Notes on neutrino Deuteron scattering

W. N. Polyzou*

Department of Physics and Astronomy

The University of Iowa

Iowa City, IA 52242, USA

(Dated: April 26, 2023)

*Electronic address: polyzou@uiowa.edu; This work supported by the U.S. Department of Energy, Office of Science, Grant #DE-SC16457

1 Single nucleons - relativistic treatment of spin **p.3**

2 Two-body bases and partial wave analysis **p.7**

3 Dynamics and the Deuteron **p.12**

4 Spinor representations of Lorentz transformations **p.17**

5 The Equivalence of two and four-component spinors **p.20**

6 Gamma matrices from SL(2,C) **p.24**

7 Dirac spinors **p.25**

8 Gamma matrix conventions - Bjorken and Drell **p.29**

9 Current matrix elements **p.23**

item[10] Covariant Decomposition **p. 33**

10 Impulse matrix elements **p.34**

11 Relativistic scattering theory **p.36**

12 Time independent multi-channel scattering theory **p.43**

13 Cross sections **p.47**

14 Two potential scattering **p.36**

15 Fields and potentials **p 54**

16 Breakup **p 60**

17 PDB conventions **p 61**

18 structure tensors **p 63**

19 inclusive scattering **p 65**

20 two-body currents **p 69**

21 EW currents **81**

22 D functions **83**

I. SINGLE NUCLEON HILBERT SPACE - RELATIVISTIC TREATMENT OF SPIN

The single nucleon on-shell four momenta (m = nucleon mass) are:

$$p_1, p_2 \qquad p_i = (\sqrt{\mathbf{p}_i^2 + m_i^2}, \mathbf{p}_i) = (\omega_{m_i}(\mathbf{p}), \mathbf{p}_i) \qquad i = 1, 2. \tag{1}$$

It is convenient to use the 2×2 matrix representation of four vectors for the relativistic treatment of spin. Four vectors can be represented by 2×2 Hermitian matrices:

$$P := p^\mu \sigma_\mu = \begin{pmatrix} \omega_m(\mathbf{p}) + p_z & p_x - ip_y \\ p_x + ip_y & \omega_m(\mathbf{p}) - p_z \end{pmatrix} \qquad \sigma_\mu := (I, \boldsymbol{\sigma}), \tag{2}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. The identity

$$\mathrm{Tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu} \tag{3}$$

can be used to extract the components of the four vector p^μ from the matrix P

$$p^\mu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu P). \tag{4}$$

Note

$$P = P^\dagger \qquad \det(P) = (p^0)^2 - \mathbf{p}^2 = m^2. \tag{5}$$

Since the determinant is the proper (time)^2 of the four vector, real Lorentz transformations are linear transformations that preserve the determinant and Hermiticity (reality) of P. They have the general form

$$P \to P' = \pm \Lambda P \Lambda^\dagger \qquad \det(\Lambda) = 1. \tag{6}$$

The $-$ sign is for space-time reflections. For $\det(\Lambda) \neq 0$:

$$\Lambda = e^M \qquad \det(\Lambda) = e^{\mathrm{Tr}(M)} = 1 \quad \to \quad \mathrm{Tr}(M) = 2\pi i n. \tag{7}$$

Since any 2×2 matrix M can be expressed as $M = M^0 I + \mathbf{M} \cdot \boldsymbol{\sigma}$ with $\mathrm{Tr}(\boldsymbol{\sigma}) = 0$ it follows that (7) requires $2M^0 = 2\pi i n$ which means

$$\Lambda = \pm e^{\mathbf{M} \cdot \boldsymbol{\sigma}}. \tag{8}$$

In this case both signs lead to the same transformation. The matrix in the exponent is complex. It can be expressed in the form

$$\mathbf{M}\cdot\boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{\rho} + i\boldsymbol{\theta})\cdot\boldsymbol{\sigma} \tag{9}$$

where $\boldsymbol{\theta}$ is the angle of a rotation and $\boldsymbol{\rho}$ is the rapidity of a canonical (rotationless) Lorentz boost:

$$e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \sigma_0\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} \tag{10}$$

$$e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \sigma_0\cosh(\frac{\rho}{2}) + \sinh(\frac{\rho}{2})\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\sigma} \tag{11}$$

where

$$\cosh(\rho) = p^0/m \qquad \sinh(\rho) = |\mathbf{p}|/m \qquad \hat{\boldsymbol{\rho}} = \hat{\mathbf{p}}. \tag{12}$$

The corresponding 4×4 Lorentz transformations $\Lambda^\mu{}_\nu$ are related to the 2×2 matrix Λ by

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu\Lambda\sigma_\nu\Lambda^\dagger). \tag{13}$$

A general 2×2 Λ has a polar decomposition as a product of a unitary matrix R followed by a positive Hermitian matrix P:

$$\Lambda = PR = (\Lambda\Lambda^\dagger)^{1/2}(\Lambda\Lambda^\dagger)^{-1/2}\Lambda \tag{14}$$

where

$$(\Lambda\Lambda^\dagger)^{1/2} = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\underline{\boldsymbol{\sigma}}} := P = P^\dagger \qquad (\Lambda\Lambda^\dagger)^{-1/2}\Lambda = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\underline{\boldsymbol{\sigma}}} := \underline{R} = (R^\dagger)^{-1} \tag{15}$$

which means that any Lorentz transformation can be expressed as a rotation followed by a rotationaless boost. Since a rotation leaves $(m,0,0,0)$ unchanged, Λ and $P = (\Lambda\Lambda^\dagger)^{1/2}$ are both boosts to the same final momentum. They differ by the rotation $R = (\Lambda\Lambda^\dagger)^{-1/2}\Lambda$. The rotationless boost $P = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\underline{\boldsymbol{\sigma}}}$ is special because Λ is a positive Hermitian matrix. It is also called the canonical boost.

Remark - these represent passive coordiante changes rather than active transformations.

Spin is defined as the angular momentum of a particle or system in its rest frame. A natural question is how to compare spins in different frames.

One way to compare the spins of particles with different momenta is to boost them to a common frame with a standard type of boost. The common frame is usually taken as the rest frame. There are many possible choices of the standard boost. Each choice of standard boost defines a different spin observable - it is the spin that would be measured in the rest frame if the particle was boosted to the rest frame with the chosen standard boost.

The inverse of the boost $\Lambda = PR$ it is $R^{\dagger}P^{-1}$ which means that the angular momentum in the rest frame depends on the choice of boost. If we let $\Lambda = B_x(p)$ be a boost parameterized by momentum, (actually they are parameterized by the 4 velocity) a spin operator can be defined by

$$\boxed{\mathbf{s}_x^i = \frac{1}{2}\epsilon^{ijk}\Lambda(B_x^{-1}(p_{op}))^i{}_{\mu}\Lambda(B_x^{-1}(p_{op}))^i{}_{\nu}J_{\mu\nu}} \tag{16}$$

where (1) $J_{\mu\nu}$ is the angular momentum tensor,

$$\Lambda(B_x^{-1}(p_{op}))^i{}_{\mu} \tag{17}$$

is the 4×4 matrix representation of the boost $B_x^{-1}(p)$ with the parameter p^{μ} replaced by the corresponding operator p_{op}^{μ}. It follows from these definitions and the Poincaré commutation relations that

$$[s_x^i, s_x^j] = i\epsilon^{ijk}s_x^k \tag{18}$$

independent of the choice of boost, where all components of $\mathbf{s}_x$ commute with p_{op}^{μ} for any choice of boost.

Using p_i, $\hat{\mathbf{z}}\cdot\mathbf{s}_x$, the total and z-component of isospin as commuting observables, the single nucleon basis vectors are:

$$|(m, \frac{1}{2}, \frac{1}{2})\mathbf{p}, m_{si}, m_{ti}\rangle \tag{19}$$

where m_{ti} is the projection of the isospin of the nucleon.

The choice of boost used to define the spin is consistent with the following Lorentz transformation property

$$U(B_x(p/m))|(m, \frac{1}{2}, \frac{1}{2})\mathbf{0}, m_{si}, m_{ti}\rangle = |(m, \frac{1}{2}, \frac{1}{2})\mathbf{p}, m_{si}, m_{ti}\rangle\sqrt{\frac{\omega_m(\mathbf{p})}{m}} \tag{20}$$

where the spin defined with the x-boost remains unchanged in boosting from the rest frame to the frame where the particle has momentum $\mathbf{p}$ with $B_x(p/m)$. The square root factors ensure that this transformation is unitary when the basis vectors have delta-function normalizations:

$$\langle(m, \frac{1}{2}, \frac{1}{2})\mathbf{p}', m'_{si}, m'_{ti}|(m, \frac{1}{2}, \frac{1}{2})\mathbf{p}, m_{si}, m_{ti}\rangle = \delta(\mathbf{p}' - \mathbf{p})\delta_{m'_{si}m_{si}}\delta_{m'_{ti}m_{ti}}. \tag{21}$$

Since rotations leave the rest vector invariant, they can only transform the spin

$$U(R)|(m, \frac{1}{2}, \frac{1}{2})\mathbf{0}, m_{si}, m_{ti}\rangle =$$

$$\sum_{m'_{si}} |(m,\frac{1}{2},\frac{1}{2})\mathbf{0}, m'_{si}, m_{ti}\rangle D^{1/2}_{m'_{si} m_{si}}[R] \tag{22}$$

where

$$D^{1/2}_{m'_{si} m_{si}}[R]; = \langle \frac{1}{2}, m'_{si}|U(R)|\frac{1}{2}, m_{si}\rangle \tag{23}$$

is the Wigner D-function. Since any Lorentz transformation can be expressed in terms of boosts to and from the rest frame and rotations in the rest frame, it follows that

$$\boxed{\begin{aligned} & U(\Lambda)|(m,\frac{1}{2},\frac{1}{2})\mathbf{k}, m_{si}, m_{ti}\rangle_x = && (24a) \\ & |(m,\frac{1}{2},\frac{1}{2})\mathbf{\Lambda}(p), m'_{si}, m_{ti}\rangle_x \sqrt{\frac{\omega_m(\mathbf{\Lambda}(\Lambda)p}{\omega_m(\mathbf{p})}} D^{1/2}_{m'_{si} m_{si}}[B_x^{-1}(\Lambda)p/m)\Lambda B_x(p/m)]. && (24b) \end{aligned}}$$

The subscript x on the states indicates the type of spin that is used as a commuting observable. Equation (102) defines a unitary representation of the Lorentz group on the single-nucleon subspace. The matrix

$$R_w(\Lambda, p) := B_x^{-1}(\Lambda p/m)\Lambda B_x(p/m) \tag{25}$$

is a $SU(2)$ Wigner rotation. The subscript x indicates that Wigner rotations depend on the choice of boost, $B_x(p/m)$.

The canonical boost is special because it has the property that the Wigner rotation of a rotation is the rotation:

$$\boxed{B_c^{-1}(Rp/m)RB_c(p/m) = \underline{R}.} \tag{26}$$

This can be rewritten as

$$RB_c(p/m)R^{-1} = B_c(Rp/m). \tag{27}$$

This follows because

$$RB_c(p/m)R^{-1} = Re^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}}R^{-1} = e^{\frac{1}{2}\boldsymbol{\rho}\cdot(R\boldsymbol{\sigma}R^{-1})} = e^{\frac{1}{2}(\boldsymbol{\rho})\cdot\Lambda(R^{-1})\boldsymbol{\sigma}} = e^{\frac{1}{2}(\Lambda(R)\boldsymbol{\rho})\cdot\boldsymbol{\sigma}} \tag{28}$$

which is the desired result. The important property is that the canonical boost Wigner rotation of the rotation is the rotation, independent of p. This is needed for adding spins in a many-body system, where each particle has a different momentum. Sinc they all rotate the same way the spins can be added with ordinary $SU(2)$ Clebsch-Gordon coefficients. The canonical spin is the only spin with this property.

If the spin is not canonical, it can be converted to canonical spin using a momentum-dependent rotation:

$$
\begin{aligned}
&|(m,\frac{1}{2},\frac{1}{2})\mathbf{p}_1, m_{sx1}, m_{t1}\rangle = \\
&= U(B_x(p_i/m))|(m,\frac{1}{2},\frac{1}{2})\mathbf{0}_1, m_{sx1}, m_{t1}\rangle \sqrt{\frac{m}{\omega_m(p_i/m)}} \\
&= U(B_c(p_i/m))U(B_c^{-1}(p_i/m))U(B_x(p_i))|(m,\frac{1}{2},\frac{1}{2})\mathbf{0}_1, m_{sx1}, m_{t1}\rangle \sqrt{\frac{m}{\omega_m(p_i/m)}} \\
&= U(B_c(p_i/m))|(m,\frac{1}{2},\frac{1}{2})\mathbf{0}_1, m'_{sc1}, m_{t1}\rangle \sqrt{\frac{m}{\omega_m(p_i/m)}} D^{1/2}_{m'_{sc1}m_{sx1}}[\underline{B}_c^{-1}(p_i/m)\underline{B}_x(p_i/m)] \\
&= |(m,\frac{1}{2},\frac{1}{2})\mathbf{p}_1, m'_{sc1}, m_{t1}\rangle D^{1/2}_{m'_{sc1}m_{sx1}}[B_c^{-1}(p_i/m)B_x(p_i/m)]
\end{aligned} \tag{29}
$$

where $[B_c^{-1}(p_i/m)B_x(p_i/m)]$ is a momentum-dependent $SU(2)$ rotation called a generalized Melosh rotation. In these expression the subscript c =canonical spin, x=spin constructed with the x boost. The rotation (29) relates the x-spin and canonical spin basses.

II. TWO-BODY BASES AND PARTIAL WAVE ANALYSIS

Most few-body calculations use a partial wave basis. In the relativistic case the relevant generalization is a Poincaré irreducible basis.

A basis for the two-free-nucleon system is the tensor product of 2 single nucleon states:

$$
|(m,\frac{1}{2},\frac{1}{2})\mathbf{p}_1, m_{s1}, m_{t1}\rangle \times |(m,\frac{1}{2},\frac{1}{2})\mathbf{p}_2, m_{s2}, m_{t2}\rangle. \tag{30}
$$

In what follows these basis vectors are expressed as linear superpositions of basis vectors that transform irreducibly. To do this first define the total four momentum of non-interacting system:

$$
P_0^\mu = p_1^\mu + p_2^\mu \tag{31}
$$

The invariant mass of non-interacting system is

$$
M_0^2 = \eta_{\mu\nu} P_0^\mu P_0^\nu \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{32}
$$

The four velocity of non-interacting system is

$$Q_0^\mu = P_0^\mu / M_0. \tag{33}$$

Also define

$$k_i^\mu = \Lambda(B_x^{-1}(Q_0))^\mu{}_\nu p_i^\nu. \tag{34}$$

which represents the momentum of particle i if it was transformed to the rest frame with an x-boost. Here $\Lambda(B_x^{-1}(Q_0))^\mu{}_\nu$ is considered to be a matrix of multiplication operators. The k_i are operators whose eigenvalues are the momentum of each nucleon (particle) when it is transformed to the 0 total momentum frame with the boost $B_x^{-1}(Q)$. Here I assume that the boost used to define k_i is the same as the one used to define the spin. Note that $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$. It is important to note that k^μ does not transform like a 4-vector. Instead it undergoes Wigner rotations:

$$k^{\mu\prime} = \Lambda(B^{-1}(\Lambda P)\Lambda p_1 = B^{-1}(\Lambda P)\Lambda B(P))^\mu{}_\nu k_1^\nu \tag{35}$$

The next step is to consider canonical spin two-body basis states (30) in the non-interacting two-body rest frame

$$|(m, \frac{1}{2}, \frac{1}{2})\mathbf{k}_1, m_{s1c}, m_{t1}\rangle \times |(m, \frac{1}{2}, \frac{1}{2}) - \mathbf{k}_1, m_{s2c}, m_{t2}\rangle \tag{36}$$

since $\mathbf{k}_2 = -\mathbf{k}_1$ this only depends on $\mathbf{k}_1$. Here these vectors represent products of ordinary single-particle states in the system rest frame. Next we use spherical Harmonics to perform a partial wave decomposition of the angle dependence of $\hat{\mathbf{k}}_1$

$$|(M, l, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \mathbf{0}, m_l, m_{s1c}, m_{t1}, m_{s2c}, m_{t2}\rangle :=$$

$$\int d\hat{\mathbf{k}}_1 |(m, \frac{1}{2}, \frac{1}{2})\mathbf{k}_1, m_{s1c}, m_{t1}\rangle \times |(m, \frac{1}{2}, \frac{1}{2}) - \mathbf{k}_1, m_{s2x}, m_{t2}\rangle Y^l_{m_l}(\hat{\mathbf{k}}_1).$$

Because the spins are canonical this vector transforms covariantly under ordinary rotations

$$U(R)|(M, l, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\mathbf{0}, m_l, m_{s1c}, m_{t1}.m_{s2c}, m_{t2}\rangle :=$$

$$\sum |(M, l, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \mathbf{0}, m_l', m_{s1c}', m_{t1} m_{s2c}', m_{t2}\rangle D^l_{m_l' m_l}[R] D^{1/2}_{m_{sc1}' m_{sc1}}[R] D^{1/2}_{m_{sc2}' m_{sc2}}[R]. \tag{37}$$

It follows that we can couple the spins and orbital angular momenta with ordinary Clebsch-Gordan coefficients

$$|(M, j, l, s_{12}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\mathbf{0}, m_j, m_{t1}, m_{t2}\rangle :=$$

$$\int d\hat{\mathbf{k}}_1 |(m,\frac{1}{2},\frac{1}{2})\mathbf{k}_1, m_{s1c}, m_{t1}\rangle \times |(m,\frac{1}{2},\frac{1}{2}) - \mathbf{k}_1, m_{s2x}, m_{t2}\rangle \times$$

$$Y^l_{m_l}(\hat{\mathbf{k}}_1) C(s_{12},\frac{1}{2},\frac{1}{2}; m_{sc12}, m_{sc1}, m_{sc2}) C(j,l,s_{12}; m_j, m_l, m_{sc12}). \tag{38}$$

Now we return to the case of a spin associated with a general boost, $B_x(p)$. We use Melosh rotations to express the single particle canonical spins in terms of the x spins

$$|(M,j,l,s_{12},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\mathbf{0}, m_j, m_{t1}, m_{t2}\rangle :=$$

$$\sum \int d\hat{\mathbf{k}}_1 |(m,\frac{1}{2},\frac{1}{2})\mathbf{k}_1, m_{s1x}, m_{t1}\rangle \times |(m,\frac{1}{2},\frac{1}{2}) - \mathbf{k}_1, m_{s2x}, m_{t2}\rangle \times$$

$$D^{1/2}_{m_{s1x}m_{s1c}}[B^{-1}_x(k_1/m)B_c(k_1/m)] D^{1/2}_{m_{s2x}m_{s2c}}[B^{-1}_x(k_2/m)B_c(k_2/m)] Y^l_{m_l}(\hat{\mathbf{k}}_1) \times$$

$$C(s_{12},\frac{1}{2},\frac{1}{2}; m_{sc12}, m_{sc1}, m_{sc2}) C(j,l,s_{12}; m_j, m_l, m_{sc12}). \tag{39}$$

Finally we boost both sides with $B_x(P/M_0)$ to get an irreducible state with an x-spin. On the left side of (39), since it is a rest state we get the x boost will leave the total x-spin unchanged

$$|(M,j,l,s_{12},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\mathbf{P}, m_j, m_{t1}, m_{t2}\rangle \sqrt{\frac{\omega_M(P)}{M}}. \tag{40}$$

The square root factors ensure unitarity. On the right hand side of (39) k_i is the momentum of particle i in the non-interacting two nucleon rest frame. The boost $B_x(P/M_0)$ acts on each of the k_i and boosts them to the single particle momenta p_i and causes the spins to Wigner rotate

$$|(M,j,l,s_{12},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\mathbf{P}, m_j, m_{t1}, m_{t2}\rangle \sqrt{\frac{\omega_M(P)}{M}} =$$

$$\sum \int d\hat{\mathbf{p}}_1 |(m,\frac{1}{2},\frac{1}{2})\mathbf{k}_1, m_{s1x}, m_{t1}\rangle \times |(m,\frac{1}{2},\frac{1}{2}) - \mathbf{k}_2, m_{s2x}, m_{t2}\rangle \times$$

$$D^{1/2}_{m_{s1x}m_{s1x'}}[B^{-1}_x(p_1/m)B_x(P/M)B_x(k_1/m)] D^{1/2}_{m_{s2x}m_{s2x'}}[B^{-1}_x(p_2/m)\underline{B}_x(P/M)B_x(k_2/m)] \times$$

$$\sqrt{\frac{\omega_{m_1}(p_1)\omega_{m_2}(p_2)}{\omega_{m_1}(k_1)\omega_{m_2}(k_2)}} D^{1/2}_{m_{s1x'}m_{s1c}}[B^{-1}_x(k_1/m)B_c(k_1/m)] D^{1/2}_{m_{s2x'}m_{s2c}}[B^{-1}_x(k_2/m)B_c(k_2/m)] \times$$

$$Y^l_{m_l}(\hat{\mathbf{k}}_1) C(s_{12},\frac{1}{2},\frac{1}{2}; m_{sc12}, m_{sc1}, m_{sc2}) C(j,l,s_{12}; m_j, m_l, m_{sc12}) \tag{41}$$

In this expression there are both Wigner and Melosh rotations. If the spin is canonical the Melosh rotations are replaced by the identity, while if $B_x(B)$ is a light-front preserving boost, there are no Wigner rotations (because the light front boosts form a subgroup). The variables p_i represent the momentum of each particle in the two-particle rest frame. Since they are eigenvalues of the operator

$$k_i = \Lambda(B_x^{-1}(P/M_0))p_i \tag{42}$$

they are related to p_i by the x-boost. The spins on the right side of (41) are the ones that couple to currents, while the state on the left transforms irreducibly. Also note

$$M_0 = \sqrt{m_1^2 + \mathbf{k}_1^2} + \sqrt{m_2^2 + \mathbf{k}_1^2} \tag{43}$$

This basis is the relativistic version of the two-body partial wave basis.

It is useful to replace the general form by the instant and light front forms

Instant form partial wave basis in terms of instant form tensor product basis:

$$|(M, j, l, s_{12}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\mathbf{P}, m_j, m_{t1}, m_{t2}\rangle = \tag{44a}$$

$$\sum \int d\hat{\mathbf{k}}_1 |(m, \frac{1}{2}, \frac{1}{2})\mathbf{p}_1, m_{s1c}, m_{t1}\rangle \times |(m, \frac{1}{2}, \frac{1}{2}) - \mathbf{p}_1, m_{s2c}, m_{t2}\rangle \times \tag{44b}$$

$$D^{1/2}_{m_{s1c}m_{s1c'}}[B_c^{-1}(p_1/m)B_c(P/M)B_c(k_1/m)] \times \tag{44c}$$

$$D^{1/2}_{m_{s2c}m_{s2c'}}[B_c^{-1}(p_2/m)B_c(P/M)B_c(k_2/m)] \times \tag{44d}$$

$$Y^l_{m_l}(\hat{\mathbf{k}}_1)C(s_{12}, \frac{1}{2}, \frac{1}{2}; m_{sc12}, m_{sc1}, m_{sc2})C(j, l, s_{12}; m_j, m_l, m_{sc12}) \times \tag{44e}$$

$$\sqrt{\frac{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)M_0}{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)\omega_{M_0}(\mathbf{P})}} \tag{44f}$$

Front form partial wave basis in terms of instant form tensor product basis:

$$|(M, j, l, s_{12}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})P^+, \mathbf{P}_\perp, m_j, m_{t1}, m_{t2}\rangle = \tag{45a}$$

$$\sum \int d\hat{\mathbf{k}}_1 |(m, \frac{1}{2}, \frac{1}{2})p_1^+, \mathbf{p}_{1\perp}, m_{s1f}, m_{t1}\rangle \times |(m, \frac{1}{2}, \frac{1}{2})p_2^+, \mathbf{p}_{2\perp}, m_{s2f}, m_{t2}\rangle \times \tag{45b}$$

$$D^{1/2}_{m_{s1f}m_{s1c}}[B_f^{-1}(k_1/m)B_c(k_1/m)]D^{1/2}_{m_{s2f}m_{s2c}}[B_f^{-1}(k_2/m)B_c(k_2/m)] \tag{45c}$$

$$Y^l_{m_l}(\hat{\mathbf{k}}_1) \times C(s_{12}, \frac{1}{2}, \frac{1}{2}; m_{sc12}, m_{sc1}, m_{sc2})C(j, l, s_{12}; m_j, m_l, m_{sc12}) \tag{45d}$$

$$\sqrt{\frac{p_1^+ p_2^+}{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)}}\sqrt{\frac{M_0}{P^+}} \tag{45e}$$

Normally calculations are performed in the partial wave basis

$$|(M, j, l, s_{12}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\mathbf{P}, m_j, m_{t1}, m_{t2}\rangle \tag{46}$$

or in the light front case

$$|(M, j, l, s_{12}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})P^+, \mathbf{P}_\perp, m_j, m_{t1}, m_{t2}\rangle \tag{47}$$

The internal variables in both of these bases are spectrally equivalent, the differences being in the treatment of the total momentum and total spin.

From these expressions we can read off the overlap coefficients

$$\boxed{\begin{aligned}
&\langle (m, \frac{1}{2}, \frac{1}{2})\mathbf{p}_1, m_{s1c}, m_{t1}; (m, \frac{1}{2}, \frac{1}{2}) - \mathbf{p}_1, m_{s2c}, m_{t2}|(M, j, l, s_{12}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\mathbf{P}, m_j, m_{t1}, m_{t2}\rangle = \\
&\sum \delta(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2)\frac{\delta(k - k(\mathbf{p}_1\mathbf{p}_2))}{\mathbf{k}_1^2}\sqrt{\frac{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)}{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}}\sqrt{\frac{\omega_{M_0}(\mathbf{P})}{M_0}}\times \\
&D^{1/2}_{m_{s1c}m_{s1c'}}[B_c^{-1}(p_1/m)B_c(P/M)B_c(k_1/m)]D^{1/2}_{m_{s2c}m_{s2c'}}[B_c^{-1}(p_2/m)B_c(P/M)B_c(k_2/m)]\times \\
&Y^l_{m_l}(\hat{\mathbf{k}}_1)C(s_{12}, \frac{1}{2}, \frac{1}{2}; m_{sc12}, m_{sc1}, m_{sc2})C(j, l, s_{12}; m_j, m_l, m_{sc12})
\end{aligned}}$$

(48a)

(48b)

(48c)

(48d)

and for the light front case

$$\langle (m,\frac{1}{2},\frac{1}{2})p_1^+,\mathbf{p}_{1\perp},m_{s1f},m_{t1};(m,\frac{1}{2},\frac{1}{2})p_2^+,\mathbf{p}_{2\perp},m_{s2f},m_{t2}|(M,j,l,s_{12},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})P^+,\mathbf{P}_\perp,m_j,m_{t1},m_{t2}\rangle$$

$$\delta(P^+-p_1^+-p_2^+)\delta^2(\mathbf{P}_\perp-\mathbf{p}_{1\perp}-\mathbf{p}_{2\perp})\frac{\delta(k-k(\mathbf{p}_1\mathbf{p}_2))}{\mathbf{k}_1^2}\sqrt{\frac{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)}{p_1^+p_2^+}}\sqrt{\frac{P^+}{M_0}}\times$$

$$D^{1/2}_{m_{s1f}m_{s1c}}[B_f^{-1}(k_1/m)B_c(k_1/m)]D^{1/2}_{m_{s2f}m_{s2c}}[B_f^{-1}(k_2/m)B_c(k_2/m)]Y^l_{m_l}(\hat{\mathbf{k}}_1)\times$$

$$C(s_{12},\frac{1}{2},\frac{1}{2};m_{sc12},m_{sc1},m_{sc2})C(j,l,s_{12};m_j,m_l,m_{sc12})$$

(49a)

(49b)

(49c)

(49d)

(49e)

III. DYNAMICS AND THE DEUTERON

In this section I discuss how non-relativistic interactions fit to data can be reinterpreted as relativistic interactions. This avoids any need to refit the parameters of the interaction.

