(1/2)(\mathbf{p}_1'-\mathbf{p}_1)+\mathbf{p}_1)} e^{i\mathbf{b}_2\cdot(-\mathbf{k}_2+(1/2)\mathbf{a}_2-\lambda\mathbf{a}_2-\lambda\mathbf{p}_2+\lambda\mathbf{p}_2'+\mathbf{p}_2)} e^{i\mathbf{x}\cdot(\mathbf{a}_2+\mathbf{p}_2-\mathbf{p}_2')}. \tag{64}$$

Replacing $\mathbf{b}_1 = i\boldsymbol{\nabla}_{k_1}$ and $\mathbf{b}_2 = i\boldsymbol{\nabla}_{k_2}$ acting on the exponent and integrating over $\mathbf{b}_1$ and $\mathbf{b}_2$ gives $(2\pi)^6$ times delta functions:

$$\langle \mathbf{p}_2', \mathbf{p}_1'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1, \mathbf{p}_2\rangle = -\frac{ie_1}{(2\pi)^3}\int d\mathbf{a}_1 d\mathbf{k}_1 d\mathbf{k}_2 \int_0^1 d\lambda \times$$

$$\langle \mathbf{k}_1 + \mathbf{a}_1/2, \mathbf{k}_2 + (\mathbf{p}_2' - \mathbf{p}_2)/2|\hat{v}|\mathbf{k}_1 - \mathbf{a}_1/2, \mathbf{k}_2 - (\mathbf{p}_2' - \mathbf{p}_2)/2\rangle \times$$

$$i\boldsymbol{\nabla}_{k_1}\delta(-\mathbf{k}_1 + \mathbf{a}_1/2 - \lambda\mathbf{a}_1 - \lambda\mathbf{p}_1 + \lambda\mathbf{p}_1' + \mathbf{p}_1)\delta(-\mathbf{k}_2 + (1/2)(\mathbf{p}_2' - \mathbf{p}_2) + \mathbf{p}_2)\times$$

$$e^{i\mathbf{x}\cdot(\mathbf{a}_1+\mathbf{p}_1-\mathbf{p}_1')}$$

$$-\frac{ie_2}{(2\pi)^3}\int d\mathbf{a}_2 d\mathbf{k}_1 d\mathbf{k}_2 \int_0^1 d\lambda \times$$

$$\langle \mathbf{k}_1 + (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 + \mathbf{a}_2/2|\hat{v}|\mathbf{k}_1 - (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 - \mathbf{a}_2/2\rangle \times$$

$$i\delta(-\mathbf{k}_1 + (1/2)(\mathbf{p}_1' - \mathbf{p}_1) + \mathbf{p}_1)\boldsymbol{\nabla}_{k_2}\delta(-\mathbf{k}_2 + (1/2)\mathbf{a}_2 - \lambda\mathbf{a}_2 - \lambda\mathbf{p}_2 + \lambda\mathbf{p}_2' + \mathbf{p}_2)\times$$

$$e^{i\mathbf{x}\cdot(\mathbf{a}_2+\mathbf{p}_2-\mathbf{p}_2')}. \tag{65}$$

Integrating the $\boldsymbol{\nabla}_{k_i}$'s by parts gives

$$\langle \mathbf{p}_2', \mathbf{p}_1'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1, \mathbf{p}_2\rangle = -\frac{e_1}{(2\pi)^3}\int d\mathbf{a}_1 d\mathbf{k}_1 d\mathbf{k}_2 \int_0^1 d\lambda \times$$

$$\boldsymbol{\nabla}_{k_1}\langle \mathbf{k}_1 + \mathbf{a}_1/2, \mathbf{k}_2 + (\mathbf{p}_2' - \mathbf{p}_2)/2|\hat{v}|\mathbf{k}_1 - \mathbf{a}_1/2, \mathbf{k}_2 - (\mathbf{p}_2' - \mathbf{p}_2)/2\rangle \times$$

$$\delta(-\mathbf{k}_1 + \mathbf{a}_1/2 - \lambda\mathbf{a}_1 - \lambda\mathbf{p}_1 + \lambda\mathbf{p}_1' + \mathbf{p}_1)\delta(-\mathbf{k}_2 + (1/2)(\mathbf{p}_2' - \mathbf{p}_2) + \mathbf{p}_2)\times$$

$$e^{i\mathbf{x}\cdot(\mathbf{a}_1+\mathbf{p}_1-\mathbf{p}_1')}$$

$$-\frac{e_2}{(2\pi)^3}\int d\mathbf{a}_2 d\mathbf{k}_1 d\mathbf{k}_2 \int_0^1 d\lambda \times$$

$$\boldsymbol{\nabla}_{k_2}\langle \mathbf{k}_1 + (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 + \mathbf{a}_2/2|V|\mathbf{k}_1 - (\mathbf{p}_1' - \mathbf{p}_1)/2, \mathbf{k}_2 - \mathbf{a}_2/2\rangle \times$$

$$\delta(-\mathbf{k}_1 + (1/2)(\mathbf{p}_1' - \mathbf{p}_1) + \mathbf{p}_1)\delta(-\mathbf{k}_2 + (1/2)\mathbf{a}_2 - \lambda\mathbf{a}_2 - \lambda\mathbf{p}_2 + \lambda\mathbf{p}_2' + \mathbf{p}_2)\times$$

$$e^{i\mathbf{x}\cdot(\mathbf{a}_2+\mathbf{p}_2-\mathbf{p}_2')}. \tag{66}$$

Now it is possible to perform both $\mathbf{k}$ integrals:

$$\langle \mathbf{p}_2', \mathbf{p}_1'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1, \mathbf{p}_2\rangle = -\frac{e_1}{(2\pi)^3}\int d\mathbf{a}_1 \int_0^1 d\lambda \times$$

$$\boldsymbol{\nabla}_{11'}\langle (1-\lambda)(\mathbf{p}_1 + \mathbf{a}_1) + \lambda\mathbf{p}_1', \mathbf{p}_2'|\hat{v}|\lambda(\mathbf{p}_1' - \mathbf{a}_1) + (1-\lambda)\mathbf{p}_1, \mathbf{p}_2\rangle e^{i\mathbf{x}\cdot(\mathbf{a}_1+\mathbf{p}_1-\mathbf{p}_1')}$$

$$-\frac{e_2}{(2\pi)^3}\int d\mathbf{a}_2 \int_0^1 d\lambda \times$$

$$\boldsymbol{\nabla}_{22'}\langle \mathbf{p}_1', (1-\lambda)(\mathbf{p}_2 + \mathbf{a}_2) + \lambda\mathbf{p}_2'|\hat{v}|\mathbf{p}_1, \lambda(\mathbf{p}_2' - \mathbf{a}_2) + (1-\lambda)\mathbf{p}_2\rangle e^{i\mathbf{x}\cdot(\mathbf{a}_2+\mathbf{p}_2-\mathbf{p}_2')} \tag{67}$$

where in these expressions

$$\nabla_{ii'}\langle \mathbf{p}_1, \mathbf{p}_2|\hat{v}|\mathbf{p}_1', \mathbf{p}_2'\rangle := \nabla_{p_i}\langle \mathbf{p}_1, \mathbf{p}_2|\hat{v}|\mathbf{p}_1', \mathbf{p}_2'\rangle + \nabla_{p_i'}\langle \mathbf{p}_1, \mathbf{p}_2|\hat{v}|\mathbf{p}_1', \mathbf{p}_2'\rangle.$$

For $\mathbf{x} = 0$ the current matrix elements become

$$\langle \mathbf{p}_2', \mathbf{p}_1'|\mathbf{J}(\mathbf{0},0)|\mathbf{p}_1, \mathbf{p}_2\rangle :=$$

$$= -\frac{e_1}{(2\pi)^3}\int d\mathbf{a}_1 \int_0^1 d\lambda \times$$

$$\boldsymbol{\nabla}_{11'}\langle (1-\lambda)(\mathbf{p}_1 + \mathbf{a}_1) + \lambda\mathbf{p}_1', \mathbf{p}_2'|\hat{v}|\lambda(\mathbf{p}_1' - \mathbf{a}_1) + (1-\lambda)\mathbf{p}_1, \mathbf{p}_2\rangle$$

$$-\frac{e_2}{(2\pi)^3}\int d\mathbf{a}_2 \int_0^1 d\lambda \times$$

$$\boldsymbol{\nabla}_{22'}\langle \mathbf{p}_1', (1-\lambda)(\mathbf{p}_2 + \mathbf{a}_2) + \lambda\mathbf{p}_2'|\hat{v}|\mathbf{p}_1, \lambda(\mathbf{p}_2' - \mathbf{a}_2 + (1-\lambda)\mathbf{p}_2\rangle. \tag{68}$$

For interactions that conserve momentum the $\boldsymbol{\nabla}_{ii'}$ it is useful to write the interaction in the form

$$\langle \mathbf{p}_1', \mathbf{p}_2'|\hat{v}|\mathbf{p}_1, \mathbf{p}_2\rangle = \delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)F_1(\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_2) + \delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)F_2(\mathbf{p}_2', \mathbf{p}_1', \mathbf{p}_1).$$

Then

$$\nabla_{ii}\langle \mathbf{p}_1, \mathbf{p}_2|\hat{v}|\mathbf{p}_1', \mathbf{p}_2'\rangle = \delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)\nabla_i' F_i(\mathbf{p}_i', \mathbf{p}_j', \mathbf{p}_j) \qquad j \neq i.$$

Define

$$\mathbf{F}_i(\mathbf{p}_i', \mathbf{p}_j', \mathbf{p}_j) := \nabla_i' F_i(\mathbf{p}_i', \mathbf{p}_j', \mathbf{p}_j) \qquad j \neq i$$

Using this in equation (68) note that in the first term the momentum conserving delta function gives $\mathbf{a}_1 = \mathbf{p}_2' - \mathbf{p}_2$ while in the second term it gives $\mathbf{a}_2 = \mathbf{p}_1' - \mathbf{p}_1$. Since $\boldsymbol{\nabla}_{ii'}$ gives 0 when acting on the momentum conserving delta functions, it is possible to use the delta functions to perform the $\mathbf{a}$ integrals.

$$\langle \mathbf{p}_2', \mathbf{p}_1'|\mathbf{J}(\mathbf{0},0)|\mathbf{p}_1, \mathbf{p}_2\rangle :=$$

$$= -\frac{1}{(2\pi)^3}\int_0^1 d\lambda\,(e_1\mathbf{F}_1((1-\lambda)(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')+\lambda\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_2)$$

$$+e_2\mathbf{F}_2((1-\lambda)(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')+\lambda\mathbf{p}_2',\mathbf{p}_1',\mathbf{p}_1))\,. \tag{69}$$

In the one-photon exchange approximation the two-body current appears in the Hamiltonian in the form

$$\int d\mathbf{x}\mathbf{A}(\mathbf{x},0)\langle\mathbf{p}_1',\mathbf{p}_2'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}_1,\mathbf{p}_2\rangle = (2\pi)^{3/2}\tilde{\mathbf{A}}(\mathbf{Q},0)\langle\mathbf{p}_2',\mathbf{p}_1'|\mathbf{J}(\mathbf{0},0)|\mathbf{p}_1,\mathbf{p}_2\rangle \tag{70}$$

where $\tilde{\mathbf{A}}(\mathbf{Q},0)$is the Fourier transform of the vector potential at $t=0$:

$$\tilde{\mathbf{A}}(\mathbf{Q},0) + \frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}\mathbf{A}(\mathbf{x},0)e^{i\mathbf{Q}\cdot\mathbf{x}}$$

VIII. ONE-BODY CURRENTS

A useful test is to apply the construction of the previous section to a relativistic kinetic energy operator. The relativistic kinetic energy for two relativistic particles has the form

$$\sqrt{\mathbf{p}_1^2+m_2^2}+\sqrt{\mathbf{p}_2^2+m_2^2}. \tag{71}$$

If the momentum operators are replaced by covariant derivatives, then the expression for the kinetic energy involves square roots of sums of non-commuting operators. One-body currents can be derived using minimal substitution in the Weyl representation following the construction used for the two-body interactions. Because the momentum operators appear in the square roots, the resulting current is no longer the familiar charge × velocity.

For each particle the method used to derive the two-body currents can be applied to each one-body kinetic energy operator:

$$e\frac{d}{de}\langle\mathbf{q}|e^{i\mathbf{b}\cdot(\mathbf{p}-e\mathbf{A}(\mathbf{q}))}|\mathbf{p}\rangle_{e=0} = \langle\mathbf{q}|\int_0^1 d\lambda e^{i\lambda\mathbf{b}\cdot\mathbf{p}}(-ie\mathbf{A}(\mathbf{q})\cdot\mathbf{b})e^{i(1-\lambda)\mathbf{b}\cdot\mathbf{p}}|\mathbf{p}\rangle =$$

$$\langle\mathbf{q}|\int_0^1 d\lambda(-ie\mathbf{A}(\mathbf{q}+\lambda\mathbf{b})\cdot\mathbf{b})e^{i\mathbf{b}\cdot\mathbf{p}}|\mathbf{p}\rangle =$$

$$\frac{1}{(2\pi)^{3/2}}\int_0^1 d\lambda(-ie\mathbf{A}(\mathbf{q}+\lambda\mathbf{b})\cdot\mathbf{b})e^{i(\mathbf{b}+\mathbf{q})\cdot\mathbf{p}} =$$

$$\frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}(-ie\mathbf{A}(\mathbf{x})\cdot\mathbf{b})\int_0^1 d\lambda\delta(\mathbf{x}-\mathbf{q}-\lambda\mathbf{b})e^{i(\mathbf{b}+\mathbf{q})\cdot\mathbf{p}} =$$

$$\frac{1}{(2\pi)^{3/2}}\int d\mathbf{x}(-ie\mathbf{A}(\mathbf{x})\cdot\mathbf{b})\int_0^1 d\lambda e^{i(\mathbf{x}+(1-\lambda)\mathbf{b})\cdot\mathbf{p}}. \tag{72}$$

Expression (72) can be used with an operator that is pure multiplication in $\mathbf{p}$:

$$o(\mathbf{a},\mathbf{b}) = \frac{1}{(2\pi)^3}\delta(\mathbf{a})\int d\mathbf{k}e^{-i\mathbf{b}\cdot\mathbf{k}}\sqrt{\mathbf{k}^2+m^2}. \tag{73}$$

Using this in the expression for the gauge covariant single particle energy gives

$$\langle\mathbf{q}|\sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2}|\mathbf{p}\rangle =$$

$$\frac{1}{(2\pi)^3}\int d\mathbf{k}d\mathbf{b}d\mathbf{x}e^{-i\mathbf{b}\cdot\mathbf{k}}\sqrt{\mathbf{k}^2+\mathbf{m}^2}\frac{1}{(2\pi)^{3/2}}(-ie\mathbf{A}(\mathbf{x})\cdot\mathbf{b})\times$$

$$\int_0^1 d\lambda\delta(\mathbf{x}-\mathbf{q}-\lambda\mathbf{b})e^{i(\mathbf{x}+(1-\lambda)\mathbf{b})\cdot\mathbf{p}} =$$

$$\int d\mathbf{k}d\mathbf{b}d\mathbf{x}\frac{1}{(2\pi)^{9/2}}(-ie\mathbf{A}(\mathbf{x})\cdot\mathbf{b})\int_0^1 d\lambda\delta(\mathbf{x}-\mathbf{q}-\lambda\mathbf{b})\sqrt{\mathbf{k}^2+\mathbf{m}^2}e^{i(\mathbf{x}+(1-\lambda)\mathbf{b})\cdot\mathbf{p}-\mathbf{b}\cdot\mathbf{k}}). \tag{74}$$