For a relativistic treatment of the Deuteron let V be any realistic non-relativistic nucleon-nucleon interaction that gives the correct Deuteron binding energy and experimental phase shifts as a function of the center of mass momentum of one particle when used in the non-relativistic Schrödinger equations.

$$\boxed{h|\psi\rangle = (\frac{\mathbf{k}^2}{2\mu}+V)|\psi\rangle = e|\psi\rangle} \tag{50}$$

where

$$\mathbf{k}_i = \mathbf{p}_i - m_i\mathbf{P}/M = (M\mathbf{p}_i - m_i(\mathbf{p}_i+\mathbf{p}_j))/M = (m_j\mathbf{p}_i - m_i\mathbf{p}_j)/M \qquad M = m_1+m_2. \tag{51}$$

is the momentum of particle 1 boosted to the two body rest frame with a Galilean boost. $\mu = m_1 m_2/(m_1+m_2)$ is the non-relativistic reduced mass of the two-body system, and

$$h = H - \frac{\mathbf{P}^2}{2M} \tag{52}$$

is the Hamiltonian in the 2 body rest frame.

Note that the experimental data that determines V is measured experimentally - it is not "non relativistic".

Define the interacting mass operator for the two nucleon system:

$$\boxed{M := \sqrt{m_1^2 + \mathbf{k}_1^2 + 2\mu V} + \sqrt{m_2^2 + \mathbf{k}_2^2 + 2\mu V}} \tag{53}$$

where $\mathbf{k}_i$ is the relativistic $\mathbf{k}_i$ which replaces the non-relativistic $\mathbf{k}_i$ in the expression for the potential. This operator is a function of the non-relativistic Hamiltonian. This means that the wave functions and phase shifts, as a function of $\mathbf{k}_i$, are identical to the non-relativistic quantities. The only caveat here is that we have to identify the operator defined by boosting the momentum of one particle to the rest frame with a Lorentz transformation with the operator defined by boosting the momentum of one-particle to the rest frame with a Galilean boost. This assumes that the cross sections are measured as a function of these variables .

Note that most experimentalists use relativistic kinematics when they measure cross sections.

Thus it is not necessary to diagonalize M directly. We can check that M gives the correct Deuteron binding energy up to a small correction:

$$M = \sqrt{m_1^2 + 2\mu h} + \sqrt{m_2^2 + 2\mu h} \to \sqrt{m_1^2 - 2\mu\epsilon} + \sqrt{m_2^2 - 2\mu\epsilon} = m_1\sqrt{1 - 2\mu\epsilon/m_1^2} + m_2\sqrt{1 - 2\mu\epsilon/m_2^2} \approx$$

$$m_1 + m_2 - \frac{m_2}{(m_1+m_2)}\epsilon - \frac{m_1}{(m_1+m_2)}\epsilon + \frac{m_2^2\epsilon^2}{m_1(m_1+m_2)^2} + \frac{m_1^2\epsilon^2}{m_2(m_1+m_2)^2} + \cdots =$$

$$m_1 + m_2 - \epsilon + \epsilon\frac{\epsilon}{2(m_1+m_2)}\left(\frac{m_2^2}{(m_1+m_2)m_1} + \frac{m_1^2}{(m_1+m_2)m_2}\right) + \cdots \tag{54}$$

This gives the observed binding energy up to corrections that are about $(1/2000)\epsilon$ (about 1 KeV).

While the phase shifts can be directly read off of the wave functions, the result can also be obtained using the invariance principle

$$S = \lim_{t\to\infty} e^{iH_{r0}t} e^{-2iH_r t} e^{iH_{r0}t} = \lim_{t\to\infty} e^{iM_{r0}t} e^{-2iM_r t} e^{iM_{r0}t} =$$

$$\lim_{t\to\infty} e^{ih_{nr0}t}e^{-2ih_{nrt}t}e^{ih_{nr0}t} = \lim_{t\to\infty} e^{iH_{nr0}t}e^{-2iH_{nrt}t}e^{iH_{nr0}t} = e^{2i\delta} \tag{55}$$

where

$$h = \frac{\mathbf{k}_i^2}{2\mu} + V \qquad H = \frac{\mathbf{P}^2}{2M} + h \tag{56}$$

which shows that the relativistic and non-relativistic scattering operators exactly reproduce the measured scattering data as a function of k_i. This means that there is no need to refit data, standard potentials can be used directly. Note that the non-relativistic limit of the relativistic calculation is not the same as the non-relativistic calculation - that because the non-relativistic calculation is not fit to the non relativistic limit of the data; it is fit to the same data as the relativistic model.

The non-relativistic calculations of transition operators can be used to calculate on shell, half shell and fully on shell relativistic transition matrix elements. For the (right) half shell transition matrix elements we use the fact that the scattering wave functions in the relativistic and non-relativistic cases are identical:

$$\langle \mathbf{k}_f \| T_n \| \mathbf{k}_i \rangle = \langle \mathbf{k}_f \| V_n \| \mathbf{k}_i^- \rangle = \langle \mathbf{k}_f | H_n - H_{0n} | \mathbf{k}_i^- \rangle =$$

$$\langle \mathbf{k}_f | (\frac{\mathbf{k}_i^2 - \mathbf{k}_f^2}{2\mu}) | \mathbf{k}_i^- \rangle$$

$$\langle \mathbf{k}_f | (\frac{\mathbf{k}_i^2 - \mathbf{k}_f^2}{2\mu}) \frac{(\omega_1(\mathbf{k}_i) + \omega_2(\mathbf{k}_i) - \omega_1(\mathbf{k}_f) - \omega_2(\mathbf{k}_f))}{\omega_1(\mathbf{k}_i) + \omega_2(\mathbf{k}_i) - \omega_1(\mathbf{k}_f) - \omega_2(\mathbf{k}_f)} | \mathbf{k}_i^- \rangle =$$

$$\frac{(\mathbf{k}_i^2 - \mathbf{k}_f^2)}{2\mu(\omega_1(\mathbf{k}_i) + \omega_2(\mathbf{k}_i) - \omega_1(\mathbf{k}_f) - \omega_2(\mathbf{k}_f)} \langle \mathbf{k}_f | ((\omega_1(\mathbf{k}_i) + \omega_2(\mathbf{k}_i) - \omega_1(\mathbf{k}_f) - \omega_2(\mathbf{k}_f) | \mathbf{k}_i^- \rangle =$$

$$= \frac{(\mathbf{k}_i^2 - \mathbf{k}_f^2)}{2\mu(\omega_1(\mathbf{k}_i) + \omega_2(\mathbf{k}_i) - \omega_1(\mathbf{k}_f) - \omega_2(\mathbf{k}_f))} \langle \mathbf{k}_f | (M_r - M_0) | \mathbf{k}_i^- \rangle =$$

$$\frac{(\mathbf{p}_i^2 - \mathbf{k}_f^2)}{2\mu(\omega_1(\mathbf{k}_i) + \omega_2(\mathbf{k}_i) - \omega_1(\mathbf{k}_f) - \omega_2(\mathbf{k}_f))} \langle \mathbf{k}_f \| T_r \| \mathbf{k}_i \rangle \tag{57}$$

In these expressions an overall 3-momentum conserving delta function has been factored out:

$$\langle \mathbf{P}', \mathbf{k}' | T | \mathbf{P}, \mathbf{k} \rangle = \delta(\mathbf{P}' - \mathbf{P}) \langle \mathbf{k}' \| T \| \mathbf{k} \rangle \tag{58}$$

The singular parts of the coefficient in front cancel:

$$= \frac{(\mathbf{k}_i^2 - \mathbf{k}_f^2)}{2\mu(\omega_1(\mathbf{k}_i) + \omega_2(\mathbf{k}_i) - \omega_1(\mathbf{k}_f) - \omega_2(\mathbf{k}_f))} =$$

$$\frac{(\mathbf{k}_i^2-\mathbf{k}_f^2)}{2\mu(\omega_1(\mathbf{k}_i)-\omega_1(\mathbf{k}_f)+\omega_2(\mathbf{k}_i)-\omega_2(\mathbf{k}_f))}$$

$$\frac{(\mathbf{k}_i^2-\mathbf{k}_f^2)}{2\mu}\frac{1}{\frac{(\mathbf{k}_i^2-\mathbf{k}_f^2)}{\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_f)}+\frac{(\mathbf{k}_i^2-\mathbf{k}_f^2)}{\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_f)}}=$$

$$\frac{1}{2\mu}\frac{(\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_f))(\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_f))}{\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_f)+\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_f)} \tag{59}$$

Thus

$$2\mu\langle\mathbf{k}_f\|T_n\|\mathbf{k}_i\rangle=\frac{(\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_f))(\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_f))}{\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_f)+\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_f)}\langle\mathbf{k}_f\|T_r\|\mathbf{k}_i\rangle. \tag{60}$$

or

$$\boxed{\langle\mathbf{k}_f\|T_r\|\mathbf{k}_i\rangle=2\mu\frac{\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_f)+\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_f)}{(\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_f))(\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_f))}}\langle\mathbf{k}_f\|T_n\|\mathbf{k}_i\rangle \tag{61}$$

This result is valid for either the right or left half shell transition matrix elements. If we evaluate these on shell $|\mathbf{k}_i|=|\mathbf{k}_f|$ this becomes

$$2\mu\langle\mathbf{k}_f\|T_n\|\mathbf{k}_i\rangle=2\frac{\omega_1(\mathbf{k}_i)\omega_2(\mathbf{k}_f))}{\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_i)}\langle\mathbf{k}_f\|T_r\|\mathbf{k}_i\rangle \tag{62}$$

or

$$\boxed{\langle\mathbf{k}_f\|T_r\|\mathbf{k}_i\rangle=\mu\frac{\omega_2(\mathbf{k}_i)+\omega_2(\mathbf{k}_i)}{\omega_1(\mathbf{k}_i)\omega_2(\mathbf{k}_f))}\langle\mathbf{k}_f\|T_n\|\mathbf{k}_i\rangle} \tag{63}$$

Note that the relation between the scattering operator and the transition operators in the relativistic and non-relativistic cases are (spin and isospin degrees suppressed)

$$\langle\mathbf{P}_f,\mathbf{k}_f|S_r|\mathbf{P}_i,\mathbf{k}_i\rangle=\delta(\mathbf{P}_f-\mathbf{P}_i)(I-2\pi i\delta(m_i-m_f)\langle\mathbf{k}_f\|T_r(m+i\epsilon)\|\mathbf{k}_i\rangle \tag{64}$$

$$T_r:=V_r+V_r\frac{1}{m-M+i\epsilon}V_v \tag{65}$$

$$\langle\mathbf{P}_f,\mathbf{k}_f|S_n|\mathbf{P}_i,\mathbf{k}_i\rangle=\delta(\mathbf{P}_f-\mathbf{P}_i)(I-2\pi i\delta(h_i-h_f)\langle\mathbf{k}_f\|T_n(h+i\epsilon)\|\mathbf{k}_i\rangle \tag{66}$$

$$T_n:=V_n+V_n\frac{1}{h-\hat{h}+i\epsilon}V_n. \tag{67}$$

The corresponding expressions for the differential cross sections are

$$d\sigma_r=\frac{(2\pi)^4}{|\mathbf{v}_r|}|\langle\mathbf{k}_f\|T_r(m+i\epsilon)\|\mathbf{k}_i\rangle|^2\delta(m-m')\mathbf{k}_f^2\frac{dk_f}{dm}d\Omega(\hat{\mathbf{k}}_f) \tag{68}$$

$$V_r=M-M_0 \tag{69}$$

$$\frac{1}{v_r}=\frac{1}{|\mathbf{k}_i/\omega_1(\mathbf{k}_i)+{}_i/\omega_2(\mathbf{k}_i)|}=\frac{\omega_1(\mathbf{k}_i)\omega_2(\mathbf{k}_i)}{|\mathbf{k}_i|(\omega_1(\mathbf{k}_i)+\omega_1(\mathbf{k}_i)} \tag{70}$$

above are initial; below are final

$$|\mathbf{k}_f|^2 \frac{dm}{dk_1} = |\mathbf{k}_f|^2 \frac{\omega_1(\mathbf{k}_f)\omega_2(\mathbf{k}_f)}{k_1(\omega_1(\mathbf{k}_f) + \omega_2(\mathbf{k}_f))} = k_1 \frac{\omega_1(\mathbf{k}_f)\omega_2(\mathbf{k}_f)}{(\omega_1(\mathbf{k}_f) + \omega_2(\mathbf{k}_f))} \tag{71}$$

combining with the velocity factor

$$d\sigma = 4\pi^2 \frac{\omega_1(\mathbf{k}_i)\omega_2(\mathbf{k}_i)}{|\mathbf{k}_i|(\omega_1(k_i) + \omega_1(k_i))} |\mathbf{k}_f| \frac{\omega_1(\mathbf{k}_f)\omega_2(\mathbf{k}_f)}{(\omega_1(\mathbf{k}_f) + \omega_2(\mathbf{k}_f))} |\langle \mathbf{k}_f \| T_r(m + i\epsilon) \| \mathbf{k}_i \rangle|^2 d\Omega(\hat{\mathbf{k}}_f) \tag{72}$$

for elastic scattering

$$d\sigma = \left(4\pi^2 \frac{\omega_1(\mathbf{k})\omega_2(\mathbf{k})}{(\omega_1(k) + \omega_1(k))} \right)^2 |\langle \mathbf{k}_f \| T_r(m + i\epsilon) \| \mathbf{k}_i \rangle|^2 d\Omega(\hat{\mathbf{k}}_f) \tag{73}$$

In the non-relativistic case the corresponding formula is

$$d\sigma = \frac{(2\pi)^4}{|\mathbf{k}_i|/m_1 + |\mathbf{k}_i|/m_2} |\langle \mathbf{k}_f \| T_n(h + i\epsilon) \| \mathbf{k}_i \rangle|^2 |\mathbf{k}_f|^2 \frac{\mu}{|\mathbf{k}_f|} d\Omega(\hat{\mathbf{k}}_f) = \tag{74a}$$

$$(4\pi^2\mu)^2 |\frac{|\mathbf{k}_f|}{|\mathbf{k}_i|}| |\langle \mathbf{k}_f \| T_n(h + i\epsilon) \| \mathbf{k}_i \rangle|^2 d\Omega(\hat{\mathbf{k}}_f) = \tag{74b}$$

$$\left(4\pi^2 \frac{\omega_1(\mathbf{k})\omega_2(\mathbf{k})}{(\omega_1(\mathbf{k}) + \omega_2(\mathbf{k}))} \right)^2 |\langle \mathbf{k}_f \| T_r(m + i\epsilon) \| \mathbf{k}_i \rangle|^2 d\Omega(\hat{\mathbf{k}}_f) \tag{74c}$$

which is identical to the relativistic expression, where we have used the on shell identity (62). This is no surprise, since we have already established that the on shell relativistic and non-relativistic cross sections are identical (this is equivalent to the wave functions and phase shifts being the same). This means that for two-particle physics you can use most of the non-relativistic results unchanged.

Remark - while it is a simple matter to multiply by the appropriate function of the momentum to express the relativistic half-shell transition matrix elements in terms of the non-relativistic half shell transition matrix elements, it is also possible to calculate the fully off shell one in terms of the non-relativistic fully off shell transition matrix elements. This is relevant for three-body scattering. To do this note

$$\frac{1}{z_1 - M} = \frac{1}{z_2 - M} + \frac{1}{z_2 - M}(z_2 - z_1)\frac{1}{z_1 - M} \tag{75}$$

which implies

$$T(z_1) = T(z_2) + V \frac{1}{z_2 - M}(z_2 - z_1)\frac{1}{z_1 - M} V. \tag{76}$$

This is equivalent to

$$T(z_1) = T(z_2) + T(z_2)\frac{1}{z_1 - M_0}(z_2 - z_1)\frac{1}{z_2 - M_0} T(z_1). \tag{77}$$

Given $T(z_2)$ on the left half shell (obtained from the non-relativistic $T(z_2)$, this equation can be used to shift the energy denominator to $T(z_1)$ is fully off shell.

$$\langle \mathbf{k}_2 \| T(z_1) \| \mathbf{k} \rangle = \langle \mathbf{k}_2 \| T(z_2) \| \mathbf{k} \rangle + \int \langle \mathbf{k}_2 \| T(z_2) \| \mathbf{k}'' \rangle \frac{1}{z_1 - M_0''} (z_2 - z_1) \frac{1}{z_2 - M_0''} d\mathbf{k}'' \langle \mathbf{k}'' \| T(z_1) \| \mathbf{k} \rangle \tag{78}$$

This requires half-shell input for all energies in the mesh, which means the equation has to be solved for each half shell momentum in the integration grid. Note that this equation has 2 integrable singularities, but because we are only interested in $z_1 \neq z_2$ they are at different places. The point is that it is possible to use non-relativistic half-shell matrices to calculate the fully off-shell relativistic T matrix. You never have to deal with the relativistic interaction that contains the square roots of the nucleon-nucleon potential.

The differences between the relativistic and non-relativistic theory arises when the partial wave basis is converted to a tensor product of single particle bases. This is important for scattering experiments involving electroweak probes.

Note that since V commutes with $\mathbf{j}$, the internal 2-body free total angular momentum, by simultaneously diagonalizing $M, \mathbf{P}, j^2, j_{zc}$ or $M, P^+, \mathbf{P}_\perp, j^2, j_{zf}$ the resulting eigenstates transform irreducibly under the Poincaré group. The dynamical unitary representation of the Lorentz group becomes

$$U(\Lambda)|(M_n, j, I)\mathbf{P}, m_j, m_t\rangle = \sum |(M_n, j, I))\mathbf{\Lambda}(\Lambda)P), m_j', m_t\rangle \sqrt{\frac{\omega_m(\mathbf{\Lambda}(\Lambda)P}{\omega_{M_n}(\mathbf{P})}} D^{1/2}_{m_{si}' m_{si}} [B_x^{-1}(\Lambda(\Lambda)P/M_n)\Lambda B_x(P/M_n)]. \tag{79a}$$

(79b)

This is different than the free tensor product representation because the dynamical mass eigenvalues, M_n, appear in the coefficients of the transformation.

What is needed to compute current matrix elements are scattering wave functions. Because the relativistic mass operator is a function of the non-relativistic center of mass Hamiltonian, the scattering wave functions are identical to the non-relativistic wave functions.

IV. SPINOR REPRESENTATIONS OF LORENTZ TRANSFORMATIONS:

Four vectors can be represented by 2×2 Hermitian matrices using the Pauli matrices as a basis for Hermitian matrices (with real coefficients):

$$X := x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(X\sigma_\mu) \tag{80}$$

where

$$\sigma_\mu := (I, \boldsymbol{\sigma}) \qquad \mathrm{Tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu} \qquad \sigma_\mu = \sigma_\mu^\dagger \tag{81}$$

are the 2×2 identity and the traceless Hermitian Pauli matrices. Note that

$$\det(X) = (x^0)^2 - (\mathbf{x})^2 = x^2. \tag{82}$$

is the Lorentz invariant proper time squared.

This means that any transformation that preserves both the determinant and Hermiticity of X is a 2×2 spinor representation of a real Lorentz transformation. The determinant and Hermiticity will be preserved if

$$\boxed{X' = \Lambda X \Lambda^\dagger} \tag{83}$$

where Λ is a complex 2×2 matrix ($SL(2, C)$) satisfying

$$\det(\Lambda) = 1. \tag{84}$$

It can be shown that all Lorentz transformations continuously connected to the identity can be put in this form. For the discrete transformations space-time reflection are given by

$$X \to X' = -X \tag{85}$$

while space reflections involve a complex conjugation

$$X' = \sigma_2 X^* \sigma_2 = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix}. \tag{86}$$

The discrete Lorentz transformations are **not** considered relativistic symmetries because they are broken by the weak interaction, however space reflections are relevant in neutrino physics.

The most general 2×2 matrix with determinant 1 can be expressed as

$$\Lambda = e^{\frac{\mathbf{z}\cdot\boldsymbol{\sigma}}{2}} \tag{87}$$

where $\mathbf{z}$ is a complex 3-vector. The factor of 1/2 is included for later convenience. The matrix Λ has a polar decomposition as a product of a positive Hermitian matrix P times a unitary matrix R:

$$\Lambda = \underbrace{(\Lambda\Lambda^\dagger\Lambda)^{1/2}}_{P}\underbrace{(\Lambda\Lambda^\dagger)^{-1/2}\Lambda}_{R} = PR \tag{88}$$

The positive Hermitian matrix,

$$P = P^\dagger > 0, \tag{89}$$

corresponds to a rotationless (canonical) boost and has the general form

$$P = e^{\frac{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}}{2}} = \sigma_0 \cosh(\frac{\rho}{2}) + \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\sigma}\sinh(\frac{\rho}{2}) \qquad \boldsymbol{\rho} = \text{rapidity} \tag{90}$$

while the unitary matrix R is a $SU(2)$ matrix

$$RR^\dagger = (\Lambda\Lambda^\dagger)^{-1/2}(\Lambda\Lambda^\dagger)(\Lambda\Lambda^\dagger)^{-1/2} = I \tag{91}$$

that can be expressed in the familiar form

$$R = e^{i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}} = \sigma_0 \cos(\frac{\theta}{2}) + i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin(\frac{\theta}{2}). \tag{92}$$

Note that from the definition of rapidity it follows that

$$\boxed{P^2 = P^\dagger P = PP^\dagger = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \sigma_0\cosh(\rho) + \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\sigma}\sinh(\rho) = \sigma_0\frac{p^0}{m} + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\sigma} = \frac{p}{m}\cdot\sigma} \tag{93}$$

which we will use later. Because of the relation to p in (93) we use the notation

$$P = P(p). \tag{94}$$

Equation (88) means that any Lorentz transformation continuously connected to the identity can be factored into a rotation R followed by a rotationless Lorentz transformation $P(p)$. Since a rotation does not change a rest 4-vector, the final momentum is determined by the positive matrix $P(p)$. This means the we can express a general boost $B(p)$ as

$$B(p) = P(p)R(p), \tag{95}$$

with the property

$$\boxed{B(p)B^\dagger(p) = P(p)R(p)R^\dagger P(p) = P^2(p) = \sigma_0\frac{p^0}{m} + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\sigma} = \frac{p}{m}\cdot\sigma} \tag{96}$$

which is independent of the type of boost.

It follows from the general representation

$$\Lambda = e^{\frac{\mathbf{z}\cdot\boldsymbol{\sigma}}{2}} \qquad \mathbf{z} = \text{complex vector} \tag{97}$$

that

$$\boxed{\tilde{\Lambda} = (\Lambda^\dagger)^{-1} = \sigma_2\Lambda^*\sigma_2.} \tag{98}$$

The related notation

$$\boxed{\tilde{\sigma}_\mu := \sigma_2\sigma_\mu^*\sigma_2 = (\sigma_0, -\boldsymbol{\sigma})} \tag{99}$$

will also be used. The definitions imply following identities that will be used in what follows

$$(\tilde{\Lambda})^\dagger = \Lambda^{-1} \qquad (\tilde{\Lambda})^{-1} = \Lambda^\dagger. \tag{100}$$

Equations (80) and (83) imply that

$$X' = x^{\mu\prime}\sigma_\mu = \Lambda^\mu{}_\nu x^\nu\sigma_\mu = \Lambda X\Lambda^\dagger = \Lambda\sigma_\nu x^\nu\Lambda^\dagger. \tag{101}$$

Equating the coefficients of x^ν gives the following transformation properties of the matrices σ_μ under Lorentz transformations

$$\boxed{\Lambda\sigma_\nu\Lambda^\dagger = \sigma_\mu\Lambda^\mu{}_\nu \qquad \Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu\Lambda\sigma_\nu\Lambda^\dagger).} \tag{102}$$

Note when there is no obvious ambiguity we will use Λ to represent both the 2×2 $SL(2,C)$ matrix and 4×4 Lorentz transformation related by (102).

If we take complex conjugates and multiply both sides of (102) by σ_2 on the right and left we get

$$\tilde{\Lambda}\sigma_2\sigma_\nu^*\sigma_2\tilde{\Lambda}^\dagger = \sigma_2\sigma_\mu^*\sigma_2\Lambda^\mu{}_\nu \tag{103}$$

or equivalently

$$\boxed{\tilde{\Lambda}\tilde{\sigma}_\nu\tilde{\Lambda}^\dagger = \tilde{\sigma}_\mu\Lambda^\mu{}_\nu} \tag{104}$$

which gives the correct transformation for a space reflected four vector.

V. THE EQUIVALENCE OF TWO AND FOUR COMPONENT SPINORS

Recall the relation between the 2×2 and 4×4 representation of Lorentz transformations

$$\Lambda^{\mu}{}_{\nu} := \frac{1}{2}\mathrm{Tr}(\sigma_{\mu}\Lambda\sigma_{\nu}\Lambda^{\dagger}). \tag{105}$$

Since the 4×4 Lorentz matrix, $\Lambda^{\mu}{}_{\nu}$, is real, taking complex conjugate of (105) gives

$$\Lambda^{\mu}{}_{\nu} := \Lambda^{\mu *}{}_{\nu} = \frac{1}{2}\mathrm{Tr}(\sigma^{*}_{\mu}\Lambda^{*}\sigma^{*}_{\nu}\Lambda^{\dagger *}) \tag{106}$$

Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ (106) can be replaced by

$$\Lambda^{\mu}{}_{\nu} := \frac{1}{2}\mathrm{Tr}(\sigma_2\sigma^{*}_{\mu}\sigma_2\sigma_2\Lambda^{*}\sigma_2\sigma_2\sigma^{*}_{\nu}\sigma_2\Lambda^{\dagger *}\sigma_2) = \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}_{\mu}(\Lambda^{-1})^{\dagger}\tilde{\sigma}_{\nu}\Lambda^{-1}). \tag{107}$$

The relevant observation is both the right and left handed (space reflected) representations give the same Lorentz transformation.

To understand the equivalence of Poincaré covariant representations and Lorentz covariant representations consider the unitary representation of the Poincaré group acting on simultaneous eigenstates of mass, spin, linear momentum, and spin projection with a delta function normalization,

$$\langle (m,j)p,\mu|(m',j')p',\mu'\rangle = \delta(\mathbf{p}-\mathbf{p}')\delta_{\mu\mu'}\delta_{jj'}\delta_{mm'},$$

$$U(\Lambda)|(m,j)p,\mu\rangle = \sum_{\nu}|\Lambda(m,j)p,\nu\rangle\sqrt{\frac{(\Lambda p)^0}{p^0}}isD^{j}_{\nu\mu}[B^{-1}(\Lambda p)\Lambda B(p)] =$$

$$\sum_{\nu}|(m,j)\Lambda p,\nu\rangle\sqrt{\frac{(\Lambda p)^0}{p^0}}D^{j}_{\nu\mu}[B^{\dagger}(\Lambda p)(\Lambda^{\dagger})^{-1}(B^{\dagger})^{-1}(p)] \tag{108}$$

where we used $R = (R^{\dagger})^{-1}$ in the Wigner rotation in the second line of (108). Here $B(p) = P(p)R(p)$ represents a general choice of $SL(2,C)$ boost.