Next transform from a mixed representation to a momentum representation

$$e\frac{d}{de}\langle\mathbf{p}'|\sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2}|\mathbf{p}\rangle =$$

$$\int d\mathbf{x}d\mathbf{k}d\mathbf{q}d\mathbf{b}\frac{1}{(2\pi)^6}(-ie\mathbf{A}(\mathbf{x})\cdot\mathbf{b})\int_0^1 d\lambda\delta(\mathbf{x}-\mathbf{q}-\lambda\mathbf{b})\sqrt{\mathbf{k}^2+\mathbf{m}^2}e^{i(\mathbf{x}+(1-\lambda)\mathbf{b})\cdot\mathbf{p}-\mathbf{k}\cdot\mathbf{b}-\mathbf{p}'\cdot\mathbf{q})} =$$

$$\int d\mathbf{x}d\mathbf{k}d\mathbf{b}\frac{1}{(2\pi)^6}(-ie\mathbf{A}(\mathbf{x})\cdot\mathbf{b})\int_0^1 d\lambda\sqrt{\mathbf{k}^2+\mathbf{m}^2}e^{i(\mathbf{x}+(1-\lambda)\mathbf{b})\cdot\mathbf{p}-\mathbf{k}\cdot\mathbf{b}-\mathbf{p}'\cdot(\mathbf{x}-\lambda\mathbf{b}))} =$$

$$\int d\mathbf{x}d\mathbf{k}d\mathbf{b}\frac{1}{(2\pi)^6}(-ie\mathbf{A}(\mathbf{x})\cdot\mathbf{b})\int_0^1 d\lambda\sqrt{\mathbf{k}^2+\mathbf{m}^2}e^{i(\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')+\mathbf{b}\cdot((1-\lambda)\mathbf{p}-\mathbf{k}+\lambda\mathbf{p}'))}. \tag{75}$$

After replacing $\mathbf{b}$ by $i\boldsymbol{\nabla}_k$ acting on the exponent the $\mathbf{b}$ integral can be performed which gives

$$(i\boldsymbol{\nabla}_k)(2\pi)^3\delta((1-\lambda)\mathbf{p}-\mathbf{k}+\lambda\mathbf{p}'). \tag{76}$$

After this the $\mathbf{k}$ integral can be evaluated giving

$$\mathbf{k} = (1-\lambda)\mathbf{p}+\lambda\mathbf{p}' \tag{77}$$

and after integrating by parts the term linear in the vector potential becomes

$$-\int d\mathbf{x}e\mathbf{A}(\mathbf{x})\frac{1}{(2\pi)^3}\int_0^1 d\lambda\cdot\frac{(1-\lambda)\mathbf{p}+\lambda\mathbf{p}'}{\sqrt{((1-\lambda)\mathbf{p}+\lambda\mathbf{p}')^2+\mathbf{m}^2}}e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}. \tag{78}$$

This gives a one-body current matrix element of the form

$$\langle\mathbf{p}'|\mathbf{J}(\mathbf{x},0)|\mathbf{p}\rangle = -\frac{1}{(2\pi)^3}\int_0^1 d\lambda\frac{(1-\lambda)\mathbf{p}+\lambda\mathbf{p}')}{\sqrt{((1-\lambda)\mathbf{p}+\lambda\mathbf{p}')^2+\mathbf{m}^2}}e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}. \tag{79}$$

If the $\mathbf{x}$ integral in (78) is used to Fourier transform the vector potential the current for two non-interacting relativistic particles becomes

$$\langle\mathbf{p}_1',\mathbf{p}_2'|\mathbf{J}(\mathbf{0},0)|\mathbf{p}_1',\mathbf{p}_2'\rangle = -\frac{1}{(2\pi)^{3/2}}\int_0^1 d\lambda$$

$$\left[e_1\frac{(1-\lambda)\mathbf{p}_1+\lambda\mathbf{p}_1')}{\sqrt{((1-\lambda)\mathbf{p}_1+\lambda\mathbf{p}_1')^2+\mathbf{m}_1^2}}\delta(\mathbf{p}_2'-\mathbf{p}_2)+\right.$$

$$\left.e_2\frac{(1-\lambda)\mathbf{p}_2+\lambda\mathbf{p}_2')}{\sqrt{((1-\lambda)\mathbf{p}_2+\lambda\mathbf{p}_2')^2+\mathbf{m}_2^2}}\delta(\mathbf{p}_1'-\mathbf{p}_1)\right]. \tag{80}$$

The non-trivial convolution arises because of the square root factor.

It is instructive to compare this to the non-relativistic case. In the non-relativistic case $\sqrt{\mathbf{k}^2+m^2}$ is replaced by $\mathbf{k}^2/2m$. In that case equation (80) becomes

$$\langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{J}(\mathbf{0},0) | \mathbf{p}_1, \mathbf{p}_2 \rangle = -\frac{1}{(2\pi)^{3/2}} \int_0^1 d\lambda [e_1 \frac{(1-\lambda)\mathbf{p}_1 + \lambda \mathbf{p}_1')}{2m_1} \delta(\mathbf{p}_2' - \mathbf{p}_2)$$

$$+ \frac{e_2(1-\lambda)\mathbf{p}_2 + \lambda \mathbf{p}_2')}{2m_2} \delta(\mathbf{p}_1' - \mathbf{p}_1)] =$$

$$-\frac{1}{(2\pi)^{3/2}} [e_1 \frac{(\mathbf{p}_1 + \mathbf{p}_1')}{2m_1} \delta(\mathbf{p}_2' - \mathbf{p}_2)$$

$$+ e_2 \frac{(\mathbf{p}_2 + \mathbf{p}_2')}{2m_2} \delta(\mathbf{p}_1' - \mathbf{p}_1)]. \tag{81}$$

Both (81) and (80) have the form of charge $\times$ velocity where the velocities involve the average of the initial and final relativistic respectively non-relativistic velocities.

As mentioned earlier, this results in a convection current for a point charge. If the relativistic kinetic energy (71) is replaced by the sum of one-body Dirac Hamiltonians, the resulting one body-currents will have both a convection and magnetic component, however they will not have nucleon form factors. This is because nucleons are composite systems with non-trivial internal charge and current distributions.

IX. CHARGE DENSITY

Replacing the momentum operators in the Weyl representation of the interaction by covariant derivatives results in a vector current, but it does not result in the full four current.

Since the 4-current transforms the same way as the four momentum under Lorentz transformations, current covariance can be used to compute the charge density operator in terms of the vector part of the current. The most straightforward way to determine the charge density from the vector part or the current is use the commutator with the dynamical rotationless boost generators

$$J^0(\mathbf{q},0) = i[K^i, J^i(\mathbf{q},0)] \qquad \text{no sum, any } i, \tag{82}$$

where $\mathbf{K}$ is the generator of rotationless boosts. This has the advantage that the result is an operator (not a matrix element) that transforms as a 4 vector under the original representation of the Poincaré group. This is consistent with the one-photon exchange approximation.

Both the boost generators and the Hamiltonian depend on the interactions. If the momenta in the expression for all of the generators were replaced by covariant derivatives, the resulting operators would no longer satisfy the Poincaré commutation relations. This is because, unlike ordinary partial derivatives, different components of the covariant derivatives do not commute. However any locally gauge invariant extension of the full Poincare Lie algebra should reduce to the original algebra in the limit of zero charge. The current is the coefficient of the term in the gauge invariant Hamiltonian that is linear in the vector potential so the transformation properties of the current in the one photon exchange approximation is determined by the original representation of the Poincaré Lie algebra. This assumes that it is possible to construct locally gauge invariant boost that is consistent with the gauge invariant Hamiltonian.

To compute $J^0(\mathbf{q},0)$ it is necessary to have explicit dynamical boost generators. The two-body Bakamjian-Thomas construction [42] leads to an explicit expression for the dynamical boost generators as a function of the dynamical Hamiltonian and non-interacting one-body generators. Formally the Bakamjian-Thomas dynamical boost generator is

$$\mathbf{K} = \frac{1}{2}(H\mathbf{X}_0 + \mathbf{X}_0 H) - \frac{\mathbf{s}_0 \times \mathbf{P}_0}{M+H} \tag{83}$$

where $\mathbf{s}_0$ is the non-interacting two-body canonical spin and $\mathbf{X}_0$ is the Newton-Wigner position operator for the non-interacting two-body system. In this case all of the interaction dependence appears in M and H. Expressions for boost generators in representations that satisfy many-body cluster separability can be constructed following [25][26].

The non-interacting two-body $\mathbf{X}_0$ and $\mathbf{s}_0$ operators are the following functions of the non-interacting one-body Poincaré generators

$$\mathbf{X}_0 = -\frac{1}{2}\{\frac{1}{H_1+H_2}, \mathbf{K}_1 + \mathbf{K}_2\}$$

$$-\frac{(\mathbf{P}_1+\mathbf{P}_2)\times((H_1+H_2)(\mathbf{J}_1+\mathbf{J}_2)-(\mathbf{P}_1+\mathbf{P}_2)\times(\mathbf{K}_1+\mathbf{K}_2))}{((H_1+H_2)M_0(H_1+H_2+M_0)} \tag{84}$$

where

$$\mathbf{s}_0 = (\mathbf{J}_1+\mathbf{J}_2) - \mathbf{X}_0 \times (\mathbf{P}_1+\mathbf{P}_2) \tag{85}$$

$$M_0 = \sqrt{(H_1+H_2)^2 - (\mathbf{P}_1+\mathbf{P}_2)^2}. \tag{86}$$

$$\mathbf{K}_i = \frac{1}{2}\{\sqrt{\mathbf{p}_i^2+m_i^2}, (i\boldsymbol{\nabla}_{p_1}\} - \frac{\mathbf{s}_i \times \mathbf{p}_i}{m_i + \sqrt{\mathbf{p}_i^2+m_i^2}}. \tag{87}$$

In these expressions $\mathbf{J}$ is an angular momentum generator rather than a current operator. The advantage is that these are operator definitions that can be consistently applied to different kinds of reactions. While these are complex expressions at the operator level, they are easy to compute in matrix elements between irreducible eigenstates, where $\mathbf{s}_0$ is the spin of the initial or final two-body state, $\mathbf{X}_0$ is (i $\times$) the partial derivative with respect to the total linear momentum holding the spin constant, and H and M are the energy and mass eigenvalues of the initial or final state. Since

$$e^{iK_x\rho}|(m,j)\mathbf{P},\mu\rangle = |(m,j)\boldsymbol{\Lambda}_x(\rho)P,\nu\rangle D^j_{\mu\nu}[B^{-1}(\Lambda_x^{-1}(\rho)p)\Lambda_x(\rho)B(p)]$$

where $\Lambda_x(\rho)$ is a rotationless boost in the x direction with rapidity ρ and $B(p)$ is a rotationless boost that transforms a particle at rest to one moving with momentum $\mathbf{p}$. Tt follows that

$$K_x|(m,j)\mathbf{P},\mu\rangle = -i\frac{\partial}{\partial\rho}|(m,j)\boldsymbol{\Lambda}_x(\rho)P,\nu\rangle D^j_{\mu\nu}[B^{-1}(\Lambda_x^{-1}(\rho)p)\Lambda_x(\rho)B(p)]_{\rho=0}$$

can be obtained by differentiating the Lorentz boosted state by the rapidity at zero rapidity.

For systems of more than two particles the Bakamjian-Thomas construction does not lead to Poincaré generators that satisfy cluster properties. This can be repaired [25][26] at the expense of introducing many-body interactions in the Hamiltonian, boost generators and spin operators.

X. TIME DERIVATIVES

The fields in the Hamiltonian are at a fixed common time, t. The gauge transformations discussed in the previous section were spatial gauge transformations at a single fixed time. The Schrödinger equation involves time derivatives. To make the time-dependent Schrödinger equation invariant under gauge transformations the time derivative also has to be replaced by a covariant derivative.

Gauge invariance can be achieved by replacing the time derivative by

$$(\partial_t - i\sum_i e_i A_0(\mathbf{q}_i,t)). \tag{88}$$

To see this note that under a local gauge transformation

$$\psi(\mathbf{q}_1,\cdots,\mathbf{q}_N,t) \to \tag{89}$$

$$e^{i\sum\chi(\mathbf{q}_i,t)}\psi(\mathbf{q}_1,\cdots,\mathbf{q}_N,t). \tag{90}$$

The time derivative of the transformed wave function is

$$e^{i\sum\phi(\mathbf{q}_i,t)}(i\partial_t - \sum_i \partial_t\chi(\mathbf{q}_i,t))\psi(\mathbf{q}_1,\cdots,\mathbf{q}_N,t). \tag{91}$$

The derivatives of the phases can be eliminated by replacing the time derivative by

$$(i\partial_t + \sum_i e_i A_0(\mathbf{q}_i,t))\Psi(\mathbf{q}_1,\cdots,\mathbf{q}_N,t). \tag{92}$$

In this way the gauge invariant Schrödinger equation becomes

$$i\partial_t\Psi(\mathbf{q}_1,\cdots,\mathbf{q}_N,t) =$$

$$-\sum_i eA_0(\mathbf{q}_i,t)\Psi(\mathbf{q}_1,\cdots,\mathbf{q}_N,t) + H(\mathbf{p}-e\mathbf{A},\mathbf{q})\Psi(\mathbf{q}_1,\cdots,\mathbf{q}_N,t) \tag{93}$$

where the gauge invariant Hamiltonian is constructed by replacing $\mathbf{p}_i$ by $\mathbf{p}_i - e_i\mathbf{A}(\mathbf{q}_i)$ in the Weyl representation of the Hamiltonian.

This prescription results in a locally gauge covariant equation that reduces to the ordinary relativistic Schrödinger equation in the limit that the charge vanishes.

XI. SPIN-DEPENDENT INTERACTIONS

The canonical pairs of operators used in Weyl representation were the single-particle momentum and the single-particle Newton-Wigner position operators, which in the momentum-canonical spin basis are $i\boldsymbol{\nabla}_{\mathbf{p}_i}$ where the partial derivatives are computed holding the single particle canonical spins constant. In this representation, since the derivatives commute with the spin, the gauge transformations are independent of the canonical spin. This is not entirely trivial since the spins undergo momentum dependent Wigner rotations under boosts. The procedure is still the same. The first step is to express the reduced interaction in terms of single-nucleon momenta and spins. Since the single-particle canonical spins are gauge invariant in this representation, the current has a structure similar to (67) where the potential kernel has both single-particle momenta and spins:

For interactions that conserve momentum the interactions can be expressed in the form:

$$\langle \mathbf{p}_1',\mu_1',\mathbf{p}_2',\mu_2',|\hat{v}|\mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2\rangle =$$

$$\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)(F_1(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_2,\mu_1',\mu_2',\mu_1,\mu_2)+F_2(\mathbf{p}_2',\mathbf{p}_1',\mathbf{p}_1,\mu_1',\mu_2',\mu_1,\mu_2).$$