The Wigner functions,

$$D^{j}_{\mu,\mu'}[R] = \langle s,\mu|U[R]|j,\mu'\rangle =$$

$$\sum_{k=0}^{j+\mu}\frac{\sqrt{(j+\mu)!(j+\mu')!(j-\mu)!(j-\mu')!}}{k!(j+\mu'-k)!(j+\mu-k)!(k-\mu-\mu')!}R^{k}_{++}R^{j+\mu'-k}_{+-}R^{j+\mu-k}_{-+}R^{k-\mu-\mu'}_{--} \tag{109}$$

where

$$R = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix} = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \sigma_0\cos(\frac{\theta}{2}) + i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin(\frac{\theta}{2}) \tag{110}$$

are degree $2j$ polynomials with real coefficients in the $SU(2)$ matrix elements R_{ij} which are entire functions of angles. This means that the group representation property and angular momentum addition laws can be analytically continued to complex angles (i.e. rapidities). The means that the group representation property and angular momentum addition laws

$$\sum_{\mu''} D^j_{\mu,\mu''}[R_2] D^j_{\mu'',\mu'}[R_1] - D^j_{\mu,\mu'}[R_2 R_1] = 0, \tag{111}$$

$$D^j_{\mu,\mu'}[R] - \sum_{j_1 j_2 \mu_1 \mu_2 \mu'_1 \mu'_2} \langle j,\mu|j_1,\mu_1,j_2,\mu_2\rangle D^{j_1}_{\mu_1,\mu'_1}[R] D^{j_2}_{\mu_2,\mu'_2}[R] \langle s_1,\mu'_1,s'_2,\mu'_2|j,\mu'\rangle = 0 \tag{112}$$

$$D^{j_1}_{\mu_1,\mu'_1}[R] D^{j_2}_{\mu_2,\mu'_2}[R] - \sum_{j\mu\mu'} \langle j_1,\mu_1,j_2,\mu_2|j,\mu\rangle D^j_{\mu,\mu'}[R] \langle j,\mu'|j_1,\mu'_1,j_2,\mu'_2\rangle = 0, \tag{113}$$

are also valid for $SL(2,C)$ matrices, i.e. $R \to \Lambda$. Using the group representation properties the Wigner rotations can be decomposed into products of D-functions of $SL(2,C)$ matrices and the boosts can be absorbed in the definitions of the states, resulting in new states that transform covariantly under $SL(2,C)$:

$$\boxed{U(\Lambda) \underbrace{\sum_\nu |(m,j)p,\nu\rangle \sqrt{p^0} D^j_{\nu\mu}[B^{-1}(p)]}_{|(m,j)p,\mu\rangle_{lc}} = \sum_\rho \underbrace{\sum_\nu |(m,j)\Lambda p,\nu\rangle \sqrt{(\Lambda p)^0} D^j_{\nu\rho}[B^{-1}(\Lambda p)]}_{|(m,j)\Lambda p,\rho\rangle_{lc}} D^j_{\rho\mu}[\Lambda]} \tag{114}$$

$$\boxed{U(\Lambda) \underbrace{\sum_\nu |(m,j)p,\nu\rangle \sqrt{p^0} D^j_{\nu\mu}[B^\dagger(p)]}_{|(m,j)p,\mu\rangle_{lc*}} = \sum_\rho \underbrace{\sum_\nu |\Lambda p,\nu\rangle \sqrt{(\Lambda p)^0} D^j_{\nu\rho}[B^\dagger(\Lambda p)}_{|(m,j)\Lambda p,\rho\rangle_{lc*}} D^j_{\rho\mu}[(\tilde{\Lambda}]} \tag{115}$$

where the lc subscript indicates that the states are Lorentz covariant.

The transformations relating $|(m,j)p,\nu\rangle$, $|(m,j)p,\nu\rangle_{lc}$, and $|(m,j)p,\nu\rangle_{lc*}$ are all invertible, so they are all **equivalent** ways of representing relativistic states. The differences are in the representations of Lorentz transformations

$$U(\Lambda)|(m,j)p,\mu\rangle_{lc} = \sum_\nu |(m,j)p,\nu\rangle_{lc} D^j_{\nu\mu}[\Lambda] \tag{116}$$

$$U(\Lambda)|(m,j)p,\mu\rangle_{lc*} = \sum_\nu |(m,j)p,\nu\rangle_{lc*} D^j_{\nu\mu}[\tilde{\Lambda}] \tag{117}$$

While it is known there are no finite dimensional unitary representations of the Lorentz group, this in not a contradiction because the Hilbert space inner product has a non-trivial momentum-dependent kernel. This can be seen by expressing the identity in terms of covariant states:

$$I = \sum_\mu \int |(m,j)p,\mu\rangle d\mathbf{p} \langle (m,j)p,\mu| =$$

$$\sum_{\mu\nu}\int |(m,j)p,\mu\rangle_{lc}\frac{d\mathbf{p}}{p^0}D^j_{\mu\nu}[B(p)B^\dagger(p)]_{lc}\langle(m,j)p,\nu| =$$

$$\sum_{\mu\nu}\int |(m,j)p,\mu\rangle_{lc}2d^4p\delta(p^2-m^2)\theta(p^0)D^j_{\mu\nu}[\sigma\cdot p]_{lc}\langle(m,j)p,\nu| =$$

$$\sum_{\mu\nu}\int |(m,j)p,\mu\rangle_{lc*}2d^4p\delta(p^2-m^2)\theta(p^0)D^j_{\mu\nu}[\tilde{\sigma}\cdot p]_{lc*}\langle(m,j)p,\nu| \qquad (118)$$

where we have used (96). The problem is that while $R=(R^\dagger)^{-1}$ for $SU(2)$ matrices, this is not true for $SL(2,C)$ matrices. In fact there is no constant similarity transformations that relates the two representations. They are called inequivalent representations. Equation (86) implies that the two representations are related by space reflection. What this means that is that the space reflected states will not transform correctly under Lorentz transformations in these representations.

One way to construct Lorentz covariant vectors that transform linearly with respect to space reflection is to replace (116) and (117) by the $4j+2$ component spinor states

$$|(m,j)p,\alpha\rangle_{cov} = \sqrt{p^0}\begin{pmatrix} \sum_\nu |(m,j)p,\nu\rangle D^j_{\nu\alpha}[B^{-1}(p)] \\ \sum_\nu |(m,j)p,\nu\rangle D^j_{\nu\alpha}[B^\dagger(p)] \end{pmatrix}. \qquad (119)$$

This is a $2j+1\times 2(2j+1)$ matrix in spin degrees of freedom. α takes on $2(2j+1)$ values; corresponding to the rows of this rectangular matrix. $\alpha_\pm$ is used to denote the first $(+)$ or last $(-)$ $2j+1$ components of α. The transformation properties are

$$U(\Lambda)\sqrt{p^0}\begin{pmatrix} \sum_\mu |(m,j)p,\mu\rangle D^j_{\mu\alpha_+}[B^{-1}(p)] \\ \sum_\mu |(m,j)p,\mu\rangle D^j_{\mu\alpha_-}[B^\dagger[(p)] \end{pmatrix} =$$

$$\sqrt{(\Lambda p)^0}\begin{pmatrix} \sum_\mu |(m,j)\Lambda p,\mu\rangle D^j_{\mu\beta}[B^{-1}(\Lambda p)] \\ \sum_\mu |(m,j)\Lambda p,\mu\rangle D^j_{\mu\beta}[B^\dagger[\Lambda p] \end{pmatrix}\begin{pmatrix} D^j_{\beta\alpha_+}[\Lambda] & 0 \\ 0 & D^j_{\beta\alpha_-}[(\tilde{\Lambda}] \end{pmatrix}. \qquad (120)$$

Note the following intertwining property

$$\sum_\nu D^j_{\mu\nu}[R_w(\Lambda,p)]\begin{pmatrix} \sum_\nu D^j_{\mu\nu}[R_w(\Lambda,p)]D^j_{\nu\beta_+}[B^{-1}(p)] \\ \sum_\nu D^j_{\mu\nu}[R_w(\Lambda,p)]D^j_{\nu\beta_-}[B^\dagger[p] \end{pmatrix} =$$

$$\begin{pmatrix} \sum_\nu D^j_{\mu\nu}[B^{-1}(\Lambda p)\Lambda B(p)]D^j_{\nu\beta_+}[B^{-1}(p)] \\ \sum_\nu D^j_{\mu\nu}[B^{-1}(\Lambda p)\Lambda B(p)]D^j_{\nu\beta_-}[B^\dagger[p] \end{pmatrix} =$$

$$\begin{pmatrix} \sum_\nu D^j_{\mu\nu}[B^{-1}(\Lambda p)\Lambda B(p)]D^j_{\nu\beta_+}[B^{-1}(p)] \\ \sum_\nu D^j_{\mu\nu}[B^{-1}(\Lambda p)\Lambda B(p)]D^j_{\nu\beta_-}[B^\dagger[(p)] \end{pmatrix} =$$

$$
\begin{pmatrix} \sum_\nu D^j_{\mu\nu}[B^{-1}(\Lambda p)\Lambda B(p)] D^j_{\nu\beta_+}[B^{-1}(p)] \\ \sum_\nu D^j_{\mu\nu}[B^\dagger(\Lambda p)(\Lambda^{-1})^\dagger (B^{-1})^\dagger(p)] D^j_{\nu\beta_-}[B^\dagger(p)] \end{pmatrix} =
$$

$$
\sum_\beta \begin{pmatrix} D^j_{\mu\beta_-}[B^{-1}(\Lambda p)] \\ D^j_{\mu\beta_+}[B^\dagger[(\Lambda p)] \end{pmatrix} \begin{pmatrix} D^j_{\beta_+\alpha_+}[\Lambda] & 0 \\ 0 & D^j_{\beta_-\alpha_-}[\tilde{\Lambda}] \end{pmatrix}. \tag{121}
$$

This shows that this combination of boosts maps Wigner rotations into representations of $SL(2,C)$.

This is for a general spin. For spin $\frac{1}{2}$ the covariant basis states have the form

$$
|(m,\frac{1}{2})p,\alpha\rangle_{cov} = \sqrt{p^0} \begin{pmatrix} \sum_\nu |(m,\frac{1}{2})p,\nu,\rangle B^{-1}(p)_{\nu\alpha_+} \\ \sum_\nu |(m,\frac{1}{2})p,\nu,\rangle(\tilde{B}^{-1}(p)_{\nu\alpha_-}) \end{pmatrix} \tag{122}
$$

and the representation of the unitary representation of the Lorentz group is

$$
\begin{pmatrix} D^{1/2}_{\beta_+\alpha_+}[\Lambda] & 0 \\ 0 & D^{1/2}_{\beta_-\alpha_-}[\tilde{\Lambda}] \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix} =: S(\Lambda) \tag{123}
$$

This defines the 4×4 spinor representation of the Lorentz group.

$$
U(\Lambda)|(m,\frac{1}{2})p,\alpha\rangle_{cov} = \sum_\beta \sqrt{(\Lambda p)^0} \begin{pmatrix} \sum_\nu |(m,\frac{1}{2})\Lambda p,\nu\rangle B^{-1}(\Lambda p) \\ \sum_\nu |(m,\frac{1}{2})\Lambda p,\nu\rangle(\tilde{B}^{-1}(\Lambda p)) \end{pmatrix}_{\nu\beta} \begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix}_{\beta\alpha} \tag{124}
$$

We remark that these 2×4 matrices transform Wigner rotations to representations of the $SL(2,C)$ group. The key observation is that the doubled 2 and four component representation are equivalent; it is possible to transform back and forth between them. The pronlem is that a consistent treatment of space reflection in the covariant representations requires having both a right and left (space reflected) representation.

We also remark that the kernels in the covariant representations are up to normalization and change of representation the 2-point Wightman functions of a free field theory.

VI. GAMMA MATRICES FROM SL(2,C)

We define the 4×4 representation of $SL(2,C)$ as the direct sum of the right and left handed representations:

$$
S(\Lambda) = \begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix} \tag{125}
$$

It follows from equations (102) and (104) that

$$S(\Lambda)\begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} S(\Lambda)^{-1} = \begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \tilde{\Lambda}^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \Lambda\sigma_\mu\Lambda^\dagger \\ \tilde{\Lambda}\tilde{\sigma}_\mu\tilde{\Lambda}^\dagger & 0 \end{pmatrix} = \sum_\nu \begin{pmatrix} 0 & \sigma_\nu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} \Lambda^\nu{}_\mu \tag{126}$$

This suggests the definition

$$\boxed{\gamma_\mu := \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} \qquad \gamma^\mu := \begin{pmatrix} 0 & \tilde{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}} \tag{127}$$

We also define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{128}$$

With definition (127) equation (126) can be expressed as

$$\boxed{S(\Lambda)\gamma_\mu S(\Lambda)^{-1} = \sum_\nu \gamma_\nu \Lambda^\nu{}_\mu} \tag{129}$$

It also follows from the definition (127) that

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad -i\sigma^{0i} = \frac{1}{2}[\gamma^0, \gamma^i] = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \qquad \sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}} \tag{130}$$

VII. DIRAC SPINORS

Consider the 2×4 matrices

$$\boxed{u(p)_{\mu,\alpha} := \frac{1}{\sqrt{2}} \begin{pmatrix} B(p)_{\mu\alpha_+} \\ \tilde{B}(p)_{\mu\alpha_-} \end{pmatrix} = S(B(p))\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 \\ \sigma_0 \end{pmatrix}} \tag{131}$$

and

$$\boxed{v(p)_{\mu,\alpha} := \frac{1}{\sqrt{2}} \begin{pmatrix} B(p)_{\mu\alpha_+} \\ -\tilde{B}(p)_{\mu\alpha_-} \end{pmatrix} = S(B(p))\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 \\ -\sigma_0 \end{pmatrix}} \tag{132}$$

Note that

$$(p^\mu\gamma_\mu - m)u(p) = (p^\mu\gamma_\mu - m)S(B(p))u(0) = S(B(p))S^{-1}(B(p))(p^\mu\gamma_\mu - m)S(B(p))u(0) =$$

$$S(B(p))(p^\mu S^{-1}(B(p))\gamma_\mu S(B(p)) - m)u(0) = S(B(p))(p^\mu\gamma_\nu B^{-1}(p)^\nu{}_\mu - m)u(0) =$$

$$S(B(p))(B^{-1}(p)^\nu{}_\mu p^\mu\gamma_\nu - m)u(0) = S(B(p))(m\gamma_0 - m)u(0) = 0 \tag{133}$$

which is the Dirac equation:

$$\boxed{(p^\mu\gamma_\mu - m)u(p) = 0.} \tag{134}$$

Similarly

$$(p^\mu\gamma_\mu + m)v(p) = (p^\mu\gamma_\mu + m)S(B(p))v(0) = S(B(p))S^{-1}(B(p))(p^\mu\gamma_\mu + m)S(B(p))v(0) =$$

$$S(B(p))(p^\mu S^{-1}(B(p))\gamma_\mu S(B(p)) + m)v(0) = S(B(p))(p^\mu\gamma_\nu B^{-1}(p)^\nu{}_\mu + m)v(0) =$$

$$S(B(p))(B^{-1}(p)^\nu{}_\mu p^\mu\gamma_\nu + m)v(0) = S(B(p))(m\gamma_0 + m)v(0) = 0 \tag{135}$$

which gives

$$\boxed{(p^\mu\gamma_\mu + m)v(p) = 0.} \tag{136}$$

Given these definitions define $u^\dagger(p)_{\alpha\mu}$, $\bar{u}(p)_{\alpha,\mu}$, $v^\dagger(p)_{\alpha\mu}$ and $\bar{v}(p)_{\alpha,\mu}$ by

$$u^\dagger(p)_{\alpha\mu} := \frac{1}{\sqrt{2}}\left((B^\dagger(p),\ \tilde{B}^\dagger(p) \right), \tag{137}$$

$$\bar{u}(p)_{\alpha\mu} := \frac{1}{\sqrt{2}}\left(\tilde{B}^\dagger(p),\ B^\dagger(p) \right) = \frac{1}{\sqrt{2}}\left(B^{-1}(p),\ \tilde{B}^{-1}(p) \right) \tag{138}$$

$$v^\dagger(p)_{\alpha\mu} := \frac{1}{\sqrt{2}}\left((B^\dagger(p),\ -\tilde{B}^\dagger(p) \right) \tag{139}$$

and

$$\bar{v}(p)_{\alpha\mu} := \frac{1}{\sqrt{2}}\left(-\tilde{B}^\dagger(p),\ B^\dagger(p) \right) = \frac{1}{\sqrt{2}}\left(-B^{-1}(p),\ \tilde{B}^{-1}(p) \right). \tag{140}$$

If we multiply these together we get

$$\bar{u}(p)u(p) = \sigma_0 \tag{141}$$

$$\bar{v}(p)v(p) = -\sigma_0. \tag{142}$$

The polar decomposition of $B(p)$ and $B^\dagger(p)$ gives

$$B(p) = P(p)R(p) \tag{143}$$

$$B^\dagger(p) = R^\dagger(p)P(p) \tag{144}$$

implies

$$B(p)B^\dagger(p) = P^2(p) = p \cdot \sigma \tag{145}$$

$$\tilde{B}(p)\tilde{B}^\dagger(p) = (B^\dagger(p))^{-1}B^{-1}(p) = p \cdot \tilde{\sigma} \tag{146}$$

These properties can be used to compute

$$u(p)\bar{u}(p) = \frac{1}{2}\begin{pmatrix} I & B(p)B^\dagger(p) \\ \tilde{B}(p)\tilde{B}^\dagger(p) & I \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I & \frac{p\cdot\sigma}{m} \\ \frac{p\cdot\tilde{\sigma}}{m} & I \end{pmatrix} =$$

$$\frac{1}{2m}(m + p^\mu \gamma_\mu) \tag{147}$$

$$v(p)\bar{v}(p) = \frac{1}{2}\begin{pmatrix} -I & B(p)B^\dagger(p) \\ \tilde{B}(p)\tilde{B}^\dagger(p) & -I \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -I & \frac{p\cdot\sigma}{m} \\ \frac{p\cdot\tilde{\sigma}}{m} & -I \end{pmatrix} =$$

$$\frac{1}{2m}(-m + p^\mu \gamma_\mu) \tag{148}$$

Using the polar decomposition and the property that for $SU(2)$ that $R = \tilde{R}$ gives

$$u^\dagger(p)u(p) = \frac{1}{2}(B^\dagger(p)B(p) + (\tilde{B}^\dagger(p)\tilde{B}(p)) = \frac{1}{2}(R^\dagger B_c^2(p)R + \tilde{R}^\dagger \tilde{B}_c^2(p)\tilde{R}) =$$

$$\frac{1}{2m}R^\dagger(\sigma \cdot p + \tilde{\sigma}\cdot p)R = \frac{p^0}{m}R^\dagger \sigma_0 R = \frac{p^0}{m}\sigma_0 \tag{149}$$

$$v^\dagger(p)v(p) = \frac{1}{2}(B^\dagger(p)B(p) + (-\tilde{B}^\dagger(p))(-\tilde{B}(p))) = \frac{1}{2}(R^\dagger B_c^2(p)R + \tilde{R}^\dagger \tilde{B}_c^2(p)\tilde{R}) =$$

$$\frac{1}{2m}R^\dagger(\sigma \cdot p + \tilde{\sigma}\cdot p)R = \frac{p^0}{m}R^\dagger \sigma_0 R = \frac{p^0}{m}\sigma_0 \tag{150}$$

$$u(p)u^\dagger(p) = \frac{1}{2}\begin{pmatrix} B(p)B^\dagger(p) & B(p)B^{-1}(p) \\ \tilde{B}(p)B^\dagger(p) & \tilde{B}(p)(\tilde{B}^\dagger(p) \end{pmatrix} = \begin{pmatrix} \frac{p\cdot\sigma}{2m} & I \\ I & \frac{p\cdot\tilde{\sigma}}{2m} \end{pmatrix} = \frac{1}{2m}(m + p^\mu\gamma_\mu)\gamma_0 \tag{151}$$

$$v(p)v^\dagger(p) = \frac{1}{2}\begin{pmatrix} B(p)B^\dagger(p) & -B(p)B^{-1}(p) \\ -\tilde{B}(p)B^\dagger(p) & \tilde{B}(p)(\tilde{B}^\dagger(p) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{p\cdot\sigma}{m} & -I \\ -I & \frac{p\cdot\tilde{\sigma}}{m} \end{pmatrix} = \frac{1}{2m}(-m + p^\mu\gamma_\mu)\gamma_0 \tag{152}$$

It is useful to summarize all of these properties

$$\boxed{\bar{u}(p)u(p) = -\bar{v}(p)v(p) = \sigma_0} \tag{153}$$

$$\boxed{u^\dagger(p)u(p) = v^\dagger(p)v(p) = \frac{p^0}{m}\sigma_0} \tag{154}$$

$$\boxed{u(p)\bar{u}(p) = \frac{1}{2m}(m + p^\mu \gamma_\mu) \qquad v(p)\bar{v}(p) = \frac{1}{2m}(-m + p^\mu \gamma_\mu)} \tag{155}$$

$$\boxed{u(p)u^\dagger(p) = \frac{1}{2m}(m + p^\mu \gamma_\mu)\gamma_0 \qquad v(p)v^\dagger(p) = \frac{1}{2m}(-m + p^\mu \gamma_\mu)\gamma_0} \tag{156}$$

It also follows that

$$\bar{u}(p)v(p) = \bar{v}(p)u(p) = 0 \tag{157}$$

Note that the rotation in the polar decomposition cancels out in all of these expressions. So while the spinors depend on the choice of boost, the quantities with the gamma matrices do not. The exception are the $u(p)$ and $v(p)$ spinors which are used to transform between the representations.

The operators

$$\boxed{u(p)\bar{u}(p) = \frac{1}{2m}(m + p^\mu \sigma_\mu) \qquad - v(p)\bar{v}(p) = \frac{1}{2m}(m - p^\mu \sigma_\mu)} \tag{158}$$

satisfy

$$u(p)\bar{u}(p) - v(p)\bar{v}(p) = I_{4\times 4} \tag{159}$$

$$(u(p)\bar{u}(p))^2 = \frac{1}{4m^2}(2m^2 + 2mp^\mu \sigma_\mu) = u(p)\bar{u}(p) \tag{160}$$

$$(-v(p)\bar{v}(p))^2 \frac{1}{4m^2}(2m^2 - 2mp^\mu \sigma_\mu) = -v(p)\bar{v}(p) \tag{161}$$

are projection operators - although note that they are not Hermitian matrices.

A useful property of $S(\Lambda)$ is

$$\gamma^0 S(\Lambda)\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & \Lambda \end{pmatrix} = (S(\Lambda)^\dagger)^{-1} \tag{162}$$

or

$$\boxed{\gamma^0 S^{-1}(\Lambda)\gamma^0 = S^\dagger(\Lambda)} \tag{163}$$

Here are some other spinor identities that are useful for dealing with current matrix elements

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\frac{\gamma^\alpha p'_\alpha}{m}\gamma^\mu \frac{\gamma^\beta p_\beta}{m} u(p) = \frac{p'_\alpha p_\beta}{m^2}\bar{u}(p')\gamma^\alpha \gamma^\mu \gamma^\beta u(p) =$$

$$\frac{p'_\alpha p_\beta}{2m^2}\bar{u}(p')(\gamma^\alpha \gamma^\mu \gamma^\beta + \gamma^\alpha \gamma^\mu \gamma^\beta)u(p) =$$

$$\frac{p'_\alpha p_\beta}{4m^2}\bar{u}(p')((\{\gamma^\alpha, \gamma^\mu\} + [\gamma^\alpha, \gamma^\mu]\gamma^\beta) + \gamma^\alpha(\{\gamma^\mu, \gamma^\beta\} + [\gamma^\mu, \gamma^\beta]))u(p) =$$

$$\frac{p'_\alpha p_\beta}{4m^2}\bar{u}(p')((2\eta^{\alpha\mu} - 2i\sigma^{\alpha\mu})\gamma^\beta) + \gamma^\alpha(2\eta^{\mu\beta} - 2i\sigma^{\mu\beta}))u(p) =$$

$$\frac{1}{2m^2}\bar{u}(p')((p'^\mu p_\beta\gamma^\beta - ip'_\alpha\sigma^{\alpha\mu})p_\beta\gamma^\beta) + p'_\alpha\gamma^\alpha(p^\mu - ip_\beta\sigma^{\mu\beta}))u(p) =$$

$$\frac{1}{2m}\bar{u}(p')((p'^\mu - ip'_\alpha\sigma^{\alpha\mu})) + (p^\mu - ip_\beta\sigma^{\mu\beta}))u(p) =$$

$$\frac{1}{2m}\bar{u}(p')((p'^\mu + p^\mu) + i(p'_\alpha - p_\alpha)\sigma^{\mu\alpha})u(p) \tag{164}$$

where we used the Dirac equation, $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ and $\gamma^0\gamma^\mu = (\gamma^\mu)^\dagger\gamma^0$. The identity (164) is called the Gordon identity, which gives the structure of current operators.

$$\boxed{\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')((p'^\mu + p^\mu) + i(p'_\alpha - p_\alpha)\sigma^{\mu\alpha})u(p)} \tag{165}$$

Similarly

$$\bar{u}(p')\gamma^\mu\gamma^5 u(p) = \bar{u}(p')\frac{\gamma^\alpha p'_\alpha}{m}\gamma^\mu\gamma^5\frac{\gamma^\beta p_\beta}{m}u(p) = -\frac{p'_\alpha p_\beta}{m^2}\bar{u}(p')\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^5 u(p) =$$

$$-\frac{p'_\alpha p_\beta}{2m^2}\bar{u}(p')(\gamma^\alpha\gamma^\mu\gamma^\beta + \gamma^\alpha\gamma^\mu\gamma^\beta)\gamma^5 u(p) =$$

$$-\frac{p'_\alpha p_\beta}{4m^2}\bar{u}(p')((\{\gamma^\alpha,\gamma^\mu\} + [\gamma^\alpha,\gamma^\mu]\gamma^\beta) + \gamma^\alpha(\{\gamma^\mu,\gamma^\beta\} + [\gamma^\mu,\gamma^\beta]))\gamma^5 u(p) =$$

$$-\frac{p'_\alpha p_\beta}{4m^2}\bar{u}(p')((2\eta^{\alpha\mu} - 2i\sigma^{\alpha\mu})\gamma^\beta) + \gamma^\alpha(2\eta^{\mu\beta} - 2i\sigma^{\mu\beta}))\gamma^5 u(p) =$$

$$-\frac{1}{2m^2}\bar{u}(p')((p'^\mu p_\beta\gamma^\beta - ip'_\alpha\sigma^{\alpha\mu})p_\beta\gamma^\beta) + p'_\alpha\gamma^\alpha(p^\mu - ip_\beta\sigma^{\mu\beta}))\gamma^5 u(p) =$$

$$-\frac{1}{2m}\bar{u}(p')(-(p'^\mu - ip'_\alpha\sigma^{\alpha\mu})) + (p^\mu - ip_\beta\sigma^{\mu\beta}))\gamma^5 u(p) =$$

$$\frac{1}{2m}\bar{u}(p')((p'^\mu - p^\mu) + i(-p'_\alpha + p_\alpha)\sigma^{\mu\alpha})\gamma^5 u(p)$$

$$\frac{1}{2m}\bar{u}(p')((p'^\mu - p^\mu) + i(p'_\alpha + p_\alpha)\sigma^{\mu\alpha})v(p) \tag{166}$$

or

$$\boxed{\bar{u}(p')\gamma^\mu\gamma^5 u(p) = \frac{1}{2m}\bar{u}(p')((p'^\mu - p^\mu) + i(p'_\alpha + p_\alpha)\sigma^{\mu\alpha})v(p)} \tag{167}$$

Since $\gamma^5 u(p) = v(p)$ and $\bar{u}(p)\gamma^5 = \bar{v}(p)$ we also have

$$\boxed{\bar{u}(p')\gamma^\mu\gamma^5 u(p) = \bar{u}(p')\gamma^\mu v(p) = \bar{v}(p')\gamma^\mu u(p)} \tag{168}$$

$$\boxed{\bar{u}(p')\gamma^\mu\gamma^5 v(p) = \bar{u}(p')\gamma^\mu u(p) = \bar{v}(p')\gamma^\mu v(p).} \tag{169}$$

We also note that the spinor Feynman propagator is

$$F(p) = \frac{p_\mu\gamma^\mu + m}{p^2 - m^2 = i0^+)} =$$

$$\frac{p_\mu \gamma^\mu + m}{(p^0 - \sqrt{\mathbf{p}^2+m^2} + i0^+)(p^0 + \sqrt{\mathbf{p}^2+m^2} - i0^+)} =$$
$$\frac{m}{\sqrt{m^2+\mathbf{p}^2}} \left(\frac{\bar{u}(p)u(p)}{p^0 - \omega_m(\mathbf{p}) + i0^+} + \frac{\bar{v}(p)v(p)}{p^0 + \omega_m(\mathbf{p}) - i0^+} \right) \tag{170}$$

where we have substituted the residue for each of the poles. This is where the v spinors can arise in the current. They arise when a charged particle couples to a time reversed Fermion in flight (the so-called Z-graphs). The expression in terms of the spinors is consistent with the normalizations used in this section.