The current is expressed in terms of the derivatives

$$\nabla_{ii}\langle \mathbf{p}_1',\mu_1',\mathbf{p}_2',\mu_2',|\hat{v}|\mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2\rangle =$$

$$\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)\nabla_i' F_i(\mathbf{p}_i',\mathbf{p}_j',\mathbf{p}_j,\mu_1',\mu_2',\mu_1,\mu_2) \qquad j\neq i.$$

which are used to define

$$\mathbf{F}_i(\mathbf{p}_i',\mathbf{p}_j',\mathbf{p}_j,\mu_1',\mu_2',\mu_1,\mu_2) := \nabla_i' F_i(\mathbf{p}_i',\mathbf{p}_j',\mathbf{p}_j,\mu_1',\mu_2',\mu_1,\mu_2) \qquad j\neq i.$$

Using this in equation (68) note that in the first term the momentum conserving delta function gives $\mathbf{a}_1 = \mathbf{p}_2' - \mathbf{p}_2$ while in the second term it gives $\mathbf{a}_2 = \mathbf{p}_1' - \mathbf{p}_1$. Since $\boldsymbol{\nabla}_{ii'}$ gives 0 when acting on the momentum conserving delta functions, it is possible to use the delta functions to perform the $\mathbf{a}$ integrals.

$$\langle \mathbf{p}_2',\mathbf{p}_1'|\mathbf{J}(\mathbf{0},0)|\mathbf{p}_1,\mathbf{p}_2\rangle :=$$

$$= -\frac{1}{(2\pi)^3}\int_0^1 d\lambda\times$$

$$(\mathbf{F}_1((1-\lambda)(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')+\lambda\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_2)+\mathbf{F}_2((1-\lambda)(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')+\lambda\mathbf{p}_2',\mathbf{p}_1',\mathbf{p}_1)). \tag{94}$$

If the interaction is expressed in terms of kinematic masses and kinematically invariant degeneracy parameters,

$$\delta(\mathbf{P}'-\mathbf{P})\delta_{j'j}\delta_{\mu'\mu}\langle k',l',s'\|\hat{v}^j\|k,l,s\rangle \tag{95}$$

then the Clebsch-Gordon coefficients for the Poincaré group [38][43] need to be used to get the equivalent expression in terms of single-particle spins and momenta that can be differentiated

The Clebsch-Gordon coefficients for the Poincaré group that relate the non-interacting irreducible representation of the Poincaré group in the momentum-canonical spin basis to the tensor product of two single-particle irreducible representations are

$$|\mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2,\rangle = |\mathbf{P},\mu,l,s,k(p_1,p_2)\rangle\langle s,\mu_s,l,m|j,\mu\rangle\times$$

$$\sqrt{\frac{(\omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_1))\omega_1(\mathbf{k})\omega_2(\mathbf{k})}{(\omega_1(\mathbf{k})+\omega_2(\mathbf{k}))\omega_1(\mathbf{p}_1)\omega_2(\mathbf{p}_2)}}\times$$

$$\langle\frac{1}{2},\mu_1'',\frac{1}{2},\mu_2''|S,\mu_s\rangle Y^{l*}_m(\hat{\mathbf{k}}(p_1,p_2))D^{1/2}_{\mu_1''\mu_1}(R_w(P,p_1))D^{1/2}_{\mu_2''\mu_2}(R_w(P,p_2)), \tag{96}$$

where

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2 \qquad \omega_i(\mathbf{p}_i)=\sqrt{m_i^2+\mathbf{p}_i^2}, \tag{97}$$

$$\mathbf{k}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$$

$$\mathbf{k}_i=\mathbf{p}_i\frac{\mathbf{P}}{\sqrt{(\omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_2))^2-\mathbf{P}^2}}\times$$

$$\left(\frac{(\mathbf{p}_1^2-\mathbf{p}_2^2)}{\sqrt{(\omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_2))^2-\mathbf{P}^2}+\omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_2)}-\omega_i(\mathbf{p}_i)\right) \tag{98}$$

$$H_0=\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2), \qquad M_0=\sqrt{H_0^2-\mathbf{P}^2}=\omega_{m_1}(\mathbf{k}_1)+\omega_{m_2}(\mathbf{k}_2), \tag{99}$$

$R_w(P,p_2)$ is the Wigner rotation

$$R_w(P,p_i)=B^{-1}(\mathbf{k}_i/m_i)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_i/m_i), \tag{100}$$

$$k_i=B^{-1}(\mathbf{P}/M_0)p_i=(\sqrt{\mathbf{k}_i^2+m^2},\mathbf{k}_i) \tag{101}$$

$$\mathbf{k}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2) \tag{102}$$

and $B(\mathbf{p}/m)$ is a rotationless Lorentz boost:

$$B^{-1}(\mathbf{p}/m)=B^{-1}(-\mathbf{p}/m) \tag{103}$$

$$B(\mathbf{p}/m)^0{}_0=\frac{\omega_m(\mathbf{p})}{m} \qquad B(\mathbf{p}/m)^0{}_i=B(\mathbf{p}/m)^i{}_0=\frac{p^i}{m} \tag{104}$$

$$B(\mathbf{p}/m)^i{}_j=\delta^{ij}+\frac{p^ip^j}{m(\omega_m(\mathbf{p}))}. \tag{105}$$

Light-front formulations of relativistic quantum mechanics are useful for studying hadronic structure because the boosts are kinematic. The general method discussed in this paper can also be extended to the light-front generators.

The light-front Poincare generators are different linear combinations of the Poincaré infinitesimal generators where seven of the new generators can be chosen to have no interactions.

The light front Hamiltonian is

$$P^- = H - P^3.$$

An irreducible set of canonical operators can be expressed in terms of kinematic generators

$$P^+ = H + P^3, \qquad \text{and} \qquad \mathbf{E}_\perp = \mathbf{K}_\perp - \hat{\mathbf{z}} \times \mathbf{J}.$$

The canonical pairs are

$$p_i^+, q_i^- = \frac{1}{2}\{\frac{1}{p_i^+}, K_i^3\} = i\frac{\partial}{\partial p_i^+}$$

$$\mathbf{p_{i\perp}}, \mathbf{q_{i\perp}} = i\frac{1}{p^+}\mathbf{E}_\perp = i\frac{\partial}{\partial p_{\perp i}}$$

and the Weyl representation for the interacting P^- operator is

$$\hat{P}^- = \int d^{3N}\mathbf{a} d^{3N}\mathbf{b} p^-(\mathbf{a},\mathbf{b}) e^{i\mathbf{a}\cdot\hat{\mathbf{q}}} e^{i\mathbf{b}\cdot\hat{\mathbf{p}}}$$

The current can be extracted using same methods used in sections seven and eight. In this case the current will be different because the partial derivatives are computed holding the light-front spin constant. The light front and canonical spins are related by momentum dependent Melosh [44] rotations.

XII. SUMMARY - CONCLUSION

In this work current operators in phenomenological relativistic models of strongly interacting systems were constructed by requiring local gauge invariance of the Hamiltonian. In these models impulse (one-body currents) are not compatible with the dynamical constraints of current covariance and current conservation. These constraints alone are insufficient to uniquely fix a current operator. Local gauge invariance was implemented by expressing the Hamiltonian in the Weyl representation, replacing the single-particle momentum operators by gauge covariant derivatives. The dynamical current was identified with the coefficient of the part of gauge invariant Hamiltonian that is linear in the vector potential. The charge density was constructed using the commutation relations with the dynamical Lorentz boost generators, which ensures that the 4-current transforms as a 4-vector density.

The advantage of this construction is that the result is a current operator that can be used in reactions with different initial and final states that is consistent with the dynamics. The operators in (36), (67) and (73) have an explicit dependence on the representation of the Hamiltonian. The construction addresses one of the primary challenges in constructing relativistic models, which is the absence of a systematic method for constructing candidates for dynamical current operators that are consistent with the interaction. While the result is not unique, it is a 4-vector current operator that in minimally consistent with the interaction.