VIII. GAMMA MATRIX CONVENTIONS - BJORKEN AND DRELL

In this section I discuss the relation of Bjorken and Drell conventions to the convention used in the previous section. The starting point is the choice representation of the gamma matrices. For Bjorken and Drell they are

$$\gamma^0 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \boldsymbol{\gamma} := \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \gamma^5 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{171}$$

These are compared to the conventions used in the previous section

$$\gamma^0 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \boldsymbol{\gamma} := \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \gamma^5 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{172}$$

For the lower indices the vector $\boldsymbol{\gamma}$ reverses sign in both expressions. These representations are related the similarity transformation

$$W := \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = W^{-1} \tag{173}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{174}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} -\boldsymbol{\sigma} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{175}$$

The starting point of the previous section from (126) is

$$\begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_2 \sigma_\mu^* \sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \tilde{\Lambda}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \Lambda \sigma_\mu \Lambda^\dagger \\ \tilde{\Lambda} \sigma_2 \sigma_\mu^* \sigma_2 \tilde{\Lambda}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_\nu \\ \sigma_2 \sigma_\nu^* \sigma_2 0 & \end{pmatrix} \Lambda^\nu{}_\mu \tag{176}$$

or

$$S(\Lambda)\begin{pmatrix} 0 & \sigma_\mu \\ \sigma_2\sigma_\mu^*\sigma_2 & 0 \end{pmatrix} S^{-1}(\Lambda) = \begin{pmatrix} \sigma_\nu & 0 \\ 0 & \sigma_2\sigma_\nu^*\sigma_2 \end{pmatrix} \Lambda^\nu{}_\mu \tag{177}$$

where recall

$$S(\Lambda) = \begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix}. \tag{178}$$

The corresponding $S(\Lambda)$ in the Bjorken and Drell representation is

$$S_{BD}(\Lambda) = WS(\Lambda)W^{-1} = \begin{pmatrix} \Lambda & 0 \\ 0 & \tilde{\Lambda} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Lambda+\tilde{\Lambda} & \Lambda-\tilde{\Lambda} \\ \Lambda-\tilde{\Lambda} & \Lambda+\tilde{\Lambda} \end{pmatrix} \tag{179}$$

The spinors in the BD representation are

$$u_{bd}(p) = \frac{1}{\sqrt{2}} S_{BD}(P(p))W \begin{pmatrix} \sigma_0 \\ \sigma_0 \end{pmatrix} = S_{BD}(P(p)) \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix} \tag{180}$$

and

$$v_{BD} = \frac{1}{\sqrt{2}} S_{BD}(P(p))W \begin{pmatrix} \sigma_0 \\ -\sigma_0 \end{pmatrix} = S_{BD}(P(p)) \begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix} \tag{181}$$

In this case $\tilde{P}(p) = P^{-1}(p)$ is the inverse of the canonical boost that is obtained by reversing the sign of the three momentum. The matrix

$$\gamma_\mu := \begin{pmatrix} 0 & \sigma_\nu \\ \sigma_2\sigma_\mu^*\sigma_2 0 & \end{pmatrix} \tag{182}$$

is consistent with my representation of γ^μ. Note the sign change when the spatial indices are raised. It shows

$$S(\Lambda)\gamma_\nu S(\Lambda)^{-1} = \gamma_\nu \Lambda_\mu^\nu \tag{183}$$

or if we raise indices

$$S(\Lambda)\gamma^\nu S(\Lambda)^{-1} = \gamma^\nu \Lambda_\nu^\mu = \Lambda^\mu{}_\nu \gamma^\nu \tag{184}$$

Multiplying by W on the left and right gives

$$S_{BD}(\Lambda)\gamma_{BD}^\nu S_{BD}(\Lambda)^{-1} = \gamma_{BD}^\nu \Lambda_\nu^\mu = \Lambda^\mu{}_\nu \gamma_{BD}^\nu \tag{185}$$

This is a representation because it is related by a similarity transformation to a representation.

Some general comments:

1. The explicit form of the u and v spinors depends on both the choice of boost $B(p)$ and the representation of $S(\Lambda)$. Bjorken and Drell use a different representation of $S(\Lambda)$ - see next section. They also use the canonical boost $B(p) = P(p)$ in the definition of the $u(p)$ and $v(p)$ spinors. Light front spinors in either representation involve replacing $P(p)$ by a light-front preserving boost.

2. While the $u(p)$ spinor relates the Poincaré covariant and Lorentz covariant representation of states the $v(p)$ spinors appear in the expression for the Feynman propagator and will contribute to exchange currents.

3. The expressions for the gamma matrices do not depend on the choice of boost. That is because the rotations in the polar decomposition cancel in the equations (96).

IX. CURRENT MATRIX ELEMENTS

In a canonical field theory plane wave current matrix elements can be expressed in terms of free Dirac field and creation and annihilation operators as:

$$\langle p',\nu'|j^\mu(x)|p,\nu\rangle = \langle 0|b(p',\nu')\bar{\Psi}(x)\gamma^\mu\Psi(x)b^\dagger(p,\nu)|0\rangle = \frac{e^{i(p'-p)x}}{(2\pi)^3}\sqrt{\frac{m^2}{p^0p'^0}}\bar{u}(p',\nu')\gamma^\mu u(p,\nu) \tag{186}$$

Here we use Bjorken and Drell delta function normalization of states and their normalization on the free fields -see BD 13.50 and 13.52. Matrix elements of the charge operator in plane wave states are obtained by integrating matrix elements of the charge density over all space. This gives matrix elements of the Noether charge in plane wave states:

$$\langle \mathbf{p}'\nu'|\int d\mathbf{x} j^0(\mathbf{x},0)|\mathbf{p},\mu\rangle =$$

$$\int \frac{e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}}{(2\pi)^3}\sqrt{\frac{m^2}{p^0p'^0}}\bar{u}(p',\nu')\gamma^\mu u(p,\mu) = \frac{m}{\sqrt{m^2+\mathbf{p}^2}}u^\dagger(p,\nu')\gamma^\mu u(p,\nu)\delta(\mathbf{p}-\mathbf{p}') =$$

$$\frac{m}{\sqrt{m^2+\mathbf{p}^2}}\frac{\sqrt{m^2+\mathbf{p}^2}}{m}\sigma_0\delta(\mathbf{p}-\mathbf{p}') = \delta_{\nu'\nu}\delta(\mathbf{p}-\mathbf{p}') = \langle \mathbf{p}',\nu'|\mathbf{p},\nu\rangle \tag{187}$$

which is what we expect for a point charge with charge $e = 1$. This shows that the one-body current matrix elements

$$\langle p', \nu' | j^\mu(x) | p, \nu \rangle = \frac{e^{i(p'-p)\cdot x}}{(2\pi)^3} \sqrt{\frac{m^2}{p^0 p'^0}} \bar{u}(p', \nu') \gamma^\mu u(p, \mu) \tag{188}$$

are consistent with the charge normalization.

The space integral over x leads to a momentum conserving delta function. This replaces the factor $\frac{e^{i(\mathbf{p}'-\mathbf{p})\cdot \mathbf{x}}}{(2\pi)^3}$ by a three-momentum conserving delta function.

Matrix elements of the 3-dimensional fourier transformation of the current operator

$$\tilde{j}^\mu(\mathbf{q}, t) = \frac{1}{(2\pi)^3} \int e^{-iq\cdot x} j^\mu(x) d\mathbf{x} \tag{189}$$

on the 0 time surface are

$$\langle p', \nu' | j^\mu(\mathbf{q}, 0) | p, \nu \rangle \delta(\mathbf{q} + \mathbf{p} - \mathbf{p}') = \bar{u}(p', \nu') \gamma^\mu u(p, \mu) \delta(\mathbf{q} + \mathbf{p} - \mathbf{p}') \tag{190}$$

X. COVARIANT DECOMPOSTION

A basis for 4×4 matrices can be expressed in terms of gamma matrices. The independent matrices are

$$\{I, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5, \gamma^5\gamma^\mu\} \tag{191}$$

It follows that 4×4 matrix can be represented as

$$M = aI + b_\mu \gamma^\mu + c_{\mu\nu}\sigma^{\mu\nu} + d\gamma^5 + e_\mu \gamma^5 \gamma^\mu \tag{192}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad c_{\mu\nu} = -c_{\nu\mu} \tag{193}$$

The coefficients can be computed using the following trace identities

$$\text{Tr}(I) = 4, \quad \text{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\text{Tr}(\{\gamma^\mu, \gamma^\nu\}) = -4\eta^{\mu\nu}, \tag{194}$$

$$\text{Tr}(\sigma^{\mu\nu}) = 0, \quad \text{Tr}(\gamma^5) = 0, \quad \text{Tr}(\gamma^5\gamma^\mu) = 0, \tag{195}$$

$$\text{Tr}(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu = 4(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}) \tag{196}$$

In addition to these identities we will need

$$\text{Tr}(\sigma^{\mu\nu}) = 0 \tag{197}$$

(follows from (**??**),

$$\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\alpha}) = 0 \tag{198}$$

(follows from odd number of gamma matrices),

$$\mathrm{Tr}(\sigma^{\mu\nu}\gamma^{5}) = 0 \tag{199}$$

follows because

$$\mathrm{Tr}(\gamma^{\nu}\gamma^{\mu}\gamma^{5}) = 0 \tag{200}$$

$$\mathrm{Tr}(M) = 4a \tag{201}$$

since

$$\mathrm{Tr}(\gamma^{\mu}) = \mathrm{Tr}(\sigma^{\mu\nu}) = \mathrm{Tr}(\gamma^{5}) = \mathrm{Tr}(\gamma^{5}\gamma^{\mu}) = 0 \tag{202}$$

$$Tr(\gamma^{\mu}M) = b_{\nu}Tr(\gamma^{\mu}\gamma^{\nu}) = -4\eta^{\mu\nu}b_{\nu} = -4b^{\mu} \tag{203}$$

since

$$\mathrm{Tr}(\gamma^{\alpha}I) = \mathrm{Tr}(\gamma^{\alpha}\sigma^{\mu\nu}) = \mathrm{Tr}(\gamma^{\alpha}\gamma^{5}) = \mathrm{Tr}(\gamma^{\alpha}\gamma^{5}\gamma^{\mu}) = 0 \tag{204}$$

$$\mathrm{Tr}(\gamma^{5}M) = 4d \tag{205}$$

$$\mathrm{Tr}(\gamma^{5}\gamma^{\mu}M)e_{\nu}\mathrm{Tr}(\gamma^{5}\gamma^{\mu}\gamma^{5}\gamma^{\nu}) = -e_{\nu}\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) == 4e_{\nu}\eta^{\mu\nu} = 4e^{\mu} \tag{206}$$

$$c_{\mu\nu}Tr(\sigma^{\alpha\beta}\sigma^{\mu\nu}) = -c_{\mu\nu}Tr(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\nu\mu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\beta\mu}\eta^{\alpha\nu} - \eta^{\beta\nu}\eta^{\alpha\mu}\eta^{\beta \tag{207}$$

$$-16c_{\mu\nu}(\eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\mu}\eta^{\beta\nu}) = 16c^{\alpha\beta} \tag{208}$$

XI. IMPULSE MATRIX ELEMENTS

To compute matrix elements it is useful to keep track of independent kinematic variables. The matrix element is a sum of terms with different spectator nucleons. In the Born terms there are no integrals. All variables can be expressed in terms of the total Deuteron momentum and the momenta of the two final nucleons.

$$\langle \mathbf{p}_1', \nu_1', \mathbf{p}_2', \nu_2' | j^{\mu}(0) | (D, j)\mathbf{P}, \mu; l, s\rangle =$$

$$\langle \mathbf{P}' - \mathbf{p}_2, \nu_1' | j_1^{\mu}(0) | \mathbf{p}_1, \nu_1\rangle D^{s_2}_{\nu_2'\mu_2}[B^{-1}(p_2)B(P)B(k_2)] D^{s_1}_{\nu_1\mu_1}[B^{-1}(p_1)B(P)B(k_1)] Y_{lm}(\hat{\mathbf{k}}(\mathbf{p}_1, \mathbf{p}_2)) \times$$

$$C(s_{12}\mu_{12}; s_1, \mu_1, s_2, \mu_2) C(j\mu; l, m, s_{12}\mu_{12}) \sqrt{\frac{\omega_1(p_1) + \omega_2(p_2)}{\omega_1(k_1) + \omega_2(k_2)}} \sqrt{\frac{\omega_1(k_1)\omega_2(k_2)}{\omega_1(p_1)\omega_2(p_2)}} \phi^{j}_{ls}(k) +$$

$$\langle \mathbf{P'} - \mathbf{p}_1, \nu_2' | j_2^\mu(0) | \mathbf{p}_2, \nu_2 \rangle D^{s_2}_{\nu_2\mu_2}[B^{-1}(p_2)B(P)B(k_2)] D^{s_1}_{\nu_1',\mu_1}[B^{-1}(p_1)B(P)B(k_1)] Y_{lm}(\hat{\mathbf{k}}(\mathbf{p}_1\mathbf{p}_2))\times$$

$$C(s_{12}\mu_{12}; s_1, \mu_1, s_2, \mu_2) C(j\mu; l, m; s_{12}\mu_{12}) \sqrt{\frac{\omega_1(p_1)+\omega_2(p_2)}{\omega_1(k_1)+\omega_2(k_2)}} \sqrt{\frac{\omega_1(k_1)\omega_2(k_2)}{\omega_1(p_1)\omega_2(p_2)}} \phi^j_{ls}(k) \tag{209}$$

where for the first term

$$\mathbf{p}_1 = \mathbf{P} - \mathbf{p}_2' \qquad \mathbf{k} = \mathbf{k}(\mathbf{P} - \mathbf{p}_2', \mathbf{p}_2') = \mathbf{p}_1 + \frac{\mathbf{P}}{M}\left(\frac{\mathbf{P}\cdot\mathbf{p}_1}{M+H} - \omega_1(p_1)\right) \tag{210}$$

and for second term

$$\mathbf{p}_2 = \mathbf{P} - \mathbf{p}_1' \qquad \mathbf{k} = \mathbf{k}(\mathbf{p}_1', \mathbf{P} - \mathbf{p}_1') = \mathbf{p}_1' + \frac{\mathbf{P}}{M}\left(\frac{\mathbf{P}\cdot\mathbf{p}_1'}{M+H} - \omega_1(p_1')\right) \tag{211}$$

where for both terms

$$H = \omega_1(p_1) + \omega_1(p_2) \qquad M = \sqrt{H^2 - \mathbf{P}^2} \tag{212}$$

$$k_2 = (\omega_2(k_1), -\mathbf{k}_1). \tag{213}$$

The boost are given by

$$D^{1/2}_{\mu\nu}[B(p)] = B(p)_{\mu\nu} = \sqrt{\frac{\sqrt{1+\mathbf{p}^2/m^2}+1}{2}}\sigma_{0\mu\nu} + \sqrt{\frac{\sqrt{1+\mathbf{p}^2/m^2}-1}{2}}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}_{\mu\nu} =$$

$$\sqrt{\frac{\sqrt{m^2+\mathbf{p}^2}+1}{2m}}\sigma_{0\mu\nu} + \sqrt{\frac{\sqrt{m^2+\mathbf{p}^2}-m}{2m}}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}_{\mu\nu} \tag{214}$$

$$D^{1/2}_{\mu\nu}[B^{-1}(p)] = B^{-1}(p)_{\mu\nu} =$$

$$\sqrt{\frac{\sqrt{1+\mathbf{p}^2/m^2}+1}{2}}\sigma_{0\mu\nu} - \sqrt{\frac{\sqrt{1+\mathbf{p}^2/m^2}-1}{2}}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}_{\mu\nu} =$$

$$\sqrt{\frac{\sqrt{m^2+\mathbf{p}^2}+m}{2m}}\sigma_{0\mu\nu} - \sqrt{\frac{\sqrt{m^2+\mathbf{p}^2}-m}{2m}}\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}_{\mu\nu}. \tag{215}$$

The current matrix elements are

$$\frac{1}{(2\pi)^3}\langle \mathbf{p}_i, \nu_i' | j_i^\mu(0) | \mathbf{p}_i, \nu_i\rangle = \sqrt{\frac{m}{\omega(\mathbf{p}_i)}}\bar{u}_{\nu_i'}(p')\Gamma^\mu u_{\nu_i}(p_i)\sqrt{\frac{m_i}{\omega(\mathbf{p}_i)}} \tag{216}$$

where Γ^μ is similar to (164 or 167) but will generally have form factors. The factors of $u(p)$ and Γ^μ must be in the same representation.

Note for the BD spinors

$$\frac{1}{2}(P(p) + P^{-1}(p)) = \sqrt{\frac{\sqrt{1+\mathbf{p}^2/m^2}+1}{2}}\sigma_{0\mu\nu} = \sqrt{\frac{m+\omega_m(\mathbf{p})}{2m}} \tag{217}$$

$$\frac{1}{2}(P(p) - P^{-1}(p)) = \sqrt{\frac{\sqrt{1+\mathbf{p}^2/m^2} - 1}{2}}\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}_{\mu\nu} = \sqrt{\frac{m+\omega_m(\mathbf{p})}{2m}} \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{m+\omega_m(\mathbf{p}} \tag{218}$$

which gives

$$S(P(p)) = \sqrt{\frac{m+\omega_m(\mathbf{p})}{2m}} \begin{pmatrix} \sigma_0 & \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{m+\omega_m(\mathbf{p}} \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{m+\omega_m(\mathbf{p}} & \sigma_0 \end{pmatrix} \tag{219}$$

which agrees with (3.7) of BD volume 1. The $u(p)$ and $v(p)$ BD spinors are defined by applying (219) to the rest spinors in the BD representation

$$u_{BD}(0) = W\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 \\ \sigma_0 \end{pmatrix} = \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix} \tag{220}$$

$$v_{BD}(0) = W\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 \\ -\sigma_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix} \tag{221}$$

They are just the columns of the matrix (219)

$$u_{BD}(p) = S(P(p) \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix} = \sqrt{\frac{m+\omega_m(\mathbf{p})}{2m}} \begin{pmatrix} \sigma_0 \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{m+\omega_m(\mathbf{p}} \end{pmatrix} = \sqrt{\frac{1+v^0}{2}} \begin{pmatrix} \sigma_0 \\ \frac{\mathbf{v}\cdot\boldsymbol{\sigma}}{1+v^0} \end{pmatrix} \tag{222}$$

$$v_{BD}(p) = S(P(p)) \begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix} = \sqrt{\frac{m+\omega_m(\mathbf{p})}{2m}} \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{m+\omega_m(\mathbf{p}} \\ \sigma_0 \end{pmatrix} = \sqrt{\frac{1+v^0}{2}} \begin{pmatrix} \frac{\mathbf{v}\cdot\boldsymbol{\sigma}}{1+v^0} \\ \sigma_0 \end{pmatrix} \tag{223}$$

where $v^\mu = p^\mu/m$ is the 4 velocity.

XII. RELATIVISTIC SCATTERING THEORY

A relativistic treatment of scattering is needed to model reactions that are sensitive to short-distance degrees of freedom. The formulation of scattering in relativistic quantum mechanics is identical essentially identical to the non-relativistic case.

The elementary quantum mechanical observable in a scattering experiment is the probability that a system prepared in an initial state is measured to be in a final state. These states are represented by normalizable solutions of the Schrödinger equation

$$|\psi_\alpha(t)\rangle = U(t)|\psi_\alpha(0)\rangle, \qquad |\psi_\beta(t)\rangle = U(t)|\psi_\beta(0)\rangle, \tag{224}$$

where $U(t) = e^{-iHt}$ is the unitary time evolution operator. For unit normalized vectors the transition probability for scattering from state β to state α is

$$\boxed{P_{\alpha\beta} := |\langle \psi_\alpha(t)|\psi_\beta(t)\rangle|^2 = |\langle \psi_\alpha(0)|U^\dagger(t)U(t)\psi_\beta(0)\rangle|^2 = |\langle \psi_\alpha(0)|\psi_\beta(0)\rangle|^2.} \tag{225}$$

The unitarity of the time evolution operator means that this probability can be evaluated at any *common* time.

If $t = 0$ represents the approximate time of collision, the initial state at a time long-before the collision is a state representing a target and a projectile where the mean positions of the target and projectile are separated beyond the range of the interaction and the mean momentum of the projectile is directed towards the target. Similarly, the final state at a time long after the collision represents mutually non-interacting asymptotically separated fragments with their mean momenta directed towards some detectors. The difficulty with computing the transition probability (225) is that there is no common time when both of these states have a simple structure. In addition, the initial and final states are not precisely known in any experiment. Both of these issues will be addressed below.

In applications it is necessary to consider a multichannel formulation of scattering theory. The formulation must allow for scattering from bound systems of particles and allow bound reaction products.

In order to evaluate the initial and final states at a common time it is useful to replace the initials condition in the Schrödinger equation by asymptotic conditions that fix both the initial and final states at times when they are simple and related to the experimental preparation. The first step is to describe the system long after or long before the scattering reaction.

In what follows the projectile, target and the fragments in the initial and final states are assumed to be particles or stable bound systems of particles. Cluster properties of the unitary representation of the Poincaré group mean that when it acts on a state of asymptotically separated subsystems that it can be approximated by a product of subsystem unitary representations of the Poincaré group:

$$(U(\Lambda, a) - \prod_i U_i(\Lambda, a))|\psi\rangle \approx 0 \tag{226}$$

where the state vector for a system of asymptotically separated particles can be represented by a product of wave packets for each particle:

$$|\psi\rangle = \prod_i |\psi_i\rangle \tag{227}$$

In (226) $\prod_i U_i(\Lambda, a))$ is obtained from $U(\Lambda, a)$ by turning off the interactions between each asymptotically separated particle or subsystem. Particles are identified with *point spectrum eigenstates* of the mass operator associated with each $U_i(\Lambda, a)$. These could be either elementary particles or subsystem bound states. Mass-spin eigenstates can be expanded in an irreducible basis. They have to be integrated against wave packets to construct normalizable states. The normalizable single-particle state vectors have the form

$$|\psi_i\rangle = \sum_{\mu_i=-j_i}^{j_i} \int |(m_i, j_i)\mathbf{p}_i, \mu_i\rangle d\mathbf{p}_i f_i(\mathbf{p}_i, \mu_i) \tag{228}$$

where $f_i(\mathbf{p}_i, \mu_i)$ is a square integrable function of the momentum and spin of the i-th particle. $|(m_i, j_i)\mathbf{p}_i, \mu_i\rangle$ is a point spectrum mass eigenstate of $U_i(\Lambda, a)$. The point spectrum mass eigenstates, $|(m_i, j_i)\mathbf{p}_i, \mu_i\rangle$, could represent elementary or composite particles with momentum $\mathbf{p}_i$ and spin projection μ_i. The functions $f_i(\mathbf{p}_i, \mu_i)$ are wave packets that determine the mean momentum, position and spin polarization of each particle. For now they can be taken as minimal uncertainty states of the form

$$f_i(\mathbf{p}_i, \mu_i) = \frac{c_{\mu_i}}{(2\pi)^{3/4}(\Delta p_i)^{3/2}} e^{-\frac{(\mathbf{p}_i - \mathbf{p}_{i0})^2}{(2\Delta p_i)^2}}. \tag{229}$$

where the c_μ are constants that determine the spin polarization. They are just Gaussian states in momentum space with a given mean momentum and momentum uncertainty. They are constructed so the uncertainty in the conjugate coordinates are determined by minimal uncertainty. Later we will construct observables that are insensitive to the structure of the wave packets.

In the absence of interactions the time evolution of these states is given by $U_i(I, t)$

$$|\psi_i(t)\rangle = \sum_{\mu_i=-j_i}^{j_i} \int U_i(I, t)|(m_i, j_i)\mathbf{p}_i, \mu_i\rangle d\mathbf{p} f_i(\mathbf{p}, \mu) =$$

$$\sum_{\mu_i=-j_i}^{j_i} \int |(m_i, j_i)\mathbf{p}_i, \mu_i\rangle e^{-i\sqrt{\mathbf{p}_i^2+m_i^2}t} d\mathbf{p} f_i(\mathbf{p}, \mu) \tag{230}$$

A two Hilbert space representation is used to formulate multi-particle scattering. A scattering channel labels a collection of initial or final particles (or bound states). The asymptotic

Hilbert space for channel α is the Hilbert space spanned by the products of square integrable functions, $f_i(\mathbf{p}_i, \mu_i)$ of the momenta and magnetic quantum numbers of particles (bound states) in the channel α. The channel-α Hilbert space is denoted by $\mathcal{H}_\alpha$.