A relativistic Hamiltonian theory with a finite number of degrees of freedom can be formulated on a many-particle Hilbert space. In the absence of interactions particles transform under irreducible representations of the Poincaré group. The momentum and coordinates in the Weyl representation are the single-particle momentum and Newton-Wigner position operators. The Newton-Wigner position operator, which in the momentum-canonical spin basis, is $i \times$ the partial derivative with respect to the momentum holding the single particle canonical spin constant. Since the dynamical relativistic model is formulated on the same many-body Hilbert space and the Weyl algebra is irreducible, the Hamiltonian and all of the Poincaré generators can be represented in the free-particle Weyl representation.

The dynamical models under consideration are defined by a dynamical unitary representation of the Poincaré group acting on the many particle Hilbert space [45][26][46]. In this application the dynamical representation is chosen so translations and rotations are independent of interactions (Dirac's instant form of dynamics [47]). Solving the dynamics is equivalent to decomposing this unitary representation into a direct integral of irreducible representations.

The Hamiltonians in these models are typically non-local. Gauge invariance requires that the kernel of the Hamiltonian in the "position representation" satisfies the intertwining properties (3-4), which follow as a consequence of replacing the momentum operator in the Weyl representation by covariant derivatives. The analysis involves evaluating functions of non-commuting variables, but the final results are explicit expressions for the kernels.

Examples of the currents that come from non-relativistic $\mathbf{L}\cdot\mathbf{S}$, $(\mathbf{L}\cdot\mathbf{S})$ and $\mathbf{L}\cdot\mathbf{L}$ interactions were given. The resulting currents are explicit function of the interactions.

While this construction is systematic and yields a covariant current operator that is consistent with the dynamics, whether the dynamical mechanism generates currents consistent with experiment needs further investigation. One problem that was not addressed is by replacing the momentum operators by covariant derivatives, the Poincaré commutation relation are no longer preserved, since the commutator of two covariant derivatives is the field strength tensor rather than 0. Fortunately in the one-photon exchange approximation this problem goes away when the higher powers of the vector potential are set to 0. Going beyond the one-photon exchange approximation requires further investigation.

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XIII. APPENDIX

Expressions (37) and (38) for the two-body currents associated with the operators $(\mathbf{L}\cdot\mathbf{S})^2$ and $(\mathbf{L}\cdot\mathbf{L})$ in non-relativistic interactions are derived in this appendix. The construction of these current operators is similar to the construction used in the $\mathbf{L}\cdot\mathbf{S}$ case. The main difference is that these contributions involve products of covariant derivatives. The current is the coefficient of the vector potential.

The interactions before replacing the derivatives by covariant derivatives are

$$v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)(\mathbf{L}\cdot\mathbf{S})^2 = v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)(\mathbf{S}\times\mathbf{q})\cdot\mathbf{p}(\mathbf{S}\times\mathbf{q})\cdot\mathbf{p} =$$

$$v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)[\sum_{ij}(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})_j p_i p_j + i(\mathbf{S}\times\mathbf{q})\cdot(\mathbf{S}\times\mathbf{p})] \tag{106}$$

and

$$v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\mathbf{L}\cdot\mathbf{L} = v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)[\mathbf{q}^2\mathbf{p}^2 - \mathbf{q}\cdot(\mathbf{q}\cdot\mathbf{p})\mathbf{p} + 2i\mathbf{q}\cdot\mathbf{p}]. \tag{107}$$

where in these expressions the canonical commutation relations are used to move the momentum operators to the right of the coordinate operators.

In order to extract the current it is useful to express (106) and (107) in terms of single-particle variables, where minimal substitution is straightforward:

$$v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)(\mathbf{L}\cdot\mathbf{S})^2 =$$

$$\sum_{ij} v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}_1-\mathbf{q}_2|)[(\mathbf{S}\times(\mathbf{q}_1-\mathbf{q}_2))_i(\mathbf{S}\times(\mathbf{q}_1-\mathbf{q}_2))_j\frac{(\mathbf{p}_1-\mathbf{p}_2)_i}{2}\frac{(\mathbf{p}_1-\mathbf{p}_2)_j}{2}$$

$$+\frac{i}{2}(\mathbf{S}\times(\mathbf{q}_1-\mathbf{q}_2))\cdot(\mathbf{S}\times(\mathbf{p}_1-\mathbf{p}_2))] \tag{108}$$

and

$$v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}_1-\mathbf{q}_2|)\mathbf{L}\cdot\mathbf{L} =$$

$$v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}_1-\mathbf{q}_2|)[\frac{1}{4}(\mathbf{q}_1-\mathbf{q}_2)^2(\mathbf{p}_1-\mathbf{p}_2)^2$$

$$-\frac{1}{4}(\mathbf{q}_1-\mathbf{q}_2)\cdot((\mathbf{q}_1-\mathbf{q}_2)\cdot(\mathbf{p}_1-\mathbf{p}_2))(\mathbf{p}_1-\mathbf{p}_2) + i(\mathbf{q}_1-\mathbf{q}_2)\cdot(\mathbf{p}_1-\mathbf{p}_2)]. \tag{109}$$

These are operator expressions. Replacing the space derivatives by covariant derivatives is equivalent to replacing $\mathbf{p}_i$ by $\mathbf{p}_i - e_i\mathbf{A}(\mathbf{q}_i)$, maintaining the correct operator ordering. This substitution results in the gauge invariant operators

$$v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)(\mathbf{L}\cdot\mathbf{S})^2 \rightarrow$$

$$v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}_1-\mathbf{q}_2|)\left[\frac{1}{4}\sum_{ij}(\mathbf{S}\times(\mathbf{q}_1-\mathbf{q}_2))_i(\mathbf{S}\times(\mathbf{q}_1-\mathbf{q}_2))_j\times\right.$$

$$(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))_i(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))_j+$$

$$\left.\frac{i}{2}(\mathbf{S}\times(\mathbf{q}_1-\mathbf{q}_2))\cdot(\mathbf{S}\times(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))\right] \tag{110}$$

and

$$v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}_1-\mathbf{q}_2|)\mathbf{L}\cdot\mathbf{L}\to v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}_1-\mathbf{q}_2|)\times$$

$$\left[\frac{1}{4}(\mathbf{q}_1-\mathbf{q}_2)^2(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))\cdot(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))-\right.$$

$$-\frac{1}{4}(\mathbf{q}_1-\mathbf{q}_2)\cdot[(\mathbf{q}_1-\mathbf{q}_2)\cdot(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))\cdot(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))$$

$$+i(\mathbf{q}_1-\mathbf{q}_2)\cdot(\mathbf{p}_1-e_1\mathbf{A}(\mathbf{q}_1)-\mathbf{p}_2+e_2\mathbf{A}(\mathbf{q}_2))]\,. \tag{111}$$