A mapping Φ_α from $\mathcal{H}_\alpha$ to the Hilbert space of the quantum theory is defined by

$$\Phi_\alpha|f_\alpha\rangle := \prod_{i\in\alpha} |(m_i, j_i)\mathbf{p}_i, \mu_i\rangle f(\mathbf{p}_1, \mu_1 \cdots \mathbf{p}_n, \mu_n\rangle. \tag{231}$$

The unitary representation of the Poincaré group on $\mathcal{H}_\alpha$ is defined by

$$\otimes U_i(\Lambda, a)\Phi_\alpha|f_\alpha\rangle = \Phi_\alpha U_{f\alpha}(\Lambda, a)|f_\alpha\rangle. \tag{232}$$

The transformation properties of $U_{f\alpha}(\Lambda, a)$ on $|f_\alpha\rangle$ follow from the definitions

$$\otimes U_i(\Lambda, a)\Phi_\alpha|f_\alpha\rangle = \Phi_\alpha U_{f\alpha}(\Lambda, a)|f_\alpha\rangle =$$

$$\prod_i \int \sum U_i(\Lambda, a)|(m_i, j_i)\mathbf{p}_i, \mu_i\rangle d\mathbf{p}_i f(\mathbf{p}_1, \mu_1 \cdots \mathbf{p}_n, \mu_n) =$$

$$\prod_i \int \sum |(m_i, j_i)\mathbf{\Lambda} p_i, \nu_i\rangle e^{i\Lambda p_i\cdot a} \sqrt{\frac{\omega_{m_i}(\mathbf{\Lambda} p_i)}{\omega_{m_i}(\mathbf{p}_i)}} D^{j_i}_{\nu_i\mu_i}[R_w(\Lambda, p_i)] f(\mathbf{p}_1, \mu_1 \cdots \mathbf{p}_n, \mu_n)$$

$$\prod_i \int \sum |(m_i, j_i)\mathbf{p}'_i, \nu_i\rangle e^{ip'_i\cdot a} \sqrt{\frac{\omega_{m_i}(\mathbf{p}'_i)}{\omega_{m_i}(\mathbf{\Lambda}^{-1} p'_i)}} D^{j_i}_{\nu_i\mu_i}[R_w(\Lambda, \Lambda^{-1} p_i)] f(\mathbf{\Lambda}^{-1} p'_1, \mu_1 \cdots \mathbf{\Lambda}^{-1} p'_n, \mu_n). \tag{233}$$

The channel time evolution operator is

$$U_{f\alpha}(t) = e^{-iH_\alpha t} = e^{-i\sum_{j\in\alpha}\sqrt{m_j^2+\mathbf{p}_j^2}t} \tag{234}$$

where

$$H_\alpha = \sum_{j\in\alpha} \sqrt{m_j^2 + \mathbf{p}_j^2} \tag{235}$$

Note that if we express

$$H = H_a + V^a \tag{236}$$

where V^a represents the interactions in H between paticles in asymptotically separated cluster of the channel α and H_a includes all of the interaction between particles in the same clusters of α then

$$H_a\Phi_\alpha = \Phi_\alpha H_\alpha. \tag{237}$$

Equation (234) describes the time evolution of the mutually non-interacting particles or bound states in channel α.

The scattering probability for multi-channel scattering can be expressed in terms of the multi-channel scattering operator

$$P_{\alpha,\beta} = |S_{\alpha\beta}|^2. \tag{238}$$

The multi-channel scattering matrix is the probability amplitude

$$S_{\alpha\beta} = \langle \Psi^+_\alpha(0)|\Psi^-_\beta(0)\rangle \tag{239}$$

where α and β are channel labels and where the initial and final scattering states $|\Psi^+_\alpha(t)$ and $\rangle|\Psi^-_\beta(t)\rangle$ are solutions of the time-dependent Schrödinger equation with initial conditions replaced by the scattering asymptotic conditions

$$\lim_{t\to\pm\infty} \|\Psi^\pm_\alpha(t)\rangle - \Phi_\alpha|f^\pm_\alpha(t)\rangle\| = \lim_{t\to\pm\infty} \|e^{-iHt}|\Psi^\pm_\alpha(0)\rangle - \Phi_\alpha e^{-iH_\alpha t}|f^\pm_\alpha(0)\rangle\| = 0. \tag{240}$$

These equations define time-dependent solutions of the Schrödinger equation that look like non-interacting particles or bound states in the asymptotic past or future. Because this is a strong limit the unitary operator e^{iHt} can be removed replacing equation (240) by

$$\lim_{t\to\pm\infty} \|e^{iHt}(e^{-iHt}|\Psi^\pm_\alpha(0)\rangle - \Phi_\alpha e^{-iH_\alpha t}|f^\pm_\alpha(0)\rangle)\| = \lim_{t\to\pm\infty} \||\Psi^\pm_\alpha(0)\rangle - e^{iHt}\Phi_\alpha e^{-iH_\alpha t}|f^\pm_\alpha(0)\rangle)\| = 0 \tag{241}$$

which gives an expression for the states that appear in the expression for the scattering probability amplitude in terms of the free wave packets and bound state vectors.

Wave operators are defined as mappings from the asymptotic channel Hilbert space $\mathcal{H}_\alpha$ to the Hilbert space $\mathcal{H}$ of the theory

$$|\Psi^\pm_\alpha(0)\rangle = \Omega_{\alpha\pm}(H, \Phi_\alpha, H_\alpha)|f^\pm_\alpha(0)\rangle \tag{242}$$

where the multichannel wave operators are defined by

$$\boxed{\Omega_{\alpha\pm} = \lim_{t\to\pm\infty} e^{iHt}\Phi_\alpha e^{-iH_\alpha t}.} \tag{243}$$

The multichannel scattering operator can then be expressed in terms of the wave operators as

$$\boxed{S_{\alpha\beta} = \Omega^\dagger_{\alpha+}(H, \Phi_\alpha, H_\alpha)\Omega_{\beta-}(H, \Phi_\beta, H_\beta).} \tag{244}$$

In these notes the $\pm$ on the scattering states and wave operators indicates the direction of the time limit ($-$ =past/$+$ =future), which is opposite to the sign of $i\epsilon$. The operator

Π_α projects on the different possible scattering channels. The asymptotic and interacting scattering states are related by the multichannel wave operators

In order to evaluate the wave operators the first step is to express the limit as the integral of a derivative: Because the limit in (243) is a strong limit it is only defined when the operators are applied to wave packets, as they are in (242).

$$\Omega_{\alpha\pm} := \Phi_\alpha + \int_0^{\pm\infty} \frac{d}{dt}(e^{iHt}\Phi_\alpha e^{-H_\alpha t})dt = \Phi_\alpha + i\int_0^{\pm\infty} e^{iHt}H^a\Phi_\alpha e^{-iH_\alpha t}dt. \tag{245}$$

where we have used (236). Convergence follows provided

$$\| \int_0^{\pm\infty} e^{iHt}H^a\Phi_\alpha e^{-iH_\alpha t}dt|f_\alpha^{\pm}(0)\rangle\| < \infty. \tag{246}$$

A sufficient condition for this to be finite is

$$\int_0^{\pm\infty} \|e^{iHt}H^a\Phi_\alpha e^{-iH_\alpha t}dt|f_\alpha^{\pm}(0)\rangle\| < \infty \tag{247}$$

or equivalently by unitarity of e^{iHt}

$$\boxed{\int_0^{\pm\infty} \|H^a\Phi_\alpha e^{-iH_\alpha t}dt|f_\alpha^{\pm}(0)\rangle\| < \infty.} \tag{248}$$

Whether this is true depends on the interactions. The product $H^a\Phi_\alpha$ is translationally invariant, but it falls off in all relative directions. Intuitively the combination of the bound states in Φ_α and the interactions between particles in different bound states in H^a lead to terms that fall off for large time off like inverse powers of t for large t. In non-relativistic quantum mechanics this is called the Cook condition.

In what follows we assume that the channel Møller wave operators exist.

The channel Møller wave operators satisfy the **intertwining relations**

$$H\Omega_{\alpha\pm} = \Omega_{\alpha\pm}H_\alpha. \tag{249}$$

To prove (249) note that

$$e^{iHs}\Omega_{\alpha\pm} = \lim_{(t+s)\to\pm\infty} e^{iH(t+s)}\Phi_\alpha e^{-iH_\alpha(t+s)}e^{iH_\alpha s} = \Omega_{\alpha\pm}e^{iH_\alpha s}. \tag{250}$$

Differentiation with respect to s, setting s to zero gives (249). This condition ensures that energy is conserved in the scattering experiment. Let $|E_\alpha\rangle$ be an eigenstate of H_α with eigenvalues E_α. Then (249) gives

$$H\Omega_{\alpha\pm}|E_\alpha\rangle = \Omega_\pm H_\alpha|E_\alpha\rangle = \Omega_{\alpha\pm}E_\alpha|E_\alpha\rangle \tag{251}$$

which shows that $\Omega_{\alpha\pm}$ maps eigenstates of H_α with energy E_α to eigenstates of H with the same energy. This is a reflection of the fact that the energy of the scattering state is conserved and agrees with its values when the particles are asymptotically separated.

It also follows from (249) that

$$|\Psi^{\pm}_{\alpha}(t)\rangle = U(t)|\Psi^{\pm}_{\alpha}(0)\rangle = U(t)\Omega_{\alpha\pm}|f^{\pm}_{\alpha}(0)\rangle = \Omega_{\alpha\pm}U_\alpha(t)|f^{\pm}_{\alpha}(0)\rangle = \Omega_{\alpha\pm}|f^{\pm}_{\alpha}(t)\rangle. \tag{252}$$

The probability for scattering from a state in channel α to one in channel β can be expressed directly in terms of the asymptotic free-particle wave packets using channel Møller operators:

$$P_{\alpha\beta} = |\langle\Psi^{+}_{\beta}(t)|\Omega^{\dagger}_{\beta+}\Omega_{\alpha-}|\Psi^{-}_{\alpha}(t)\rangle|^2 = |\langle f^{+}_{\beta}(0)|\Omega^{\dagger}_{\beta+}\Omega_{\alpha-}|f^{-}_{\alpha}(0)\rangle|^2 \tag{253}$$

which is independent of t by (225).

The channel **scattering operator**, $S_{\beta\alpha}$, is defined by

$$\boxed{S_{\beta\alpha} := \Omega^{\dagger}_{\beta+}\Omega_{\alpha-}.} \tag{254}$$

The scattering probability can be expressed in terms of the channel asymptotic states and $S_{\beta\alpha}$ as

$$P_{\alpha\beta} = |\langle f^{+}_{\beta}(0)|S_{\beta\alpha}|f^{-}_{\alpha}(0)\rangle|^2 = |\langle f^{+}_{\beta}(t)|S_{\beta\alpha}|f^{-}_{\alpha}(t)\rangle|^2. \tag{255}$$

If we denote the set of all possible scattering channels by $\mathcal{A}$ then we can define multi-channel versions of the equations above. The asymptotic Hilbert space is the orthogonal direct sum of the channel Hilbert spaces

$$\mathcal{H}_{\mathcal{A}} := \oplus_{\alpha\in\mathcal{A}}\mathcal{H}_\alpha. \tag{256}$$

$$\Pi_\alpha\mathcal{H}_{\mathcal{A}} \to \mathcal{H}_\alpha \tag{257}$$

denotes the orthogonal projector from the asymptotic Hilbert to each channel Hilbert space $\mathcal{H}_\alpha$. The asymptotic Hamiltonian is defined by

$$H_{\mathcal{A}} := \sum_{\alpha\in\mathcal{A}} H_\alpha\Pi_\alpha \tag{258}$$

A Multichannel injection operator that maps the asymptotic Hilbert space to the Hilbert space of the theory is defined by

$$\Phi_{\mathcal{A}} := \sum_{\alpha\in\mathcal{A}} \Phi_\alpha\Pi_\alpha \tag{259}$$

Multichannel wave operators that map the asymptotic Hilbert space to the Hilbert space of the theory are defined by

$$\boxed{\Omega_\pm = \Omega_\pm(H,\Phi_\alpha,H_\alpha) := \lim_{t\to\pm\infty} e^{iHt}\Phi_{\mathcal{A}} e^{-iH_{\mathcal{A}}t},} \tag{260}$$

$$S := \Omega_+^\dagger \Omega_- : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}} \tag{261}$$

If the initial and final asymptotic states are chosen in channels β and α this becomes (255).

If the bound states are included in $\mathcal{A}$ and the incoming and outgoing scattering states both span the subspace orthogonal to the space spanned by the bound states, then the multi-channel wave operators are unitary mappings from $\mathcal{H}_{\mathcal{A}} \to \mathcal{H}$. This condition is called asymptotic completeness, which will be assumed in what follows.

The advantage of expressing the probability in terms of the asymptotic states is that they have a simple form where the asymptotic momenta an polarizations are controlled by experiment. The problem is that the probability in principle may depend on the detailed on the structure of the wave packets.

XIII. TIME INDEPENDENT MULTI-CHANNEL SCATTERING THEORY

In this section the definitions from the previous section are used to formulate the more familiar time-independent formulation of scattering. This will be used to remove the sensitivity to the choice of wave packets. To do this start with the time dependent expression for the channel scattering operator, expressing the time limit as the integral of a derivative

$$\begin{gathered}
\langle f_\beta^+(0)|S_{\beta\alpha}|f_\alpha^-(0)\rangle = \langle f_\beta^+(0)|\Omega_{\beta+}^\dagger \Omega_{\alpha-}|f_\alpha^-(0)\rangle = \lim_{t\to\infty} \langle f_\beta^+|e^{iH_\beta t}\Phi_\beta^\dagger e^{-2iHt}\Phi_\alpha e^{iH_\alpha t}|f_\alpha^-\rangle = \\
\langle f_\beta^+|\Phi_\beta^\dagger\Phi_\alpha|f_\alpha^-\rangle + \int_0^\infty dt \frac{d}{dt}\langle f_\beta^+|e^{iH_\beta t}\Phi_\beta^\dagger e^{-2iHt}\Phi_\alpha e^{iH_\alpha t}|f_\alpha^-\rangle = \\
\langle f_\beta^+|\Phi_\beta^\dagger\Phi_\alpha|f_\alpha^-\rangle + \int_0^\infty dt \langle f_\beta^+|e^{iH_\beta t}(-i\Phi_\beta^\dagger H^b)e^{-2iHt}\Phi_\alpha e^{iH_\alpha t}|f_\alpha^-\rangle + \\
\int_0^\infty dt \langle f_\beta^+|e^{iH_\beta t}\Phi_\beta^\dagger e^{-2iHt}(-iH^a\Phi_\alpha)e^{iH_\alpha t}|f_\alpha^-\rangle = \\
\langle f_\beta^+|\Phi_\beta^\dagger\Phi_\alpha|f_\alpha^-\rangle + \lim_{\epsilon\to 0^+}\int_0^\infty dt \langle f_\beta^+|e^{(iH_\beta-\epsilon)t}(-i\Phi_\beta^\dagger H^b)e^{-2iHt}\Phi_\alpha e^{(iH_\alpha-\epsilon)t}|f_\alpha^-\rangle + \\
\lim_{\epsilon\to 0^+}\int_0^\infty dt \int_0^\infty dt \langle f_\beta^+|e^{(iH_\beta-\epsilon)t}\Phi_\beta^\dagger e^{-2iHt}(-iH^a\Phi_\alpha)e^{(iH_\alpha-\epsilon)t}|f_\alpha^-\rangle
\end{gathered} \tag{262}$$

where we have used (236) again. Introducing the factors of ϵ leaves the result unchanged *provided that the integrals over the wave packets are performed before the time integral.* If the factors of ϵ remain then it is possible to change the order of the time integration and the integration over wave packets. Thus with the epsilon factors the time integral can be done by replacing the wave packets by energy eigenstates, and only after the limit $\epsilon \to 0$ integrating the wave packets over plane wave energy eigenstates,

In what follows we keep the factors of ϵ and replace $|f_\alpha^+\rangle$ and $|f_\beta^-\rangle$ by energy eigenstates $|E_\alpha^+\rangle$ and $|E_\beta^-\rangle$ of H_α and H_β. This has the advantage we can work with "plane-wave energy eigenstates. At the end of the calculation the wave packets have to be put back in.

With the ϵ factor (262) becomes, after removing the wave packets,

$$\Phi_\beta^\dagger \Phi_\alpha + \lim_{\epsilon\to 0}\int_0^\infty dt \Phi_\beta^\dagger e^{(iE_\beta-\epsilon)t} e^{-2iHt}(-iH^a)e^{(iE_\alpha+\epsilon)t}\Phi_\alpha +$$

$$\lim_{\epsilon\to 0}\int_0^\infty dt|\Phi_\beta^\dagger e^{(iH_\beta-\epsilon)t}(-iH^b)e^{-2iHt}e^{(iE_\alpha-\epsilon)t}\Phi_\alpha =$$

$$\Phi_\beta^\dagger \Phi_\alpha + \lim_{\epsilon\to 0}\Phi_\beta^\dagger \frac{1}{2i((E_\beta+E_\alpha)/2 - H + i\epsilon/2)}(iH^a)\Phi_\alpha +$$

$$\lim_{\epsilon\to 0}\Phi_\beta^\dagger (iH^b)\frac{1}{2i((E_\beta+E_\alpha)/2 - H + i\epsilon/2)}\Phi_\alpha . \tag{263}$$

We define the average energy by

$$\bar{E}_{\beta\alpha} := \frac{E_\beta + E_\alpha}{2}. \tag{264}$$

With definition (264) equation (263) becomes

$$\Phi_\beta^\dagger \Phi_\alpha + \lim_{\epsilon\to 0}\Phi_\beta^\dagger \frac{1}{2i(\bar{E}_{\beta\alpha} - H + i\epsilon/2)}(iH^a)\Phi_\alpha + \lim_{\epsilon\to 0}\Phi_\beta^\dagger (i(H^b)\frac{1}{2i(\bar{E}_{\beta\alpha} - H + i\epsilon/2)}\Phi_\alpha =$$

$$\Phi_\beta^\dagger \Phi_\alpha + \frac{1}{2}\lim_{\epsilon\to 0}\Phi_\beta^\dagger \frac{1}{(\bar{E}_{\beta\alpha} - H + i\epsilon/2)}H^a\Phi_\alpha + \frac{1}{2}\lim_{\epsilon\to 0}\Phi_\beta^\dagger H^b \frac{1}{(\bar{E}_{\beta\alpha} - H + i\epsilon/2)}\Phi_\alpha . \tag{265}$$

The second resolvent identities in the equivalent forms

$$\frac{1}{(\bar{E}_{\beta\alpha} - H + i\epsilon/2)} = \frac{1}{(\bar{E}_{\beta\alpha} - H_b + i\epsilon/2)}\left(I + H^b \frac{1}{(\bar{E}_{\beta\alpha} - H + i\epsilon/2)}\right) =$$

$$\left(I + \frac{1}{(\bar{E}_{\beta\alpha} - H + i\epsilon/2)}H^a\right)\frac{1}{(\bar{E}_{\beta\alpha} - H_a + i\epsilon/2)} \tag{266}$$

are inserted in (265) to get

$$= \Phi_\beta^\dagger \Phi_\alpha + \frac{1}{2}\lim_{\epsilon\to 0}\Phi_\beta^\dagger \frac{1}{(\bar{E}_{\beta\alpha} - H_b + i\epsilon/2)}\left(I + H^b \frac{1}{(\bar{E}_{\beta\alpha} - H + i\epsilon/2)}\right)H^a\Phi_\alpha +$$

$$\frac{1}{2}\lim_{\epsilon\to 0}\Phi_\beta^\dagger H^b\left(I+\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon/2)}H^a\right)\frac{1}{(\bar{E}_{\beta\alpha}-H_a+i\epsilon/2)}=$$

$$\Phi_\beta^\dagger\Phi_\alpha+\frac{1}{2}\lim_{\epsilon\to 0}\Phi_\beta^\dagger\frac{1}{(\bar{E}_{\beta\alpha}-E_\beta+i\epsilon/2)}\left(H^a+H^b\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon/2)}H^a\right)\Phi_\alpha+$$

$$\frac{1}{2}\lim_{\epsilon\to 0}\Phi_\beta^\dagger\left(H^b+H^b\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon/2)}H^a\right)\frac{1}{(\bar{E}_{\beta\alpha}-E_\alpha+i\epsilon/2)}=$$

$$\Phi_\beta^\dagger\Phi_\alpha+\lim_{\epsilon\to 0}\frac{1}{E_\alpha-E_\beta+i\epsilon/2}\Phi_\beta^\dagger\left(H^a+H^b\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon/2)}H^a\right)\Phi_\alpha+$$

$$\lim_{\epsilon\to 0}\frac{1}{E_\beta-E_\alpha+i\epsilon/2}\Phi_\beta^\dagger\left(H^b-H^a+H^a+H^b\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon/2)}H^a\right)\frac{1}{(\bar{E}_{\beta\alpha}-E_\alpha+i\epsilon/2)}=$$

$$\Phi_\beta^\dagger\Phi_\alpha+\lim_{\epsilon\to 0}\Phi_\beta^\dagger\frac{H-H_b-H+H_a}{E_\beta-E_\alpha+i\epsilon/2}\Phi_\alpha+$$

$$\lim_{\epsilon\to 0}\left(\frac{1}{E_\alpha-E_\beta+i\epsilon/2}+\frac{1}{E_\beta-E_\alpha+i\epsilon/2}\right)\Phi_\beta^\dagger\left(H^a+H^b\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon')}H^a\right)\Phi_\alpha=$$

$$\Phi_\beta^\dagger\Phi_\alpha\lim_{\epsilon\to 0}\Phi_\beta^\dagger\frac{E_\beta-E_\alpha}{E_\beta-E_\alpha+i\epsilon/2}\Phi_\alpha+$$

$$\lim_{\epsilon\to 0}\left(\frac{1}{E_\beta-E_\alpha+i\epsilon/2}+\frac{1}{E_\beta-E_\alpha+i\epsilon/2}\right)\Phi_\beta^\dagger\left(H^a+H^b\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon/2)}H^a\right)\Phi_\alpha=$$

$$\lim_{\epsilon\to 0}\Phi_\beta^\dagger\frac{i\epsilon/2}{E_\beta-E_\alpha+i\epsilon/2)}\Phi_\alpha$$

$$-2\lim_{\epsilon\to 0}\frac{\epsilon/2}{(E_\alpha-E_\beta)^2+(\epsilon/2)^2}\Phi_\beta^\dagger\left(H^a+H^b\frac{1}{(\bar{E}_{\beta\alpha}-H+i\epsilon/2)}H^a\right)\Phi_\alpha. \tag{267}$$

Taking the limit using

$$\lim_{\epsilon\to 0}\frac{\epsilon/2}{(E_\alpha-E_\beta)^2+(\epsilon/2)^2}=\pi\delta(E_\alpha-E_\beta) \tag{268}$$

gives

$$\boxed{S_{\alpha\beta}=\delta_{\beta\alpha}\Phi_\beta^\dagger\Phi_\alpha-2\pi i\delta(E_\alpha-E_\beta)\Phi_\beta^\dagger\left(H^a+H^b\frac{1}{(E_\alpha-H+i\epsilon/2)}H^a\right)\Phi_\alpha} \tag{269}$$

where the first term vanishes in the limit $\epsilon\to 0$ when the channels are different. When the channels are the same, we have two eigenstates of H_a that will be orthogonal unless the energies are identical, which results in the factor $\delta_{\beta\alpha}\Phi_\beta^\dagger$. Given the normalization condition this is just the identity in the channel Hilbert space $\mathcal{H}_\beta$. Because of the delta functions $\bar{E}_{\beta\alpha}=E_\alpha=E_\beta$.

To calculate the scattering probability these states have to be integrated over unit normalized initial and final wave packets. The operator that is needed to compute the non-trivial part of

$$S_{\alpha\beta} = \delta_{\beta\alpha}\Phi^\dagger_\beta\Phi_\alpha - 2\pi i\delta(E_\alpha - E_\beta)\Phi^\dagger_\beta T^{ab}\Phi_\alpha \tag{270}$$

is the transition operator

$$\boxed{T^{ba}(z) = H^a + H^b \frac{1}{z-H} H^a \qquad z = \bar{E}_\alpha + i\epsilon.} \tag{271}$$

The assumption that the unitary representation of the Poincaré group, $U(\Lambda, a)$, satisfies cluster properties means that the wave operators satisfy intertwining relations (see the discussion on page 120 of volume 1 of Weinberg's book on quantum field theory)

$$U(\Lambda, a)\Omega_{\pm\alpha} = \Omega_{\pm\alpha} U_\alpha(\Lambda, a) \tag{272}$$

A sufficient condition is that all of the Poincaré generators G_i satisfy a Cook-like condition

$$\int dt \|(G_i\Phi_\alpha - \Phi_\alpha G_{\alpha i})e^{-iH_\alpha t} f^\pm_\alpha(0)\rangle\| < \infty \tag{273}$$

This implies

$$U_\beta(\Lambda, a)S_{\beta\alpha} = U_\beta(\Lambda, a)\Omega^\dagger_{\beta+}\Omega_{\alpha-} = \Omega^\dagger_{\beta+}U(\Lambda, a)\Omega_{\alpha-} = \Omega^\dagger_{\beta+}\Omega_{\alpha-}U_\alpha(\Lambda, a). \tag{274}$$

For the full multi-channel scattering operator this means that

$$[U_{\mathcal{A}}(\Lambda, a), S] = 0 \tag{275}$$

which means that the scattering operator is Poincaré invariant.

In the non-relativistic case the Hamiltonian has the form

$$H = \frac{\mathbf{P}^2}{2M} + h. \tag{276}$$

Because of momentum conservation all calculations can be done by replacing H by h, which is equivalent to working in the rest frame of the system. The Poincaré invariance of the multi-channel scattering operator (275) means that the relativistic calculations can also be done in the rest frame of the system. Specifically boost invariance means that we can boost S to the rest frame without changing the operator. Mathematically if the interactions are well behaved the Kato-Birman theorem implies

$$\lim_{t\to\pm\infty} \|(e^{iHt}\Phi_\alpha e^{iH_\alpha t} - e^{ig(H)t}\Phi_\alpha e^{ig(H_\alpha)t})|f^\pm_\alpha(0)\rangle\| = 0 \tag{277}$$

for sufficiently nice functions $g(H)$. This is because everything Reimann -Lebesgues to death unless the energy is conserved - which results in operators satisfy intertwining relations. In the relativistic case choosing $g = \sqrt{H^2 - \mathbf{P}^2}$ means that we can replace H by M and H_α by $M_\alpha = \sqrt{H_\alpha^2 - \mathbf{P}^2\alpha}$ etc.. Mathematically the result

$$\lim_{t\to\pm\infty} \|(e^{iHt}\Phi_\alpha e^{iH_\alpha t} - e^{iMt}\Phi_\alpha e^{iM_\alpha t})|f_\alpha^\pm(0)\rangle\| = 0 \tag{278}$$

means that $S_{\alpha\beta}$ can also be expressed in terms of the mass operator, which is the relativistic analog of the non-relativistic h:

$$S_{\alpha\beta} = \delta_{\beta\alpha}\Phi_\beta^\dagger\Phi_\alpha - 2\pi i\delta(M_\alpha - M_\beta)\Phi_\beta^\dagger \left(M^a + M^b \frac{1}{(M_\alpha - M + i\epsilon/2)} M^a \right) \Phi_\alpha \tag{279}$$

In an instant form dynamics the interactions are translationally invariant which allows one to factor out a 3-momentum conserving delta function in addition to the energy conserving delta function.

In the particle data book reduced transition matrix elements are defined by

$$(2\pi)\delta(E' - E)\langle\mathbf{p}'|T^{ba}|\mathbf{p}\rangle = (2\pi)^4\delta(p' - p)\langle\mathbf{p}'\|T^{ba}\|\mathbf{p}\rangle \tag{280}$$

In this work I define the reduced matrix element by simply factoring our the momentum conserving delta fucntion:

$$\langle\mathbf{p}'|T^{ba}|\mathbf{p}\rangle = \delta(\mathbf{p}' - \mathbf{p})\langle\mathbf{p}'\|T^{ba}\|\mathbf{p}\rangle \tag{281}$$

without the factor of $(2\pi)^3$. This difference appears when we relate our formulas to those that appear in the particle data book which uses the convention (280).

XIV. CROSS SECTIONS

The problem with the scattering probability is that it depends on the structure of the initial and final wave packets. While there is some experimental control of the momenta and spins of the incident and scattered particles, it is not at the level of wave packets. The purpose of this section is to eliminate the sensitivity of the scattering observables to the choice of wave packet.

The relevant observable for the final states is the cross section. I develop it following methods used by Brenig and Haag. An initial state consisting of a target t in a state $|f_t\rangle$

and beam b in a state $|f_b\rangle$ leads to the asymptotic differential probability amplitude for a n-particle final state in channel α:

$$\langle \mathbf{p}_1, \mu_1, \cdots, \mathbf{p}_n, \mu_n | f_t f_b \rangle := \int \langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n | S_{\alpha\beta} | \mathbf{p}_b, \mu_b, \mathbf{p}_t, \mu_t \rangle d\mathbf{p}_b d\mathbf{p}_t \langle \mathbf{p}_b, \mu_b | f_b \rangle \langle \mathbf{p}_t, \mu_t | f_t \rangle . \tag{282}$$

The differential probability for observing each final particle to be within $d\mathbf{p}_i$ of $\mathbf{p}_i$ with spin polarization μ_i for this initial state is

$$dP = |\langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n | f_t f_b \rangle|^2 d\mathbf{p}_1 \cdots d\mathbf{p}_n . \tag{283}$$

Inserting the expression (270 and 272) for S in terms of the wave packets in (283), assuming either different initial channels or non-forward scattering, so there is no contribution from the $\Phi_\alpha^\dagger \Phi_\beta \delta_{\alpha\beta}$ part of the scattering operator, gives

$$dP = d\mathbf{p}_1 \cdots d\mathbf{p}_n \int (2\pi)^2 \langle \mathbf{p}_1, \mu_1, \cdots, \mathbf{p}_n, \mu_n \| \Phi_\alpha^\dagger T^{\alpha\beta} \Phi_\beta \| \mathbf{p}'_b, \mu_b, \mathbf{p}'_t, \mu_t \rangle \tag{284}$$

$$\times \ \langle \mathbf{p}_1, \mu_1, \cdots, \mathbf{p}_n, \mu_n \| T^{\alpha\beta} \| \mathbf{p}''_b, \mu_b, \mathbf{p}''_t, \mu_t \rangle^* \delta \left(\sum_i \mathbf{p}_i - \mathbf{p}'_b - \mathbf{p}'_t \right) \delta \left(\sum_j \mathbf{p}_j - \mathbf{p}''_b - \mathbf{p}''_t \right)$$

$$\times \ \delta(\mathsf{E}_\alpha - \mathsf{E}'_{bt}) \, \delta(\mathsf{E}_\alpha - \mathsf{E}''_{bt}) d\mathbf{p}'_b d\mathbf{p}'_t d\mathbf{p}''_b d\mathbf{p}''_t \langle \mathbf{p}'_b, \mu'_b | f_b \rangle \langle \mathbf{p}''_b, \mu''_n | f_b \rangle^* \langle \mathbf{p}'_t, \mu'_t | f_t \rangle \langle \mathbf{p}''_t, \mu''_t | f_t \rangle^* \tag{285}$$

where we have factored out three-dimensional momentum conserving delta functions assuming that the interactions are translationally invariant following (281).

The products of the 4-momentum conserving delta functions in (283) can be replaced by products of the equivalent four momentum conserving delta functions:

$$\delta^4 (\sum_i p_i - p'_b - p'_t) \, \delta^4 (\sum_j p_j - p''_b - p''_t) = \delta^4 (\sum_i p_i - p'_b - p'_t) \, \delta^4 (p'_b + p'_t - p''_b - p''_t). \tag{286}$$

If the initial wave packets are sharply peaked about the target and beam momenta and the transition operator varies slowly on the support of these wave packets, then the transition operators can be factored out of the integral, replacing the beam and target momenta in the transition matrix elements with the mean target and beam momenta, $\bar{\mathbf{p}}_b, \bar{\mathbf{p}}_t$. When this approximation is justified the cross section will be independent of the shape of the wave packets. The result, after expressing the second four momentum conserving delta function in (286) using a Fourier representation

$$\delta^4 (p'_b + p'_t - p''_b - p''_t) = (\frac{1}{2\pi^4} \int e^{i x \cdot (p'_b + p'_t - p''_b - p''_t)} d^4 x \tag{287}$$

is

$$dP = (2\pi)^4 d\mathbf{p}_1, \cdots d\mathbf{p}_n \int \left| \langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n \| \Phi^\dagger_\alpha T^{\alpha\beta} \Phi_\beta \| \bar{\mathbf{p}}_b, \mu_b \bar{\mathbf{p}}_t, \mu_t \rangle \right|^2 \times$$
$$\times \; |\langle \mathbf{x}, t, \mu_b | f_b \rangle|^2 |\langle \mathbf{x}, t, \mu_t | f_t \rangle|^2 d\mathbf{x}\, dt \delta \left(\sum_i \mathbf{p}_i - \bar{\mathbf{p}}_b - \bar{\mathbf{p}}_t \right) \delta \left(\sum_i \mathsf{E}_{p_i} - \bar{\mathsf{E}}_b - \bar{\mathsf{E}}_t \right) \tag{288}$$

where

$$\langle \mathbf{x}, t, \mu_b | f_b \rangle := \int \frac{d\mathbf{p}_b}{(2\pi)^{3/2}} e^{i\mathbf{p}_b \cdot \mathbf{x} - iE_b(p)t} \langle \mathbf{p}, \mu | f_t \rangle \qquad \langle \mathbf{x}, t, \mu_t | f_t \rangle := \int \frac{d\mathbf{p}_t}{(2\pi)^{3/2}} e^{i\mathbf{p}_t \cdot \mathbf{x} - iE_t(p)t} \langle \mathbf{p}, \mu | f_t \rangle \tag{289}$$

are time-dependent wave packets for the beam and target particles. This is the differential probability for a single scattering event. The space-time integral picks up a contribution whenever the beam and target are in the same place at the same time. For a single event this space-time volume is finite.

In a real experiment there is a beam of particles with current

$$\mathbf{j}_b = \mathbf{v}_{bt} n_b \tag{290}$$

where $\mathbf{v}_{bt}$ is the relative velocity between the beam and target particles and n_b is the number of beam particles per unit volume. The beam is normally incident on a target with n_t target particles per unit volume. Assuming that each beam particle scatters at most once the number of particles scattered per unit volume per unit time is proportional to both the target density and the normal component of the beam current

$$\frac{dN_{sc}}{dV dt} = n_t j_b d\sigma = n_t n_n v_{bt} d\sigma \tag{291}$$

where the constant of proportionality $d\sigma$ defines the different cross section, which by dimensional analysis has units of area.

On the other hand the total number of scattering events is equal to the probability of a scattering event per unit time per unit volume, times the total number of beam and total numbers of target particles integrated over all space and time

$$N_{sc} = N_b N_t \int \frac{dP}{dV dt} dV dt = \int \frac{dN_{sc}}{dV dt} dV dt. \tag{292}$$

It follows that

$$\frac{dN_{sc}}{dV dt} = N_b N_t \frac{dP}{dV dt} \qquad n_b = N_b |\langle \mathbf{x}, \mu_b | f_b \rangle|^2 \qquad n_t = N_t |\langle \mathbf{x}, \mu_t | f_t \rangle|^2 \tag{293}$$

Using (288) in (293)

$$\frac{dN_{sc}}{dVdt} = (2\pi)^4 d\mathbf{p}_1, \cdots d\mathbf{p}_n \int \left|\langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n \| \Phi^\dagger_\alpha T^{\alpha\beta} \Phi_\beta \| \bar{\mathbf{p}}_b, \mu_b, \bar{\mathbf{p}}_t, \mu_t \rangle\right|^2 \times$$

$$\underbrace{N_b |\langle \mathbf{x}, t, \mu_b | f_b \rangle|^2}_{n_b} \underbrace{N_t |\langle \mathbf{x}, t, \mu_t | f_t \rangle|^2}_{nt} \delta\left(\sum_i \mathbf{p}_i - \bar{\mathbf{p}}_b - \bar{\mathbf{p}}_t\right) \delta\left(\sum_i \mathsf{E}_{p_i} - \bar{\mathsf{E}}_b - \bar{\mathsf{E}}_t\right). \qquad (294)$$

comparing (294) to (291) results in the following expression for the differential cross section

$$\boxed{d\sigma = \frac{(2\pi)^4}{v_{bt}} d\mathbf{p}_1, \cdots d\mathbf{p}_n \int \left|\langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n \| \Phi^\dagger_\alpha T^{\alpha\beta} \Phi_\beta \| \bar{\mathbf{p}}_b, \mu_b, \bar{\mathbf{p}}_t, \mu_t \rangle\right|^2 \times \delta\left(\sum_i \mathbf{p}_i - \bar{\mathbf{p}}_b - \bar{\mathbf{p}}_t\right) \delta\left(\sum_i \mathsf{E}_{p_i} - \bar{\mathsf{E}}_b - \bar{\mathsf{E}}_t\right).}$$

(295a)

(295b)

Where everything we have done is fully relativistic.

The important observation is that this expression is not sensitive to the choice of initial wave packets provided they are sufficiently narrow. For the final states the cross section is proportional to the number of particles detected within $d\mathbf{p}_1$ of $\mathbf{p}_1$, $\cdots$ $d\mathbf{p}_n$ of $\mathbf{p}_n$ with spin polarizations $\mu_1, \cdots \mu_n$ assuming the beam and target particles had polarization μ_b and μ_t. This is again insensitive to the structure of the final wave packets and corresponds to a quantity that can be measured in the laboratory.

Integrating the cross section over the area of the surface of a large sphere give the total cross section which is dimensionless. It is a Lorentz invariant quantity. This means that we should be able to express the cross section in terms of Lorentz covariant quantities.

To extract the standard expression for the invariant amplitude the single particle states are replaced by states with the covariant normalization used in the particle data book [**?**]:

The first step is to replace the phase space factors by the corresponding invariant phase space factors

$$\delta\left(\sum_i \mathbf{p}_i - \bar{\mathbf{p}}_b - \bar{\mathbf{p}}_t\right) \delta\left(\sum_i \mathsf{E}_{p_i} - \bar{\mathsf{E}}_b - \bar{\mathsf{E}}_t\right) d\mathbf{p}_1, \cdots d\mathbf{p}_n \to$$

$$\delta^4(p_b + p_t - \sum p_i) \prod_{i=1}^N \frac{d\mathbf{p}_i}{(2\pi)^3 2E_i}. \qquad (296)$$

These are invariant since each factor of $\frac{d\mathbf{p}_i}{2E_i}$ is manifestly invariant,

$$\int \frac{d\mathbf{p}_i}{2E_i} = \int \delta(p^2 - m^2)\theta(p^0)d^4p_i, \tag{297}$$

while the factors of $(2\pi)^{-3}$ a particle data book convention.

In order to cancel these factors the inverses are put in the expression for the transition matrix elements

$$\langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n \| \Phi_\alpha^\dagger T^{\alpha\beta} \Phi_\beta \| \bar{\mathbf{p}}_b, \mu_b, \bar{\mathbf{p}}_t, \mu_t \rangle \to \langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n \| M^{\alpha\beta} \| \bar{\mathbf{p}}_b, \mu_b, \bar{\mathbf{p}}_t, \mu_t \rangle := \tag{298a}$$

$$(2\pi)^3 \prod_{i=1}^n \sqrt{(2\pi)^3 2E_i} \langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n \| \Phi_\alpha^\dagger T^{\alpha\beta} \Phi_\beta^\dagger \| \bar{\mathbf{p}}_b, \mu_b, \bar{\mathbf{p}}_t, \mu_t \rangle \sqrt{(2\pi)^3 2E_b} \sqrt{(2\pi)^3 2E_t} \tag{298b}$$

The additional factor of $(2\pi)^3$ is becasue the particle data book convention defines the reduced transition matrix elements using (280) rather than (281).

The energy factors make the plane wave states into states that transform covariantly (see 114-115). In this case the various factors of $(2\pi)^{3/2}$ are part of the particle data book conventions - they account for the corresponding factors in the invariant phase space. What remains is

$$\frac{(2\pi)^4}{4E_bE_tv_{bt}} \tag{299}$$

It is easy to check that $E_bE_tv_{bt}$ is an invariant quantity

$$E_bE_tv_{bt} = \sqrt{(p_t \cdot p_b)^2 - m_b^2 m_t^2} \tag{300}$$

The resulting formula for the differential cross section becomes

$$d\sigma = \frac{(2\pi)^4}{4\sqrt{(p_t \cdot p_b)^2 - m_b^2 m_t^2}} |\langle \mathbf{p}_1, \mu_1 \cdots, \mathbf{p}_n, \mu_n \| M^{\alpha\beta} \| \bar{\mathbf{p}}_b, \mu_b, \bar{\mathbf{p}}_t, \mu_t \rangle|^2 \delta^4(p_b + p_t - \sum p_i) \prod_{i=1}^N \frac{d\mathbf{p}_i}{(2\pi)^3 2E_i}. \tag{301}$$

which is exactly the expression in the particle data book, where here the reduced $T^{\alpha\beta}$ has the normalization (281). In terms of the standard definition of $T^{\alpha\beta}$ and states with delta function normalization the cross section is

$$d\sigma = \frac{(2\pi)^4 \omega_b(\mathbf{p}_b)\omega_t(\mathbf{p}_t)}{\sqrt{(p_t \cdot p_b)^2 - m_b^2 m_t^2}} |\langle \mathbf{p}_1', \mu_1', \cdots \mathbf{p}_n', \mu_n' \| T^{\alpha\beta} \| \bar{\mathbf{p}}_b, \mu_b, \bar{\mathbf{p}}_t, \mu_t \rangle|^2 \delta^4(p_b + p_t - \sum p_i) d\mathbf{p}_1' \cdots d\mathbf{p}_n'. \tag{302}$$

This expression is an idealization with respect to the polarizations. No experiment has perfectly polarized targets, beams or can perfectly identify polarizations in detectors. The uncertainties can be treated using density matrices.

We assume that the polarizations in the target and beam have a classical probability distribution given by the density matrices

$$\rho_b := |\mu_b\rangle P_{b\mu_b}\langle\mu_b| \times I_t \tag{303}$$

$$\rho_t := |\mu_t\rangle P_{t\mu_t}\langle\mu_t| \times I_b \tag{304}$$

where $P_{b\mu_b}$ and $P_{t\mu_t}$ represent the classical probability for a particle in the beam (target) to have spin polarization μ_b (μ_t). Averaging over initial over initial initial spin and target states gives

$$d\sigma = \sum_{\mu_b\mu_t} \frac{(2\pi)^4}{4\sqrt{(p_t\cdot p_b)^2 - m_b^2 m_t^2}} \langle \mathbf{p}_1,\mu_1 \cdots, \mathbf{p}_n,\mu_n \| M^{\alpha\beta} \| \bar{\mathbf{p}}_b,\mu_b,\bar{\mathbf{p}}_t,\mu_t\rangle \rho_{b\mu_b\mu_b}\rho_{t\mu_t\mu_t}\times$$

$$\langle \bar{\mathbf{p}}_b,\mu_b,\bar{\mathbf{p}}_t,\mu_t \| M^{\alpha\beta\dagger} \| \mathbf{p}_1,\mu_1\cdots,\mathbf{p}_n,\mu_n\rangle \delta^4(p_b+p_t-\sum p_i)\prod_{i=1}^N \frac{d\mathbf{p}_i}{(2\pi)^3 2E_i}. \tag{305}$$

In these expression we chose spin bases where the density matrices are diagonal; in general the density matrices are Hermitian matrices with unit trace. In a general spin basis the above expression is equivalent to

$$d\sigma = \sum_{\mu_b\mu_t,\mu_b',\mu_t'} \frac{(2\pi)^4}{4\sqrt{(p_t\cdot p_b)^2 - m_b^2 m_t^2}} \langle \mathbf{p}_1,\mu_1 \cdots, \mathbf{p}_n,\mu_n \| M^{\alpha\beta} \| \bar{\mathbf{p}}_b,\mu_b,\bar{\mathbf{p}}_t,\mu_t\rangle \rho_{b\mu_b\mu_b'}\rho_{t\mu_t\mu_t'}\times$$

$$\langle \bar{\mathbf{p}}_b,\mu_b',\bar{\mathbf{p}}_t,\mu_t' \| M^{\alpha\beta\dagger} \| \mathbf{p}_1,\mu_1\cdots,\mathbf{p}_n,\mu_n\rangle \delta^4(p_b+p_t-\sum p_i)\prod_{i=1}^N \frac{d\mathbf{p}_i}{(2\pi)^3 2E_i}. \tag{306}$$

Computationally the final state information can be encoded a final state density matrix. This is normally treated by constructing a basis of independent $((2j_1+1)(2j_2+1)\cdots(2j_n+1))^2$ Hermitian matrices S_i with the property

$$\mathrm{Tr}(S_i S_j) = \delta_{ij} \tag{307}$$

The relevant polarization observable is

$$P^i = \frac{\mathrm{Tr}(S^i M\rho_t\rho_b M^\dagger)}{\mathrm{Tr}(M\rho_t\rho_b M^\dagger)} \tag{308}$$

which be used to treat any kind of final state polarization. In this representation a general polarization observable can be computed as

$$\langle O\rangle = \frac{\mathrm{Tr}(OM\rho_t\rho_b M^\dagger)}{\mathrm{Tr}(M\rho_t\rho_b M^\dagger)} = \mathrm{Tr}(OS^i)P^i. \tag{309}$$

XV. TWO POTENTIAL SCATTERING

When the Hamiltonian is a linear combination of an interaction that must be treated non-perturbatively one that can be treated perturbatively it is useful to use the so called two-potential formalism of Gell-Mann and Goldberger. To illustrate how this works assume a Hamiltonian of the form

$$H = H_0 + V_s + V_w \tag{310}$$

where V_s is strong and V_w is weak. First consider the case of two-body scattering where both interactions are short range interactions. In this H_0 is the asymptotic Hamiltonian and the scattering operator can be expressed as

$$S = \lim_{t\to\infty} e^{iH_0t}e^{-2iHt}e^{iH_0t} =$$

$$\lim_{t\to\infty} e^{iH_0t}e^{-i(H_0+V_s)t}e^{i(H_0+V_s)t}e^{-2iHt}e^{i(H_0+V_s)t}e^{-i(H_0+V_s)t}e^{iH_0t}. \tag{311}$$

This can be replaced by the product of three limits

$$\lim_{t\to\infty} e^{iH_0t}e^{-i(H_0+V_s)t} \lim_{t\to\infty} e^{i(H_0+V_s)t}e^{-2iHt}e^{i(H_0+V_s)t} \lim_{t\to\infty} e^{-i(H_0+V_s)t}e^{iH_0t} \tag{312}$$

This is valid if all three limits exist. This gives

$$S = \Omega_+^\dagger(H_0+V_s,H_0) \lim_{t\to\pm\infty} e^{i(H_0+V_s)t}e^{-2iHt}e^{i(H_0+V_s)t}\Omega_-(H_0+V_s,H_0). \tag{313}$$

Since V_w is weak it can be treated by perturbation theory. To do this define interaction picture evolution operator

$$U(t,t') = e^{i(H_0+V_s)t}e^{-iH(t-t')}e^{-i(H_0+V_s)t'}. \tag{314}$$

$U(t,t')$ the solution of the integral equation

$$U(t,t') = I - i\int_{t'}^{t} V_W(t'')U(t'',t')dt'' \qquad V_W(t) := e^{i(H_0+V_s)t}V_W e^{-i(H_0+V_s)t}. \tag{315}$$

Using the Dyson trick to remove the iterated integrals the iterative solution of this equation can be expressed as a series of time ordered products of $V_W(t)$ integrated over a single time interval:

$$U(t,t') = I + \sum_n \frac{(-i)^n}{n!} \int_{t'}^{t} dt_1 \cdots dt_n T(V_W(t_1)\cdots V_W(t_n)) \tag{316}$$

where T is the time ordering operator. To use this in (315) let $t \to \infty$, $t' \to -\infty$ which gives the following expression for the scattering operator

$$S = \Omega_+^\dagger(H_0+V_s,H_0)[I + \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots dt_n T(V_W(t_1)\cdots V_W(t_n))]\Omega_-(H_0+V_s,H_0) \tag{317}$$

The leading three terms in this perturbative series for the scattering operator are

$$\begin{gathered} S = \Omega_+^\dagger(H_0+V_s,H_0)\Omega_-(H_0+V_s,H_0) \\ -i\int_{-\infty}^{\infty} dt_1 \Omega_+^\dagger(H_0+V_s,H_0)V_w(t_1)\Omega_-(H_0+V_s,H_0) \\ -\frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \Omega_+^\dagger(H_0+V_s,H_0)[V_W(t_1)V_W(t_2)\theta(t_1-t_2)+V_W(t_2)V_W(t_1)\theta(t_2-t_1))+\cdots] \\ \Omega_-(H_0+V_s,H_0). \end{gathered} \tag{318}$$

If we express this and expansions in eigenstates of H_0+V_s and use the representation for the Heaviside function this becomes

$$\theta(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dse^{ist}}{s-i\epsilon^+} \tag{319}$$

$$\begin{gathered} S = \\ \sum_n \Omega_+^\dagger(H_0+V_s,H_0)|n\rangle\langle n|\Omega_-(H_0+V_s,H_0) \\ -i\sum_{nm}\int_{-\infty}^{\infty} dt_1 \Omega_+^\dagger(H_0+V_s,H_0)|n\rangle e^{i(E_n-E_m)t_1}\langle n|V_w|m\rangle\langle m|\Omega_-(H_0+V_s,H_0) \\ -\frac{1}{2}\sum_{mnk}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \Omega_+^\dagger(H_0+V_s,H_0)|n\rangle[\langle n|V_w|k\rangle\langle k|V_w|m\rangle e^{i(E_n-E_k)t_1}e^{i(E_k-E_m)t_2}\theta(t_1-t_2)+ \\ \langle n|V_w|k\rangle\langle k|V_w|m\rangle e^{i(E_n-E_k)t_2}e^{i(E_k-E_m)t_1}\theta(t_2-t_1)+\cdots] \\ \langle m|\Omega_-(H_0+V_s,H_0)+\cdots = \end{gathered} \tag{320}$$

$$\begin{gathered} S = \Omega_+^\dagger(H_0+V_s,H_0)\Omega_-(H_0+V_s,H_0) \\ -i2\pi\delta(E_f-E_i)\Omega_+^\dagger(H_0+V_s,H_0)V_w\Omega_-(H_0+V_s,H_0)] \end{gathered}$$

$$-\frac{1}{2}\sum_{mnk}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\int_{-\infty}^{\infty}ds\Omega_+^\dagger(H_0+V_s,H_0)|n\rangle[\langle n|V_w|k\rangle\langle k|V_w|m\rangle\times$$

$$e^{i(E_n-E_k)t_1}e^{i(E_k-E_m)t_2}\frac{e^{is(t_1-t_2)}}{(2\pi i)(s-i\epsilon)}$$

$$+\langle n|V_w|k\rangle\langle k|V_w|m\rangle e^{i(E_n-E_k)t_2}e^{i(E_k-E_m)t_1}\frac{e^{is(t_2-t_1)}}{(2\pi i)(s-i\epsilon)}]\langle m|\Omega_-(H_0+V_s,H_0)+\cdots= \tag{321}$$

$$S=\Omega_+^\dagger(H_0+V_s,H_0)\Omega_-(H_0+V_s,H_0)$$

$$-i2\pi\delta(E_f-E_i)\Omega_+^\dagger(H_0+V_s,H_0)V_w\Omega_-(H_0+V_s,H_0)]$$

$$-\frac{2}{2}\frac{(2\pi)^2}{2\pi i}\sum_{mnk}\delta(E_n-E_m)\Omega_+^\dagger(H_0+V_s,H_0)|n\rangle[\frac{\langle n|V_w|k\rangle\langle k|V_w|m\rangle}{(2\pi i)(E_k-E_m-i\epsilon)}\Omega_-(H_0+V_s,H_0) \tag{322}$$

$$S=\Omega_+^\dagger(H_0+V_s,H_0)\Omega_-(H_0+V_s,H_0)$$

$$-i2\pi\delta(E_f-E_i)\Omega_+^\dagger(H_0+V_s,H_0)V_W\Omega_-(H_0+V_s,H_0)]$$

$$-2\pi i\delta(E_f-E_i)\Omega_+^\dagger(H_0+V_s,H_0)V_W\frac{1}{E_i-H_0-V_s+i\epsilon}V_W\Omega_-(H_0+V_s,H_0)+\cdots \tag{323}$$

This is exactly the second Born approximation in the strongly interacting eigenstates.

For the multi-channel case it is enough to replace the two-body wave operators by the channel wave operators

$$\begin{aligned}S_{\alpha\beta}=&\Omega_{\alpha+}^\dagger(H_0+V_s,\Phi_\alpha,H_\alpha)\Omega_{\beta-}(H_0+V_s,\Phi_\beta,H_\beta)\\&-i2\pi\delta(E_f-E_i)\Omega_{\alpha+}^\dagger(H_0+V_s,\Phi_\alpha,H_\alpha)V_W\Omega_{\beta-}(H_0+V_s,\Phi_\beta,H_\beta)\\&-2\pi i\delta(E_f-E_i)\Omega_{\alpha+}^\dagger(H_0+V_s,\Phi_\alpha,H_\alpha)V_W\frac{1}{E_i-H_0-V_s+i\epsilon}V_W\Omega_{-\beta}(H_0+V_s,\Phi_\beta,H_\beta)+\cdots\end{aligned}$$

(324a)

(324b)

(324c)

XVI. FIELDS AND POTENTIALS

This section examines the case of electron scattering. The main purpose for considering this example is that structurally it is similar to neutrino scattering. The main difference is in the structure of the current operators and the exchange bosons. In this case there are 2 currents, an electron current, a strong current, and an exchanged photon.

Using the two potential formulation - where the strong interaction is the strong nuclear force and the weak interaction is the electromagnetic interaction. In the interaction picture it has the form

$$V_W(t) = e^{i(H_e+H_s+H_\gamma)t} e \int d\mathbf{x}(J^\mu_s(\mathbf{x},0) + J^\mu_e(\mathbf{x},0)A_\mu(\mathbf{x},0))e^{-i(H_e+H_s+H_\gamma)t} \tag{325}$$

where H_s is the Hamiltonian for the strongly interacting system, H_e is the Hamiltonian free electrons, and free photons without the electric current term. When these operators are applied to the currents the current operators become Heisenberg picture operators with respect to the strong interaction and QED. They are interaction picture operators when all interactions are turned on. In what follows the weak interaction in the interaction picture is

$$V_W(t) = \int d\mathbf{x}(J^\mu_s(\mathbf{x},t) + J^\mu_e(\mathbf{x},t))A_\mu(\mathbf{x},t) \tag{326}$$

Using (318) the scattering operator to second order in $V_W(t)$ becomes

$$S = \Omega^\dagger_+(H_s,H_0)\Omega_-(H_s,H_0) - ie\int d^4x\Omega^\dagger_+(H_s,H_0)(J^\mu_s(x) + J^\mu_e(x))A_\mu(x)\Omega_-(H_s,H_0)$$

$$-\frac{e^2}{2!}\int d^4x_1 d^4x_2 \Omega^\dagger_+(H_s,H_0)T[((J^\mu_s(x_1)+J^\mu_e(x_1))((J^\nu_s(x_2)+J^\nu_e(x_2))]T(A_\mu(x_1)A_\nu(x_2))\Omega_-(H_s,H_0)+\cdots \tag{327}$$

where the fact that the vector potential commutes with the currents was used in this approximation. This gives a photon propagator coupled to the time ordered product of the currents. In this approximation the electron and strong currents commute, so the time ordering does not affect the product of those currents.