Keeping only the terms that are linear in $\mathbf{A}$ in (110) and (111) and expressing the result in terms of the total and relative momenta and their conjugate coordinates gives:

$$(110)\to v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}_1-\mathbf{q}_2|)\times$$

$$\left[-\frac{1}{2}(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})_j\{\mathbf{p}_i(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}((\mathbf{Q}-\frac{\mathbf{q}}{2}))_j+\right.$$

$$(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_i\mathbf{p}_j\}+$$

$$\left.-\frac{i}{2}(\mathbf{S}\times\mathbf{q})\cdot(\mathbf{S}\times(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\right] \tag{112}$$

and

$$(111)\to v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\left\{-\frac{1}{2}\mathbf{q}^2\left[(\mathbf{p}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))+\right.\right.$$

$$\left.(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\cdot\mathbf{p})\right]+$$

$$+\frac{1}{2}\{\mathbf{q}\cdot(\mathbf{q}\cdot\mathbf{p})\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))+\frac{1}{2}\mathbf{q}\cdot(\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})\mathbf{p}\}$$

$$\left.\left.-i\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})\right]\right\} \tag{113}$$

where in (112) $\mathbf{p}_i$ represents the i-th component of the relative momentum rather than the momentum of particle i The next step is to use the commutation relations to move the $\mathbf{p}$ factors to the right of all of the coordinate factors. There are three terms in (112) and (113) where the momenta are on the left of some of the coordinate operators. These terms are:

$$\mathbf{p}_i(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}((\mathbf{Q}-\frac{\mathbf{q}}{2}))_j=$$

$$(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}((\mathbf{Q}-\frac{\mathbf{q}}{2}))_j\mathbf{p}_i-\frac{i}{2}\partial_{Qi}(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_j \tag{114}$$

$$-\frac{1}{2}\mathbf{q}^2\mathbf{p}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))=$$

$$-\frac{1}{2}\mathbf{q}^2(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\cdot\mathbf{p}+i\frac{1}{4}\mathbf{q}^2\boldsymbol{\nabla}_Q\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})), \tag{115}$$

$$\frac{1}{2}\{\mathbf{q}\cdot(\mathbf{q}\cdot\mathbf{p})\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))=$$

$$\frac{1}{2}\{\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))(\mathbf{q}\cdot\mathbf{p})-i14\{\mathbf{q}\cdot(\mathbf{q}\cdot\boldsymbol{\nabla}_Q)(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})). \tag{116}$$

Using these identities in the operator expressions (112) and (113) above gives

$$(112)=$$

$$v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)\Big[-\frac{1}{2}(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})_j\{(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_j\mathbf{p}_i+$$

$$(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_i\mathbf{p}_j$$

$$-\frac{i}{2}((e_1\partial_{iQ}\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\partial_{iQ}\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_j\}$$

$$-\frac{i}{2}(\mathbf{S}\times\mathbf{q})\cdot(\mathbf{S}\times(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\Big] \tag{117}$$

and

$$(113)=$$

$$v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\Big[-\frac{1}{2}\mathbf{q}^2(2(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\cdot\mathbf{p})$$

$$+i\frac{1}{4}\mathbf{q}^2(e_1\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))+$$

$$+\{\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))(\mathbf{q}\cdot\mathbf{p})-i\frac{1}{4}\{\mathbf{q}\cdot(\mathbf{q}\cdot\boldsymbol{\nabla}_Q)(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))$$

$$-i\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})\Big]. \tag{118}$$

Since the momentum operators are to the right of the coordinate operators, the operators become numbers in a mixed coordinate-momentum basis. The mixed matrix elements of (117) and (118) are

$$\langle\mathbf{Q},\mathbf{q}|(117)|\mathbf{P},\mathbf{p}\rangle=$$

$$v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)\frac{1}{(2\pi)^3}e^{i\mathbf{Q}\cdot\mathbf{P}+i\mathbf{q}\cdot\mathbf{p}}\Big[-(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})_j\Big\{p_i(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}((\mathbf{Q}-\frac{\mathbf{q}}{2}))_j$$

$$\frac{i}{4}(e_1\partial_{iQ}\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\partial_{iQ}\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_j\Big\}$$

$$-\frac{i}{2}(\mathbf{S}\times\mathbf{q})\cdot(\mathbf{S}\times(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\Big] \tag{119}$$

and

$$\langle\mathbf{Q},\mathbf{q}|(118)|\mathbf{P},\mathbf{p}\rangle =$$

$$v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\frac{1}{(2\pi)^3}e^{i\mathbf{Q}\cdot\mathbf{P}+i\mathbf{q}\cdot\mathbf{p}}\Big[-\mathbf{q}^2(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2}-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\cdot\mathbf{p}$$

$$+\frac{i}{4}\mathbf{q}^2(e_1\boldsymbol{\nabla}_Q\cdot\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\boldsymbol{\nabla}_Q\cdot\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))+$$

$$+(\mathbf{q}\cdot\mathbf{p})\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))$$

$$-\frac{i}{4}\{\mathbf{q}\cdot(\mathbf{q}\cdot\boldsymbol{\nabla}_Q)(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))$$

$$-i\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})\Big] \tag{120}$$

Next change the final variables to momentum variables. With this change these expressions become

$$\langle\mathbf{P}',\mathbf{p}'|117)|\mathbf{P}',\mathbf{p}'\rangle =$$

$$\int d\mathbf{q}d\mathbf{Q}v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)\frac{1}{(2\pi)^6}e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\times$$

$$\Big[-(\mathbf{S}\times\mathbf{q})\cdot\mathbf{p}(\mathbf{S}\times\mathbf{q})\cdot\{(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}((\mathbf{Q}-\frac{\mathbf{q}}{2}))+$$

-

$$\sum_{ij}\frac{i}{4}(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})_j(e_1\partial_i\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\partial_i\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_j$$

$$-\frac{i}{2}(\mathbf{S}^2\mathbf{q}-(\mathbf{q}\cdot\mathbf{S})\mathbf{S})\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\Big]. \tag{121}$$

and

$$\langle\mathbf{P}',\mathbf{p}'|(118)|\mathbf{P}',\mathbf{p}'\rangle =$$

$$\int d\mathbf{q}d\mathbf{Q}v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\frac{1}{(2\pi)^6}e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\times$$

$$\Big[-\mathbf{q}^2(\mathbf{p}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))+\frac{i}{4}\mathbf{q}^2(e_1\boldsymbol{\nabla}_Q\cdot\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\boldsymbol{\nabla}_Q\cdot\mathbf{A})\mathbf{Q}-\frac{\mathbf{q}}{2}))+$$

$$+\{\cdot(\mathbf{q}\cdot\mathbf{p})\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))\frac{i}{4}\{\mathbf{q}\cdot(\mathbf{q}\cdot\boldsymbol{\nabla})(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))$$

$$-i\mathbf{q}\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})\Big] \tag{122}$$

In (121) and (122) the derivatives can be removed from the vector potential by integrating by parts, assuming no contribution from the boundary terms.

The three terms in these expressions with derivatives on the vector potential are

$$\frac{i}{4}\frac{1}{(2\pi)^6}\sum_{ij}v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})_j\times$$

$$(e_1\partial_i\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\partial_i\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_j\} \tag{123}$$

$$\frac{i}{4}\frac{1}{(2\pi)^6}\int d\mathbf{q}d\mathbf{Q}v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\mathbf{q}^2\times$$

$$(e_1\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})) \tag{124}$$

and

$$-\frac{i}{4}\frac{1}{(2\pi)^6}\int d\mathbf{q}d\mathbf{Q}v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\{\mathbf{q}\cdot(\mathbf{q}\cdot\boldsymbol{\nabla})$$

$$(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})+e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})). \tag{125}$$

The partial derivatives in (123-125) are derivatives of the argument of the vector potential, equivalently with respect to the $\mathbf{Q}$ variables. They can be replaced by $\pm 2\partial_{q_i}$

$$-\frac{ie}{2}\frac{1}{(2\pi)^6}\sum_{ij}\int d\mathbf{q}d\mathbf{Q}(\partial_{q_i}[v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})_j]\times$$

$$(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))_j \tag{126}$$

$$-\frac{i}{2}\frac{1}{(2\pi)^6}\int d\mathbf{q}d\mathbf{Q}\boldsymbol{\nabla}_q[v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\frac{1}{(2\pi)^6}e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\mathbf{q}^2]\cdot(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})) \tag{127}$$

$$\frac{i}{2}\frac{1}{(2\pi)^6}\int d\mathbf{q}d\mathbf{Q}\boldsymbol{\nabla}_q[\cdot\mathbf{q}v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\mathbf{q}]\cdot(e_1\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2})). \tag{128}$$