If the initial state is a Deuteron and an electron, the initial wave operator is replaced by the product of the Deuteron bound state wave function and the initial electron state. If the final state is a Deuteron and an electron, the final wave operator is replaced by the product of the Deuteron bound state wave function and the final electron state. If the final strong state is np or $np\pi$ then the final strong state is the strong outgoing scattering state vector.

In this example the 0-th order term ignores the perturbative part of the interaction. It involves just the strong interaction dynamics with the electron as a spectator.

The first order term is relevant for photo absorption, photoproduction or photo disintegration. For these three cases the photon couples to the strong current, and $J^\mu_e(x)$ term does not contribute.

The second order term is relevant for electron scattering; elastic or electrodisintegration in the one-photon exchange approximation. The discussion below is limited to that case. In this approximation the electromagnetic interactions of the strong system with itself and the electron with itself are ignored.

For electron scattering reactions the two terms in the interaction involving the product of the electron and strong current operators $J_e^\mu(x)$ and $J_s^\nu(y)$ are relevant. Again in this approximation the electron current, strong current, and vector potential all commute so the time ordering is only relevant for the photon propagator. The second order term for the case of elastic scattering is

$$-\frac{e^2}{2!}\int d^4x_1 d^4x_2 \langle \mathbf{p}'_D, \mu'_D, D; \mathbf{p}'_e, \mu'_e|(J_s^\mu(x_1)J_e^\nu(x_2)+J_e^\mu(x_1)J_s^\nu(x_2))T(A_\mu(x_1)A_\nu(x_2))|\mathbf{p}_D, \mu_D, D; \mathbf{p}_e, \mu_e\rangle \tag{328}$$

Since the photon propagator is symmetric under $x_1 \leftrightarrow x_2$ Using

$$\langle 0|T(A_\mu(x), A_\nu(y))|0\rangle = \langle 0|T(A_\mu(y), A_\nu(x))|0\rangle = -i\eta_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}G(k)e^{-ik(x-y)} \qquad G(k) = \frac{1}{k^2+i\epsilon} \tag{329}$$

in the above gives

$$S_{fi} =$$

$$i\frac{2e^2}{2}\int d^4x_1 d^4x_2 \frac{d^4k}{(2\pi)^4}\langle \mathbf{p}'_D, \mu'_D, D|J_s^\mu(x_1)|D, |\mathbf{p}_D, \mu_D\rangle e^{-ik(x_1-x_2)}\frac{\eta_{\mu\nu}}{k^2+i\epsilon}\langle \mathbf{p}'_e, \mu'_e|J_e^\nu(x_2)|\mathbf{p}_e, \mu_e\rangle =$$

$$\frac{ie^2}{(2\pi)^4}\int d^4x_1 d^4x_2 d^4k \langle \mathbf{p}'_D, \mu'_D, D|J_s^\mu(0)||\mathbf{p}_D, \mu_D, D\rangle e^{i(p'_s-p_D)\cdot x_1}e^{i(p_{e'}-p_e)\cdot x_2}e^{-ik(x_1-x_2)}\times$$

$$\frac{\eta_{\mu\nu}}{k^2+i\epsilon}\langle \mathbf{p}'_e, \mu'_e|J_e^\nu(0)|\mathbf{p}_e, \mu_e\rangle =$$

$$ie^2(2\pi)^4\delta^4(p'_D+p'_e-p_e-p_D)\langle \mathbf{p}'_D, \mu'_D, D|J_s^\mu(0)|\mathbf{p}_D, \mu_D, D\rangle \frac{\eta_{\mu\nu}}{(p'_e-p_e)^2+i\epsilon}\langle \mathbf{p}'_e, \mu'_e|J_e^\nu(0)|\mathbf{p}_e, \mu_e\rangle \tag{330}$$

This is the dynamical contribution to the scattering matrix

We can read off the transition matrix elements (270), with the momentum conserving delta functions factored out like (281). It is the coefficient of $-2\pi i\delta^4(p'_D+p'_e-p_e-p_D)$

$$\langle \mathbf{p}'_D, \mu'_D, D, \mathbf{p}'_e, \mu'_e|T|\mathbf{p}_D, \mu_D, D, \mathbf{p}_e, \mu_e\rangle =$$

$$-e^2(2\pi)^3\langle \mathbf{p}'_D, \mu'_D, D|J_s^\mu(0)|\Phi_D|\mathbf{p}_D, \mu_D, D\rangle \frac{\eta_{\mu\nu}}{(p'_e-p_e)^2+i\epsilon}\langle \mathbf{p}'_e, \mu'_e|J_e^\nu(0)|\mathbf{p}_e, \mu_e\rangle \tag{331}$$

The expression for the differential cross section in terms of (331) is

$$d\sigma = \frac{(2\pi)^4\omega_{m_D}(\mathbf{p}_D)\omega_{m_e}(\mathbf{p}_e)}{\sqrt{(p_D\cdot p_e)^2-m_D^2m_e^2}}|\langle \mathbf{p}'_D, \mu'_D, D, \mathbf{p}'_e, \mu'_e|T|\mathbf{p}_D, \mu_D, D, \mathbf{p}_e, \mu_e\rangle|^2\times$$

$$\omega_{m_D}(\mathbf{p}'_D)\omega_{m_e}(\mathbf{p}'_e)\delta^4(p_D + p_e - P'_D - p'_e)\frac{d\mathbf{p}_D}{\omega_{m_D}(\mathbf{p}_D)'}\frac{d\mathbf{p}_e}{\omega_{m_e}(\mathbf{p}'_e)} \qquad (332)$$

This is product of the following three covariant parts

$$\frac{(2\pi)^4}{\sqrt{(p_D \cdot p_e)^2 - m_D^2 m_e^2}} \qquad (333)$$

$$\omega_{m_D}(\mathbf{p}_D)\omega_{m_e}(\mathbf{p}_e)|\langle \mathbf{p}'_D, \mu'_D, D, \mathbf{p}'_e, \mu'_e|T|\mathbf{p}_D, \mu_D, D, \mathbf{p}_e, \mu_e\rangle|^2\omega_{m_D}(\mathbf{p}'_D)\omega_{m_e}(\mathbf{p}'_e) \qquad (334)$$

$$\delta^4(p_D + p_e - P'_D - p'_e)\frac{d\mathbf{p}'_D}{\omega_{m_D}(\mathbf{p}'_D)}\frac{d\mathbf{p}'_e}{\omega_{m_e}(\mathbf{p}'_e)} \qquad (335)$$

so it can be easily computed in any frame of reference. Next consider the computation of the elements of the transition matrix element.

Note from (186) the electron current matrix elements can be expressed in term of Dirac spinors

$$\langle \mathbf{p}'_e, \mu'_e|J_e^\nu(0)|\mathbf{p}_e, \mu_e\rangle = \frac{1}{(2\pi)^3}\sqrt{\frac{m_e^2}{\omega_{m_e}(\mathbf{p}_e)\omega_{m_e}(\mathbf{p}'_e)}}\bar{u}_e(\mathbf{p}'_e, \mu'_e)\gamma^\nu u_e(\mathbf{p}_e, \mu_3) \qquad (336)$$

here we use the normalization (141) on the Dirac spinors, $\bar{u}(p)u(p) = \sigma_0$.

The other element is the Deuteron current matrix elements.

$$\langle \mathbf{p}'_D, \mu'_D, D|J_s^\mu(0)|\mathbf{p}_D, \mu_D, D\rangle \qquad (337)$$

The formal structure of the matrix element is

$$\int \langle \mathbf{p}', D|\mathbf{p}''', \mathbf{k}';\rangle d\mathbf{p}'''d\mathbf{k}'\langle \mathbf{p}''', \mathbf{k}'|\mathbf{p}'_1, \mathbf{p}'_2\rangle d\mathbf{p}'_1, d\mathbf{p}'_2 \times$$

$$[\delta(\mathbf{p}'_1 - \mathbf{p}_1)\langle \mathbf{p}'_2|J_2^\mu(0)|\mathbf{p}_2\rangle + \delta(\mathbf{p}'_2 - \mathbf{p}_2)\langle \mathbf{p}'_1|J_1^\mu(0)|\mathbf{p}_1\rangle]d\mathbf{p}_1 d\mathbf{p}_2\langle \mathbf{p}_1, \mathbf{p}_2|\mathbf{p}'', \mathbf{k}\rangle d\mathbf{p}''d\mathbf{k}\langle \mathbf{p}'', \mathbf{k}|D, \mathbf{p}\rangle \qquad (338)$$

This expression has 7 delta functions and 8 integration variables. We want to use the following sequence of delta functions

$$\int \delta(\mathbf{p}' - \mathbf{p}''')d\mathbf{p}'''d\mathbf{k}'\delta(\mathbf{k}' - \mathbf{k}'(\mathbf{p}'''_1, \mathbf{p}'''_2))\delta(\mathbf{p}''' - \mathbf{p}'_1 - \mathbf{p}'_2)d\mathbf{p}'_1 d\mathbf{p}'_2(\delta(\mathbf{p}'_2 - \mathbf{p}_2) + \delta(\mathbf{p}'_1 - \mathbf{p}_1))\times$$

$$d\mathbf{p}_1 d\mathbf{p}_2\delta(\mathbf{p}_1 - \mathbf{p}_1(\mathbf{p}'', \mathbf{k}))\delta(\mathbf{p}_2 - \mathbf{p}_2(\mathbf{p}'', \mathbf{k}))\delta(\mathbf{p}'' - \mathbf{p})d\mathbf{p}''d\mathbf{k}_1 \qquad (339)$$

What remains, after integrating over all of these delta functions, in addition to the kinematic constraints, is the integral over the initial proton rest momentum, $\mathbf{k}_1$. The choices of delta functions fix the Jacobians that appear in the current Recall that

$$\langle \mathbf{p}_1, \mathbf{p}_2|\mathbf{P}, \mathbf{k}\rangle =$$

$$\delta(\mathbf{P}-\mathbf{p}_1-\mathbf{p}_2)\delta(\mathbf{k}-k(\mathbf{p}_1\mathbf{p}_2))\sqrt{\frac{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)}{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}}\sqrt{\frac{\omega_{M_0}(\mathbf{P})}{M_0}} =$$

$$\delta(\mathbf{p}_1-\mathbf{p}_1(\mathbf{P},\mathbf{k}))\delta(\mathbf{p}_2-\mathbf{p}_2(\mathbf{P},\mathbf{k}))\sqrt{\frac{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)}}\sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}} \tag{340}$$

we are now in a position to write down the formula for the proton current in the Deuteron. There will be a similar expression for the neutron current, it will have different spectator constraint

$$\begin{gathered}
\langle \mathbf{p}', \mu', D | J_s^\nu(0) | \mathbf{p}, \mu, D\rangle = \\
\int d\mathbf{k}_p \sum \phi^{j=1*}_{D,l',s'=1}(k') Y^{l'*}_{m'}(\mathbf{k}(\mathbf{p}'_1,\mathbf{p}'_2) C(j,l',s'|\mu',m',m'_{12}) C(s',\frac{1}{2},\frac{1}{2},m'_{12},\mu'_1,\mu'_2) \\
D^{1/2}_{\mu'_1\mu'''_1}[B_c^{-1}(\mathbf{k}'_1/m_1)B_c^{-1}(\mathbf{p}/M_0)B_c(\mathbf{p}'_1/m_1)] D^{1/2}_{\mu'_2\mu'''_2}[B_c^{-1}(\mathbf{k}'_2/m_2)B_c^{-1}(\mathbf{p}'/M'_0)B_c(\mathbf{p}_2/m_2)] \\
\sqrt{\frac{\omega_{m_1}(\mathbf{k}'_1)\omega_{m_2}(\mathbf{k}'_2)}{\omega_{m_1}(\mathbf{p}'_1)\omega_{m_2}(\mathbf{p}_2)}}\sqrt{\frac{\omega_{M_0}(\mathbf{p}')}{M'_0}}\sqrt{\frac{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)}}\sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{p})}} \times \\
\langle \mathbf{p}'_1, \mu'''_1 | J_1^\nu(0) \mathbf{p}_1, \mu''_1\rangle \delta_{\mu'''_2\mu''_2} \\
D^{1/2}_{\mu''_1\mu_1}[B_c^{-1}(\mathbf{p}_1/m_1)B_c(\mathbf{p}/M_0)B_c(\mathbf{k}_1/m_1)] D^{1/2}_{\mu''_2\mu_2}[B_c^{-1}(\mathbf{p}_2/m_2)B_c(\mathbf{p}/M_0)B_c(\mathbf{k}_2/m_2)] \times \\
C(s,\frac{1}{2},\frac{1}{2},m_{12},\mu_1,\mu_2) C(j,l,s|\mu,m,m_{12}) Y^l_m(\mathbf{k}(\mathbf{p}_1,\mathbf{p}_2)\phi^{j=1}_{D,l',s'=1}(k) + \\
\text{particle 1 spectator terms} + \text{2 body terms}
\end{gathered} \tag{341}$$

In this expression all variables are expressed in terms of $\mathbf{k}_1$ (the integration variable), the initial Deuteron momentum $\mathbf{p}$, and the momentum $\mathbf{q}$ transferred to the Deuteron from the electron.

$$\mathbf{p}' = \mathbf{p} + \mathbf{q} \tag{342}$$

$$\mathbf{p}_1 = \mathbf{p}_1(\mathbf{p}, \mathbf{k}_1) \tag{343}$$

$$\mathbf{p}_2 = \mathbf{p} - \mathbf{p}_1 \tag{344}$$

$$\mathbf{p}'_2 = \mathbf{p}_2 \tag{345}$$

$$\mathbf{p}'_1 = \mathbf{p}' - \mathbf{p}'_2 = \mathbf{p} + \mathbf{q} - \mathbf{p}_2 = \mathbf{p}_1 + \mathbf{q} \tag{346}$$

when the proton is spectator then

$$\mathbf{p}'_2 = \mathbf{p}_2 \rightarrow \mathbf{p}'_1 = \mathbf{p}_1 \tag{347}$$

What remains is the single nucleon current matrix element

$$\langle \mathbf{p}', \mu' | J^\nu(0) | \mathbf{p}, \mu\rangle =$$

$$\frac{1}{(2\pi)^3}\sqrt{\frac{m^2}{\omega_m(\mathbf{p}')\omega_m(\mathbf{p})}}\bar{u}(p',\mu')\left(\gamma^\mu F_1(Q)^2 + i\frac{(p'_\alpha - p_\alpha)\sigma^{\mu\alpha}}{2m}F_2(Q^2)\right)u(p,\mu) \tag{348}$$

where F_1 and F_2 are the Dirac nucleon form factors

For electro-disintegration the final deuteron wave function is replaced by an outgoing scattering wave function.

It that is solved in partial waves like the deuteron bound state wave function the solution will have the form

$$\langle k^+, j, l, s | k', j', l', s'\rangle \delta(\mathbf{p} - \mathbf{p}') =$$
$$\frac{\delta(k-k')}{k^2}\delta(\mathbf{p}-\mathbf{p}')\delta_{\mu\mu'}\delta_{ll'}\delta_{ss'}\delta_{jj'} +$$
$$\delta(\mathbf{p}-\mathbf{p}')\delta_{jj'}\frac{\langle k', l', s' | T^j | k, l, s\rangle}{\omega_{m_1}(\mathbf{k}_1) + \omega_{m_2}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{k}'_2) - \omega_{m_1}(\mathbf{k'_1}) + i0^+} \tag{349}$$

which replaces the deuteron wave function. This can be converted to single particle variables using (295).

XVII. BREAKUP

For electrodisintegration the final deuteron state is replaced by an outgoing wave scattering state. This appears in the current matrix element. It is normally treated using partial waves. Both the arguments and variables need to be converted to single particle degrees of freedom.

What is needed is the following

$$\langle (\mathbf{p}'_1, \mu'_1, \tau'_1 \mathbf{p}'_2, \mu'_2, \tau'_2)^+ | (\mathbf{p}_1, \mu_1, \tau_1, \mathbf{p}_2, \mu_2, \tau_2\rangle =$$

$$\delta(\mathbf{p}'_1 - \mathbf{p}_1)\delta_{\mu'_1\mu_1}\delta_{\tau'_1\tau_1}\delta(\mathbf{p}'_2 - \mathbf{p}_2)\delta_{\mu'_2\mu_2}\delta_{\tau'_1\tau_1} +$$
$$\frac{\langle \mathbf{p}'_1, \mu'_1, \tau'_1, \mathbf{p}'_2, \mu'_2, \tau'_2 \| T \| \mathbf{p}_1, \mu_1, \tau_1, \mathbf{p}_2, \mu_2, \tau_2\rangle \delta(\mathbf{p}'_1 + \mathbf{p}'_2 - \mathbf{p}_1 - \mathbf{p}_2)}{\omega_{m_1}(\mathbf{p}'_1) + \omega_{m_2}(\mathbf{p}'_2) - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2) - i\epsilon} \tag{350}$$

The first term (**??**) gives the born term. The second term has all of the final state interaction contributions. This replaces

$$\langle \phi_d \mathbf{P}, \mu, | (\mathbf{p}_1, \mu_1, \tau_1, \mathbf{p}_2, \mu_2, \tau_2\rangle \tag{351}$$

in the expression (**??**) for the current matrix elements. In this case the dynamical part has to be expressed in terms of partial waves. This term has the structure

$$\int \langle \mathbf{p}_1', \mu_1', \tau_1', \mathbf{p}_2', \mu_2', \tau_2' | (k', j), \mathbf{P}, \mu, l', s', \tau', \mu_\tau' \rangle d\mathbf{P} k'^2 dk d\mathbf{P} \times$$

$$\frac{\langle k', l', s', \tau', \tau_z' \| T^j \| k, l, s, \tau, \tau_z \rangle}{\omega_{m_1}(\mathbf{k}_1') + \omega_{m_2}(\mathbf{k}_2') - \omega_{m_1}(\mathbf{k}_1) - \omega_{m_2}(\mathbf{k}_2) - i\epsilon} k^2 dk \times$$

$$\langle (k, j), \mathbf{P}, \mu, l, s, \tau, \tau_z | \mathbf{p}_1, \mu_1, \tau_1, \mathbf{p}_2, \mu_2, \tau_2 \rangle \tag{352}$$

The coefficient (295) has delta function in both $\mathbf{P}$, k and k', which means that they can be expressed in terms of the external momenta - this means that the second term is

$$\frac{\langle \mathbf{p}_1', \mu_1', \tau_1', \mathbf{p}_2', \mu_2', \tau_2' \| T \| \mathbf{p}_1, \mu_1, \tau_1, \mathbf{p}_2, \mu_2, \tau_2 \rangle}{\omega_{m_1}(\mathbf{p}_1') + \omega_{m_2}(\mathbf{p}_2') - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2) - i\epsilon} =$$

$$\sqrt{\frac{\omega_{m_1}(\mathbf{k}_1')\omega_{m_2}(\mathbf{k}_2')}{\omega_{m_1}(\mathbf{p}_1')\omega_{m_2}(\mathbf{p}_2')}} \sqrt{\frac{\omega_{M_0'}(\mathbf{P}')}{M_0'}} \times$$

$$D^{1/2}_{\mu_1'\mu_1'''}[B_c^{-1}(p_1'/m_1')B_c(P'/M_0')B_c(k_1/m_1)] D^{1/2}_{\mu_2'\mu_2'''}[B_c^{-1}(p_2'/m_2')B_c(P'/M_0')B_c(k_2'/m)] \times$$

$$Y^{l'}_{m'}(\hat{\mathbf{k}}_1') C(s', \frac{1}{2}, \frac{1}{2}; \mu_1''', \mu_2''', m_s''') C(j, l', s'; \mu, m_l, m_{s'''} C(\tau, \frac{1}{2}, \frac{1}{2}; \tau_1', \tau_2', \tau_z') \times$$

$$\frac{\langle k', l', s', \tau', \tau_z' \| T^j \| k, l, s, \tau, \tau_z \rangle}{\omega_{m_1}(\mathbf{k}_1') + \omega_{m_2}(\mathbf{k}_2') - \omega_{m_1}(\mathbf{k}_1) - \omega_{m_2}(\mathbf{k}_2) - i\epsilon} \times$$

$$Y^{l*}_{m}(\hat{\mathbf{k}}_1) C(\tau, \frac{1}{2}, \frac{1}{2}; \tau_1, \tau_2, \tau_z) C(j, l, s; \mu, m_l'', m_s'') C(s, \frac{1}{2}, \frac{1}{2}; \mu_1'', \mu_2'', m_s'')$$

$$D^{1/2}_{\mu_1''\mu_1}[B_c^{-1}(k_1/m_1)B_c^{-1}(P/M_0)B_c(p_1/m_1)] D^{1/2}_{\mu_2''\mu_2}[B_c^{-1}(k_2/m_2)B_c^{-1}(P/M_0)B_c(p_2/m)] \times$$

$$\sqrt{\frac{\omega_{m_1}(\mathbf{k}_1)\omega_{m_2}(\mathbf{k}_2)}{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}} \sqrt{\frac{\omega_{M_0}(\mathbf{P})}{M_0}} \tag{353}$$

XVIII. COMPARISON WITH THE PARTICLE DATA BOOK CONVENTIONS

Starting with the relation of my transition matrix elements to the scattering matrix elements

$$S = I - i(2\pi) T_{fi} \delta^4(p_1 + p_2 - p_1' - p_2') \tag{354}$$

and these transition matrix elements to the differential cross section

$$d\sigma = \frac{(2\pi)^4 \omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |T_{fi}|^2 \delta(p_1 + p_2 - p_2' - p_2') d^3p_1 d^3p_2 = \tag{355}$$

I multiply by 1:

$$\frac{(2\pi)^4}{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} E_1 E_2 |T_{fi}|^2 \omega_{m_1}(\mathbf{p}_1')\omega_{m_2}(\mathbf{p}_2')\delta(p_1+p_2-p_2'-p_2')\frac{d^3p_1'}{\omega_{m_1}(\mathbf{p}_1')}\frac{d^3p_2'}{\omega_{m_2}(\mathbf{p}_2')} =$$

$$\frac{(2\pi)^4}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} 2\omega_{m_1}(\mathbf{p}_1)2\omega_{m_2}(\mathbf{p}_2)|T_{fi}|^2 2\omega_{m_21}(\mathbf{p}_1')'2\omega_{m_2}(\mathbf{p}_2')\times$$

$$\delta(p_1+p_2-p_2'-p_2')\frac{d^3p_1'}{2\omega_{m_1}(\mathbf{p}_1')}\frac{d^3p_2'}{2\omega_{m_2}(\mathbf{p}_2')} =$$

$$\frac{(2\pi)^4}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} (2\pi)^6 2\omega_{m_1}(\mathbf{p}_1)2\omega_{m_2}(\mathbf{p}_2)|\frac{T_{fi}}{(2\pi)^3}|^2 (2\pi)^6 2\omega_{m_1}(\mathbf{p}_1')2\omega_{m_2}(\mathbf{p}_2')\times$$

$$\delta(p_1+p_2-p_2'-p_2')\frac{d^3p_1'}{(2\pi)^3 2\omega_{m_1}(\mathbf{p}_1')}\frac{d^3p_2'}{(2\pi)^3 2\omega_{m_2}(\mathbf{p}_2')} = \quad (356)$$

Comparing to 48.27 in the 2020 particle data book the gives the following relation to my T_{fi}.

The factors of $(2\pi)^{3/2}$ disappear if we assume the PDB normalization

$$\langle p'|p\rangle = (2\pi)^3\delta^3(p'-p) \quad (357)$$

then we get

$$M_{fi} = \sqrt{2\omega_{m_1}(\mathbf{p}_1')}(2\pi)^{3/2}\sqrt{2\omega_{m_2}(\mathbf{p}_2')}(2\pi)^{3/2}\frac{T_{fi}}{(2\pi)^3}\sqrt{2\omega_{m_1}(\mathbf{p}_1)}(2\pi)^{3/2}\sqrt{2\omega_{m_2}(\mathbf{p}_2)}(2\pi)^{3/2} \quad (358)$$

The factor of $(2\pi)^3$ in the denominator is because I use

$$S = I - 2\pi i\delta^4(p_f - p_i)T_{fi} \quad (359)$$

while PDB uses

$$S = I - (2\pi)^4 i\delta^4(p_f - p_i)T_{fi\,pdb} \quad (360)$$

$$(2\pi)^3 T_{fi\,pdb} = T_{fi} \quad (361)$$

This shows that my conventions are consistent with the PBD conventions.