Using (126-128) and the identities

$$(e_1\mathbf{A}(\mathbf{Q}+\frac{\mathbf{q}}{2})-e_2\mathbf{A}(\mathbf{Q}-\frac{\mathbf{q}}{2}))=\int d\mathbf{x}(e_1\delta(\mathbf{x}-\mathbf{Q}-\frac{\mathbf{q}}{2})-\delta(\mathbf{x}-e_2\mathbf{Q}+\frac{\mathbf{q}}{2}))A(\mathbf{x},0) \tag{129}$$

and

$$\int d\mathbf{Q} e^{i\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}')}(e_1\delta(\mathbf{x}-\mathbf{Q}-\frac{\mathbf{q}}{2}) - e_2\delta(\mathbf{x}-\mathbf{Q}+\frac{\mathbf{q}}{2})) =$$

$$e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')}(e^{i\mathbf{q}\cdot(\mathbf{P}-\mathbf{P}')/2} - e^{-i\mathbf{q}\cdot(\mathbf{P}-\mathbf{P}')/2}) =$$

$$e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')}[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right) + i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right)] \tag{130}$$

in the expressions for the currents gives the result below for the three derivative terms. Since the $\mathbf{q}$ in these expressions comes from the vector potential, it does not get touched by derivatives. Since there is no other $\mathbf{Q}$ -dependence these factors multiply everything.

$$-\frac{i}{4}\frac{1}{(2\pi)^6}\int d\mathbf{q}(\partial_{q_i}[v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}(\mathbf{S}\times\mathbf{q})_i(\mathbf{S}\times\mathbf{q})]\times$$

$$[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right) + i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right)] \tag{131}$$

$$-\frac{i}{2}\frac{1}{(2\pi)^6}\int d\mathbf{q}\boldsymbol{\nabla}_q\cdot[v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\mathbf{q}^2]\times$$

$$[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right) + i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right)] \tag{132}$$

$$\frac{i}{2}\frac{1}{(2\pi)^6}\int d\mathbf{q}\boldsymbol{\nabla}_{q_i}[\cdot\mathbf{q}v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\mathbf{q}]\times$$

$$[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right) + i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right)] \tag{133}$$

These derivative expressions go in the expression for the current matrix elements, which is the coefficient of the vector potential.

$$\langle\mathbf{P}',\mathbf{p}'|\mathbf{J}_{(\mathbf{L}\cdot\mathbf{S})^2}(\mathbf{x},0)|\mathbf{P},\mathbf{p}\rangle = \int d\mathbf{q}v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)\frac{1}{(2\pi)^6}e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\times$$

$$[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right) + i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right)]\times$$

$$\left[-(\mathbf{S}\times\mathbf{q})\cdot\mathbf{p}(\mathbf{S}\times\mathbf{q}) - \frac{i}{2}(\mathbf{S}^2\mathbf{q}-(\mathbf{S}\cdot\mathbf{q})\mathbf{S})\right] +$$

$$-i\frac{1}{2}\frac{1}{(2\pi)^6}\int d\mathbf{q}(\sum_i\boldsymbol{\nabla}\boldsymbol{q}_i[\cdot(\mathbf{S}\times\mathbf{q})_i v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|)e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}(\mathbf{S}\times\mathbf{q})]\times$$

$$[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right) + i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}'-\mathbf{P}}{2}\right)] \tag{134}$$

For this interaction the term with the derivative acting on the interaction vanishes. This is because the coefficient is $\mathbf{q} \cdot (\mathbf{S} \times \mathbf{q}) = 0$. This means that then the $(\mathbf{L} \cdot \mathbf{S})^2$ current matrix elements become

$$\langle \mathbf{P}', \mathbf{p}' | \mathbf{J}(\mathbf{x}, 0)_{(\mathbf{L}\cdot\mathbf{S})^2} | \mathbf{P}, \mathbf{p} \rangle = \int d\mathbf{q} v_{(\mathbf{L}\cdot\mathbf{S})^2}(|\mathbf{q}|) \frac{1}{(2\pi)^6} e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')} \times$$

$$[(e_1 - e_2) \cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}' - \mathbf{P}}{2}\right) + i(e_1 + e_2) \sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}' - \mathbf{P}}{2}\right)] \times$$

$$\left[-\frac{1}{2}(\mathbf{S} \times \mathbf{q}) \cdot (\mathbf{p} + \mathbf{p}')(\mathbf{S} \times \mathbf{q}) - \frac{i}{2}(\mathbf{S}^2 \mathbf{q} - (\mathbf{S} \cdot \mathbf{q})\mathbf{S})\right] \tag{135}$$

which is equivalent to equation (37).

The part of the $\mathbf{L}^2$ current involving derivatives of the interaction also vanishes. To see this first note the expression for the current is

$$\langle \mathbf{P}', \mathbf{p}' | \mathbf{J}(\mathbf{x}, 0)_{\mathbf{L}\cdot\mathbf{L}} | \mathbf{P}, \mathbf{p} \rangle = \int d\mathbf{q} v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|) \frac{1}{(2\pi)^6} e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')} \times$$

$$[(e_1 - e_2) \cos\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right) + i(e_1 + e_2) \sin\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right)]$$

$$\left[-\mathbf{q}^2 \mathbf{p} + (\mathbf{q} \cdot \mathbf{p})\mathbf{q} + 2\mathbf{q}\right] +$$

$$\frac{1}{(2\pi)^6} \int d\mathbf{q} [(e_1 - e_2) \cos\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right) + i(e_1 + e_2) \sin\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right)] \times$$

$$\nabla_q \cdot [v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|) e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')} \mathbf{q}^2] \tag{136}$$

$$-\frac{1}{(2\pi)^6} \int d\mathbf{q} [(e_1 - e_2) \cos\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right) + i(e_1 + e_2) \sin\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right)]$$

$$\nabla[\cdot(\mathbf{q} v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|) e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')})\mathbf{q}]. \tag{137}$$

For a rotationally invariant $v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)$

$$(\nabla v)\mathbf{q}^2 = v' \hat{\mathbf{q}} \mathbf{q}^2 = v' q \mathbf{q} \tag{138}$$

while

$$\mathbf{q} \cdot (\nabla v)\mathbf{q} = v' q \mathbf{q}. \tag{139}$$

These terms come with opposite signs in the expression above so they exactly cancel. What remains after eliminating these terms is

$$\langle \mathbf{P}', \mathbf{p}' | \mathbf{J}(\mathbf{x}, 0)_{\mathbf{L}\cdot\mathbf{L}} | \mathbf{P}, \mathbf{p} \rangle = \int d\mathbf{q} v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|) \frac{1}{(2\pi)^6} e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')} \times$$

$$[(e_1 - e_2) \cos\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right) + i(e_1 + e_2) \sin\left(\frac{\mathbf{q}\cdot(\mathbf{P} - \mathbf{P}'}{2}\right)] \times$$

$$\left[-\mathbf{q}^2\mathbf{p}+(\mathbf{q}\cdot\mathbf{p})\mathbf{q}-i\mathbf{q}+\frac{1}{2}\mathbf{q}^2(\mathbf{p}-\mathbf{p}')-\frac{1}{2}\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')\mathbf{q}+i\mathbf{q}\right]=$$

$$\int d\mathbf{q} v_{\mathbf{L}\cdot\mathbf{L}}(|\mathbf{q}|)\frac{1}{(2\pi)^6}e^{i\mathbf{x}\cdot(\mathbf{P}-\mathbf{P}')+i\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}\times$$

$$[(e_1-e_2)\cos\left(\frac{\mathbf{q}\cdot(\mathbf{P}-\mathbf{P}'}{2}\right)+i(e_1+e_2)\sin\left(\frac{\mathbf{q}\cdot(\mathbf{P}-\mathbf{P}'}{2}\right)]\times$$

$$\frac{1}{2}\mathbf{q}\times(\mathbf{q}\times(\mathbf{p}+\mathbf{p}')) \tag{140}$$

which is equivalent to equation (38) Again the derivative of the potential cancels in this expression as well. This means that the potentials can be factored.

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