For the cross section in terms of the currents matrix elements

$$|T_{fi}|^2 = \frac{(2\pi)^3 e^2}{(p_e' - p_e)^2 + i0}\langle i|J_s^\mu(0)|f\rangle\langle f|J_s^\nu(0)|i\rangle \frac{(2\pi)^3 e^2}{(p_e' - p_e)^2 - i0}\langle i|J_{e\,\mu}(0)|f\rangle\langle f|J_{e\,\nu}(0)|i\rangle \quad (362)$$

Using

$$\langle \mathbf{p}', \mu'|J^\nu(0)|\mathbf{p}, \mu\rangle =$$

$$\frac{1}{(2\pi)^3}\sqrt{\frac{m^2}{\omega_m(\mathbf{p}')\omega_m(\mathbf{p})}}\bar{u}(p',\mu')\left(\gamma^\mu F_1(Q^2)+i\frac{(p'_\alpha-p_\alpha)\sigma^{\mu\alpha}}{2m}F_2(Q^2)\right)u(p,\mu) \tag{363}$$

and

$$\langle \mathbf{p}',\mu'|J^\nu(0)|\mathbf{p},\mu\rangle =$$
$$\frac{1}{(2\pi)^3}\sqrt{\frac{m^2}{\omega_m(\mathbf{p}')\omega_m(\mathbf{p})}}\bar{u}(p',\mu')\gamma^\mu u(p,\mu) \tag{364}$$

$$|T_{fi}|^2 = \frac{m^2e^2}{(2\pi)^3(p'_e-p_e)^2+i0}\frac{1}{(2\pi)^3}\sqrt{\frac{m^2}{\omega_m(\mathbf{p}')\omega_m(\mathbf{p})}}\times$$
$$\bar{u}(p',\mu')\left(\gamma^\mu F_1(Q)^2+i\frac{(p'_\alpha-p_\alpha)\sigma^{\mu\alpha}}{2m}F_2(Q^2)\right)u(p,\mu)$$
$$\frac{1}{(2\pi)^3}\sqrt{\frac{m^2}{\omega_m(\mathbf{p}')\omega_m(\mathbf{p})}}\bar{u}(p',\mu')\left(\gamma^\mu F_1(Q)^2+i\frac{(p'_\alpha-p_\alpha)\sigma^{\mu\alpha}}{2m}F_2(Q^2)\right)u(p,\mu)$$
$$\langle i|J^\mu_s(0)|f\rangle\langle f|J^\nu_s(0)|i\rangle\frac{(2\pi)^3e^2}{(p'_e-p_e)^2-i0}$$
$$\frac{1}{(2\pi)^3}\sqrt{\frac{m^2}{\omega_m(\mathbf{p}')\omega_m(\mathbf{p})}}\bar{u}(p',\mu')\gamma^\mu\frac{1}{(2\pi)^3}\sqrt{\frac{m^2}{\omega_m(\mathbf{p}')\omega_m(\mathbf{p})}}\bar{u}(p',\mu')\gamma^\mu$$
$$\langle i|J_{e\,\mu}(0)|f\rangle\langle f|J_{e\,\nu}(0)|i\rangle \tag{365}$$

XIX. CROSS SECTION IN TERMS OF STRUCTURE TENSORS

We start with the expression (302) for the differential cross section for the case $n=2$, $b=e$

$$d\sigma = \frac{(2\pi)^4\omega_b(\mathbf{p}_b)\omega_t(\mathbf{p}_t)}{\sqrt{(p_t\cdot p_e)^2-m_e^2m_t^2}}|\langle \mathbf{p}'_e,\mu'_e,\mathbf{p}'_t,\mu'_t\|T^{\alpha\beta}\|\bar{\mathbf{p}}_e,\mu_e,\bar{\mathbf{p}}_t,\mu_t\rangle|^2\delta^4(p_e+p_t-p'_e-p'_t)d\mathbf{p}'_1d\mathbf{p}'_1. \tag{366}$$

Using the expression (331) for $\langle \mathbf{p}'_1,\mu'_1,\mathbf{p}'_2,\mu'_2\|T^{\alpha\beta}\|\bar{\mathbf{p}}_b,\mu_b,\bar{\mathbf{p}}_t,\mu_t\rangle$ based in the two potential formalism gives

$$d\sigma = \frac{(2\pi)^4\omega_e(\mathbf{p}_e)\omega_t(\mathbf{p}_t)}{\sqrt{(p_t\cdot p_e)^2-m_e^2m_t^2}}\times$$
$$|(-e^2(2\pi)^3\langle \mathbf{p}'_t,\mu'_t,t|J^\mu_t(0)|\Phi_t|\mathbf{p}_t,\mu_t,t\rangle\frac{\eta_{\mu\nu}}{(p'_e-p_e)^2+i\epsilon}\times$$
$$\langle \mathbf{p}'_e,\mu'_e|J^\nu_e(0)|\mathbf{p}_e,\mu_e\rangle)|^2\delta^4(p_e+p_t-p'_e-p'_t)d\mathbf{p}'_ed\mathbf{p}'_t = . \tag{367}$$

Using the expression of the electron current matrix elemets (336)

$$d\sigma = \frac{(2\pi)^4 \omega_e(\mathbf{p}_e)\omega_t(\mathbf{p}_t)}{\sqrt{(p_t\cdot p_e)^2 - m_e^2 m_t^2}} \times$$

$$|(-e^2(2\pi)^3 \langle \mathbf{p}_t', \mu_t', t| J_t^\mu(0)|\Phi_t|\mathbf{p}_t, \mu_t, t\rangle \frac{\eta_{\mu\nu}}{(p_e' - p_e)^2 + i\epsilon} \times$$

$$\frac{1}{(2\pi)^3}\sqrt{\frac{m_e^2}{\omega_{m_e}(\mathbf{p}_e)\omega_{m_e}(\mathbf{p}_e')}} \bar{u}_e(\mathbf{p}_e', \mu_e')\gamma^\nu u_e(\mathbf{p}_e, \mu_3)|^2 \delta^4(p_e + p_t - p_e' - p_t') d\mathbf{p}_e' d\mathbf{p}_t' =$$

$$\frac{(2\pi)^4 \omega_t(\mathbf{p}_t)}{\sqrt{(p_t\cdot p_e)^2 - m_e^2 m_t^2}} \frac{e^4}{(q^2)^2} \frac{m_e^2}{\omega_e(\mathbf{p}_e')}$$

$$\langle \mathbf{p}_t', \mu_t', t| J_t^\mu(0)|\Phi_t|\mathbf{p}_t, \mu_t, t\rangle \langle \mathbf{p}_t, \mu_t, t| J_t^\nu(0)|\Phi_t|\mathbf{p}_t', \mu_t', t\rangle \times$$

$$\bar{u}_e(\mathbf{p}_e', \mu_e')\gamma_\mu u_e(\mathbf{p}_e, \mu_e)\bar{u}_e(\mathbf{p}_e, \mu_e)\gamma_\nu u_e(\mathbf{p}_e', \mu_e')\delta^4(p_e + p_t - p_e' - p_t') d\mathbf{p}_e' d\mathbf{p}_t' = \tag{368}$$

If we sum over final electron spins and average over initial electron spins using (385)

$$= \frac{(2\pi)^4 \omega_t(\mathbf{p}_t)}{\sqrt{(p_t\cdot p_e)^2 - m_e^2 m_t^2}} \frac{e^4}{(q^2)^2} \frac{m_E^2}{\omega_e(\mathbf{p}_e')}$$

$$\langle \mathbf{p}_t', \mu_t', t| J_t^\mu(0)|\Phi_t|\mathbf{p}_t, \mu_t, t\rangle \langle \mathbf{p}_t, \mu_t, t| J_t^\nu(0)|\Phi_t|\mathbf{p}_t', \mu_t', t\rangle \times$$

$$\frac{1}{2m_e^2}(p_{e\mu}' p_{e\nu} + p_{e\mu} p_{e\nu}' + (m_e^2 - p_e\cdot p_e')\eta_{\mu\nu})\delta^4(p_e + p_t - p_e' - p_t') d\mathbf{p}_e' d\mathbf{p}_t')$$

$$= \frac{1}{2}\frac{(2\pi)^4}{\sqrt{(p_t\cdot p_e)^2 - m_e^2 m_t^2}} \frac{e^4}{(q^2)^2} \frac{\omega_t(\mathbf{p}_t)}{\omega_e(\mathbf{p}_e')}$$

$$\langle \mathbf{p}_t', \mu_t', t| J_t^\mu(0)|\Phi_t|\mathbf{p}_t, \mu_t, t\rangle \langle \mathbf{p}_t, \mu_t, t| J_t^\nu(0)|\Phi_t|\mathbf{p}_t', \mu_t', t\rangle \times$$

$$(p_{e\mu}' p_{e\nu} + p_{e\mu} p_{e\nu}' + (m_e^2 - p_e\cdot p_e')\eta_{\mu\nu})\delta^4(p_e + p_t - p_e' - p_t') d\mathbf{p}_e' d\mathbf{p}_t' =)$$

$$= \frac{1}{2}\frac{(2\pi)^4}{\sqrt{(p_t\cdot p_e)^2 - m_e^2 m_t^2}} \frac{e^4}{(q^2)^2}$$

$$\sqrt{\omega_t(\mathbf{p}_t')}\langle \mathbf{p}_t', \mu_t', t| J_t^\mu(0)|\Phi_t|\mathbf{p}_t, \mu_t, t\rangle \sqrt{\omega_t(\mathbf{p}_t)}\sqrt{\omega_t(\mathbf{p}_t)}\langle \mathbf{p}_t, \mu_t, t| J_t^\nu(0)|\Phi_t|\mathbf{p}_t', \mu_t', t\rangle \sqrt{\omega_t(\mathbf{p}_t')}\times$$

$$(p_{e\mu}' p_{e\nu} + p_{e\mu} p_{e\nu}' + (m_e^2 - p_e\cdot p_e')\eta_{\mu\nu})\delta^4(p_e + p_t - p_e' - p_t') \frac{d\mathbf{p}_e'}{\omega_e(\mathbf{p}_e')}\frac{d\mathbf{p}_t'}{\omega_t(\mathbf{p}_t')}) \tag{369}$$

If we sum over the final target spins and average over the initial target spins, and replaceing the delta function by its integral representation

$$= \frac{1}{2n_t}\frac{1}{\sqrt{(p_t\cdot p_e)^2 - m_e^2 m_t^2}} \frac{e^4}{(q^2)^2}$$

$$\int d^4x e^{i(p_e - p_e')\cdot x} \sum_{\mu_t \mu_t'} \sqrt{\omega_t(\mathbf{p}_t')}\langle \mathbf{p}_t', \mu_t', t| J_t^\mu(x)|\Phi_t|\mathbf{p}_t, \mu_t, t\rangle \sqrt{\omega_t(\mathbf{p}_t)}\times$$

$$\sqrt{\omega_t(\mathbf{p}_t)}\langle \mathbf{p}_t,\mu_t,t|J_t^\nu(0)|\Phi_t|\mathbf{p}_t',\mu_t',t\rangle\sqrt{\omega_t(\mathbf{p}_t')}\times$$

$$(p'_{e\mu}p_{e\nu}+p_{e\mu}p'_{e\nu}+(m_e^2-p_e\cdot p'_e)\eta_{\mu\nu})\frac{d\mathbf{p}'_e}{\omega_e(\mathbf{p}'_e)}\frac{d\mathbf{p}'_t}{\omega_t(\mathbf{p}'_t)}) \tag{370}$$

Using the definition of the structure tensor (**??**) the unpolarized differential cross section is

$$d\sigma = \frac{1}{16\pi^2\sqrt{(p_t\cdot p_e)^2-m_e^2m_t^2}}\frac{e^4}{(q^2)^2}$$

$$W^{\mu\nu}(p_t,q)(p'_{e\mu}p_{e\nu}+p_{e\mu}p'_{e\nu}+(m_e^2-p_e\cdot p'_e)\eta_{\mu\nu})\frac{d\mathbf{p}'_e}{\omega_e(\mathbf{p}'_e)}\frac{d\mathbf{p}'_t}{\omega_t(\mathbf{p}'_t)}) =$$

$$\frac{1}{\sqrt{(p_t\cdot p_e)^2-m_e^2m_t^2}}\frac{\alpha^2}{(q^2)^2}$$

$$W^{\mu\nu}(p_t,q)(p'_{e\mu}p_{e\nu}+p_{e\mu}p'_{e\nu}+(m_e^2-p_e\cdot p'_e)\eta_{\mu\nu})\frac{d\mathbf{p}'_e}{\omega_e(\mathbf{p}'_e)}\frac{d\mathbf{p}'_t}{\omega_t(\mathbf{p}'_t)}) \tag{371}$$

where

$$W^{\mu\nu}(p_t,q)=\frac{2(2\pi)^2}{n_t}\int e^{iq\cdot x}\sqrt{\omega_{m_t}(\mathbf{p}_t)}\sum_{\mu_t}\langle\Phi_t,\mathbf{p}_t,\mu_t,t|J_t^\mu(x)|\Phi_t,\mathbf{p}'_t,\mu_t,t\rangle\langle\Phi_t,\mathbf{p}'_t,\mu_t,t|J_t^\nu(0)]|\Phi_t,\mathbf{p}_t,\mu_t\rangle\sqrt{\omega \tag{372}$$

where $p'_t = p_t + q$

(this follows Itzykson and Zuber - converting to delta function normalization)

XX. INCLUSIVE SCATTERING

In inclusive scattering the starting point is the expression for the cross section. In this case there is a sum over all final hadronic states; all that is measured are the state of the initial and final electron and the initial target hadron. The final hadronic momenta and spins are summed. The differential cross section has the standard form

$$d\sigma = \int \frac{(2\pi)^4\omega_{m_D}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}{\sqrt{(p_1\cdot p_2)^2-m_1^2m_2^2}}|T_{fi}|^2\delta^4(p_D+p_e-p'_e-\sum_n p'_n)d^3p'_ed^3p'_1\cdots d^3p'_N \tag{373}$$

where for inclusive scattering the integral and spin sums are over the final hadronic states. The dynamical contributions to cross section have the form

$$|T_{fi}|^2\delta^4(p_D+p_e-p'_e-\sum_n p'_n)d^3p'_ed^3p'_1\cdots d^3p'_N =$$

$$e^4(2\pi)^6\langle\Phi_D,\mathbf{p}_D,\mu_D,D|J_s^\mu(0)|\mathbf{p}'_1,\mu'_1,\cdots,\mathbf{p}'_N,\mu'_N\rangle\langle\mathbf{p}'_1,\mu'_1,\cdots,\mathbf{p}'_N,\mu'_N|J_s^\nu(0)|\Phi_D,\mathbf{p}_D,\mu_D\rangle\times$$

$$\frac{\delta^4(p_D + p_e - p'_e - \sum_n p'_n) d^3p'_e d^3p'_1 \cdots d^3p'_N}{((p'_e - p_e)^2 + i\epsilon)((p'_e - p_e)^2 - i\epsilon)} \times$$
$$\langle \mathbf{p}_e, \mu_e | J_{e\mu}(0) | \mathbf{p}'_e, \mu'_e \rangle \langle \mathbf{p}'_e, \mu'_e | J_{e\nu}(0) | \mathbf{p}_e, \mu_e \rangle. \tag{374}$$

where $e^2 = 4\pi\alpha$. It is useful to replace the energy-momentum conserving delta function by its Fourier representation

$$\delta^4(p_D + p_e - p'_e - p'_f) = \frac{1}{(2\pi)^4} \int d^4x e^{i(p_D + p_e - p'_e - \sum_n p'_n)\cdot x}. \tag{375}$$

With this replacement

$$\int \langle \Phi_D, \mathbf{p}_D, \mu_D, D | J^\mu_s(0) | \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_N, \mu'_N \rangle \langle \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_N, \mu'_N | J^\nu_s(0) | \Phi_D, \mathbf{p}_D, \mu_D \rangle \times$$
$$\frac{\delta^4(p_D + p_e - p'_e - \sum_n p'_n) d^3p'_e d^3p'_1 \cdots d^3p'_N}{((p'_e - p_e)^2 + i\epsilon)((p'_e - p_e)^2 - i\epsilon)} \times$$
$$\langle \mathbf{p}_e, \mu_e | J_{e\mu}(0) | \mathbf{p}'_e, \mu'_e \rangle \langle \mathbf{p}'_e, \mu'_e | J_{e\nu}(0) | \mathbf{p}_e, \mu_e \rangle =$$
$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | J^\mu_s(x) J^\nu_s(0) | \Phi_D, \mathbf{p}_D, \mu_D \rangle \times$$
$$\frac{d^4x}{(2\pi)^4((p'_e - p_e)^2 + i\epsilon)((p'_e - p_e)^2 - i\epsilon)} \times$$
$$e^{i(p_e - p'_e)\cdot x} \langle \mathbf{p}_e, \mu_e | J_{e\mu}(0) | \mathbf{p}'_e, \mu'_e \rangle d^3p'_e \langle \mathbf{p}'_e, \mu'_e | J_{e\nu}(0) | \mathbf{p}_e, \mu_e \rangle \tag{376}$$

We remark that the quantity with x in the final term arises from the integral representation of the delta function

$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | J^\mu_s(x) | \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_N, \mu'_N \rangle =$$
$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | e^{ipx} J^\mu_s(0) e^{-ipx} | \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_N, \mu'_N \rangle =$$
$$e^{ip_D x} \langle \Phi_D, \mathbf{p}_D, \mu_D, D | J^\mu_s(0) | \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_N, \mu'_N \rangle e^{-ip'_N x'} \tag{377}$$
$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | J^\mu_s(0) J^\nu_s(x) | \Phi_D, \mathbf{p}_D, \mu_D \rangle = \tag{378}$$

Here the electron momenta and initial deuteron momenta are fixed by experiment. In the lab frame the deuteron is at rest. If there is any momentum transfer the initial electron looses momentum on collision, which means that the final electron has less energy (in the lab frame) than the initial electron. In this case the energy of the final hadrons and initial electron in the lab frame is greater than the sum of the energy of the final electron and initial target. This means that this delta function vanishes. It follows that

$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | J^\mu_s(0) J^\nu_s(x) | \Phi_D, \mathbf{p}_D, \mu_D \rangle = 0 \tag{379}$$

and

$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | J_s^\mu(x) J_s^\nu(0) | \Phi_D, \mathbf{p}_D, \mu_D \rangle =$$

$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | [J_s^\mu(x), J_s^\nu(0)] | \Phi_D, \mathbf{p}_D, \mu_D \rangle. \tag{380}$$

The only time it can contribute is when there is no energy or momentum transfer and the final state is identical to the intitial state. Note that the four dimensional delta function is invariant which means that if it vanishes in the lab frame it vanishes everywhere.

Putting everything together the expression for the differential cross section becomes

$$d\sigma = \int \frac{(2\pi)^6 \omega_{m_D}(\mathbf{p}_D)\omega_{m_e}(\mathbf{p}_e)}{\sqrt{(p_D \cdot p_e)^2 - m_D^2 m_e^2}} (\frac{e^2}{q^2})^2 \int d^4x \times$$

$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | [J_s^\mu(x), J_s^\nu(0)] | \Phi_D, \mathbf{p}_D, \mu_D \rangle \times e^{i(p_e - p'_e)\cdot x} \langle \mathbf{p}_e, \mu_e | J_{e\mu}(0) | \mathbf{p}'_e, \mu'_e \rangle d^3p'_e \langle \mathbf{p}'_e, \mu'_e | J_{e\nu}(0) | \mathbf{p}_e, \mu_e \rangle d^3p'_e. \tag{381}$$

Using the expression

$$\langle \mathbf{p}'_e, \mu'_e | J_{e\mu}(0) | \mathbf{p}_e, \mu_e \rangle =$$

$$\frac{1}{(2\pi)^3} \sqrt{\frac{m_e^2}{\omega_m(\mathbf{p}'_e)\omega_m(\mathbf{p})}} \bar{u}(p'_e, \mu'_e)\gamma_\mu u(p_3, \mu_e)$$

gives

$$e^{i(p_e - p'_e)\cdot x} \langle \mathbf{p}_e, \mu_e | J_{e\mu}(0) | \mathbf{p}'_e, \mu'_e \rangle d^3p'_e \langle \mathbf{p}'_e, \mu'_e | J_{e\nu}(0) | \mathbf{p}_e, \mu_e \rangle d^3p'_e =$$

$$e^{i(p_e - p'_e)\cdot x} \frac{m_e^2}{(2\pi)^6 \omega_{m_e}(\mathbf{p}_e)\omega_{m_e}(\mathbf{p}'_e)} \bar{u}(p_e, \mu_e)\gamma_\mu u(p'_e, \mu'_e)\bar{u}(p'_e, \mu'_e)\gamma_\nu u(p_e, \mu_e). \tag{382}$$

The expression for the cross section becomes

$$d\sigma = \frac{\omega_{m_D}(\mathbf{p}_D)}{\sqrt{(p_D \cdot p_e)^2 - m_D^2 m_e^2}} (\frac{m_e e^2}{q^2})^2 \int d^4x \times$$

$$\langle \Phi_D, \mathbf{p}_D, \mu_D, D | [J_s^\mu(x), J_s^\nu(0)] | \Phi_D, \mathbf{p}_D, \mu_D \rangle \times$$

$$e^{i(p_e - p'_e)\cdot x} \bar{u}(p_e, \mu_e)\gamma_\mu u(p'_e, \mu'_e)\bar{u}(p'_e, \mu'_e)\gamma_\nu u(p_e, \mu_e) \frac{d^3p'_e}{\omega_{m_e}(\mathbf{p}'_e)} \tag{383}$$

Summing over the final electron spins, μ'_e gives

$$\sum_{\mu'_e} \bar{u}(p_e, \mu_e)\gamma_\mu u(p'_e, \mu'_e)\bar{u}(p'_e, \mu'_e)\gamma_\nu u(p_e, \mu_e) =$$

$$\bar{u}(p_e, \mu_e)\gamma_\mu \frac{m + p'_e \cdot \gamma}{2m} \gamma_\nu u(p_e, \mu_e) \tag{384}$$

Averaging over the initial electron spins gives

$$\to \frac{1}{2}\sum \gamma_\mu \frac{m_e + p'_e \cdot \gamma}{2m_e}\gamma_\nu u(p_e,\mu_e)\bar{u}(p_e,\mu_e) = \frac{1}{8m_e^2}\mathrm{Tr}\left(\gamma_\mu(m_e + p'_e\cdot\gamma)\gamma_\nu(m_e + p_e\cdot\gamma)\right) = \tag{385}$$

$$\frac{1}{8}\mathrm{Tr}(\gamma_\mu\gamma_\nu) + \frac{1}{8m_e^2}\mathrm{Tr}(\gamma_\mu(p'_e\cdot\gamma)\gamma_\nu(p_e\cdot\gamma))$$

Because the γ's anticommute with γ_5 the trace of the product of one or three gamma matrices vanishes.

$$\frac{1}{2m_e^2}(p'_{e\mu}p_{e\nu} + p_{e\mu}p'_{e\nu} + (m_e^2 - p_e\cdot p'_e)\eta_{\mu\nu})$$

Inserting this in the expression for the inclusive spin averaged differential cross section and also average over deuteron spins

$$d\sigma = \frac{1}{2\sqrt{(p_D\cdot p_e)^2 - m_D^2 m_e^2}}(\frac{4\pi\alpha}{q^2})^2\times$$

$$\int d^4x e^{i(p_e-p'_e)\cdot x}\sqrt{\omega_{m_D}(\mathbf{p}_D)}\frac{1}{3}\sum_{\mu_D}\langle\Phi_D,\mathbf{p}_D,\mu_D,D|[J_s^\mu(x),J_s^\nu(0)]|\Phi_D,\mathbf{p}_D,\mu_D\rangle\sqrt{\omega_{m_D}(\mathbf{p}_D)}\times$$

$$\left(m_e^2\eta_{\mu\nu} + (p'_{e\mu}p_{e\nu} + p_{e\mu}p'_{e\nu} + (m_e^2 - p_e\cdot p'_e)\eta_{\mu\nu})\right)\frac{d^3p'_e}{\omega_{m_e}(\mathbf{p}'_e)} \tag{386}$$

Each term in this expression is Lorentz covariant. The integral is the Fourier transform of the current commutator term that depends on the momentum transfer and the initial deuteron momentum. The factors 1/2 and 1/3 are due to the spin averaging.

Measuring the final electron energy gives the phase space factor

$$d^3p'_e = p_e'^2\frac{dp'_e}{d\omega_{m_e}(p'_e)}d^2\Omega'_2 dE'_e = p'_e\omega_{m_e}(p'_e)d^2\Omega'_e dE'_e$$

The differential cross section becomes

$$\frac{d\sigma}{dE'_e d^2\Omega'_e} = \frac{1}{2\sqrt{(p_D\cdot p_e)^2 - m_D^2 m_e^2}}(\frac{4\pi\alpha}{q^2})^2\times$$

$$\int d^4x e^{i(p_e-p'_e)\cdot x}\sqrt{\omega_{m_D}(\mathbf{p}_D)}\frac{1}{3}\sum_{\mu_D}\langle\Phi_D,\mathbf{p}_D,\mu_D,D|[J_s^\mu(x),J_s^\nu(0)]|\Phi_D,\mathbf{p}_D,\mu_D\rangle\sqrt{\omega_{m_D}(\mathbf{p}_D)}\times$$

$$\left(m_e^2\eta_{\mu\nu} + (p'_{e\mu}p_{e\nu} + p_{e\mu}p'_{e\nu} + (m_e^2 - p_e\cdot p'_e)\eta_{\mu\nu})\right)\frac{p'_e\omega_{m_e}(\mathbf{p}'_e)}{\omega_{m_e}(\mathbf{p}'_e)} \tag{387}$$

We can identify the terms in this expression with standard definitions of the deuteron structure function and leptonic structure function:

The only tricky parts are the normalizations. Using Itzykson-Zuber (13.101) with their normalization - converting to my normalization gives the following definition of the structure function

$$W_{\mu\nu}(p,q) := \frac{(2\pi)^3}{3\cdot 2\pi}\int e^{iq\cdot x}\sqrt{2\omega_{m_D}(\mathbf{p}_D)}\sum_{\mu_D}\langle \Phi_D,\mathbf{p}_D,\mu_D,D|[J_s^\mu(x),J_s^\nu(0)]|\Phi_D,\mathbf{p}_D,\mu_D\rangle\sqrt{2\omega_{m_D}(\mathbf{p}_D)}$$

$$\frac{(2\pi)^2 2}{3}\int e^{iq\cdot x}\sqrt{\omega_{m_D}(\mathbf{p}_D)}\sum_{\mu_D}\langle \Phi_D,\mathbf{p}_D,\mu_D,D|[J_s^\mu(x),J_s^\nu(0)]|\Phi_D,\mathbf{p}_D,\mu_D\rangle\sqrt{\omega_{m_D}(\mathbf{p}_D)}$$

(for protons $1/3 \to 1/2$) and

$$L_{\mu\nu} := \left(m_e^2\eta_{\mu\nu} + (p'_{e\mu}p_{e\nu} + p_{e\mu}p'_{e\nu} + (m_e^2 - p_e\cdot p'_e)\eta_{\mu\nu})\right) \tag{388}$$

with this definition the expression for the cross section becomes

$$\frac{d\sigma}{dE'_e d^2\Omega'_e} = \frac{1}{2\sqrt{(p_D\cdot p_e)^2 - m_D^2 m_e^2}}(\frac{4\pi\alpha}{q^2})^2\times$$

$$\int d^4x e^{i(p_e-p'_e)\cdot x}\sqrt{\omega_{m_D}(\mathbf{p}_D)}\frac{1}{3}\sum_{\mu_D}\langle \Phi_D,\mathbf{p}_D,\mu_D,D|[J_s^\mu(x),J_s^\nu(0)]|\Phi_D,\mathbf{p}_D,\mu_D\rangle\sqrt{\omega_{m_D}(\mathbf{p}_D)}\times$$

$$\left(m_e^2\eta_{\mu\nu} + (p'_{e\mu}p_{e\nu} + p_{e\mu}p'_{e\nu} + (m_e^2 - p_e\cdot p'_e)\eta_{\mu\nu})\right)\frac{p'_e\omega_{m_e}(\mathbf{p}'_e)}{\omega_{m_e}(\mathbf{p}'_e)} \tag{389}$$

$$\frac{d\sigma}{dE' d^2\Omega'_e} = \frac{p'_e}{2(2\pi)^2\sqrt{(p_D\cdot p_e)^2 - m_D^2 m_e^2}}(\frac{4\pi\alpha}{q^2})^2 W^{\mu\nu}(q,p_D)L_{\mu\nu}$$

$$\frac{2p'_e}{\sqrt{(p_D\cdot p_e)^2 - m_D^2 m_e^2}}(\frac{\alpha}{q^2})^2 W^{\mu\nu}(q,p_D)L_{\mu\nu} \tag{390}$$

XXI. TWO-BODY CURRENTS

There are several sources of two-body currents. Both current conservation and current covariance cannot be satisfied in an interacting theory without two-body currents. These come from current conservation and the commutation relations with the boost generators

$$\eta_{\mu\nu}[P^\mu, J_\nu] = 0$$

$$[K^i, J^j] = i\delta_{ij}J^0 \qquad [K^i, J^0] = i\delta_{ij}J^i$$

where $P^0 = H$ and $\mathbf{K}$ include interactions in the instant form.

These equations formally allow one to construct the current from the charge density and the boost generators, but this requires an explicit representation of the boost and the charge density operator (which means all matrix elements of the charge operator are needed).

In addition to this kind of two-body current there are dynamical processes. For example a virtual charged exchanged π could couple to a photon producing virtual rho. Another process would be a excitation of a virtual nucleon-antnucleon pair coupling to an exchanged pion.

Including these few body currents does not guarantee current conservation or current covariance.

In the instant form of the dynamics current matrix elements in a given pair of frames related by Λ, $\langle \psi_f | U_s^\dagger(\Lambda) J^\mu(0) | \psi_i \rangle$ are rotationally covariant. Boost covariance and current conservation of the matrix elements in a given pair of frames can be used to generate matrix elements in other frames that are consistent with a subset of conserved covariant currents (this does not fix current matrix elements where the initial and/or final states have a different particle content). The conserved covariant current matrix elements generated this way assume that the matrix elements in the initial pair of frames are correct. If we repeat this process with a different pair of frames, these is no reason to expect that the current matrix elements generated in these frames would be the same as the ones generated in the other pair of frames. In general many different currents are compatible with covariance and current conservation. If the differences vanish that means that the currents are consistent - but not necessarily correct.

The most straightforward strategy is to construct a model with rotationally covariant current matrix elements in a given pair of frames and generate the current matrix elements in other frames using covariance and current conservation of the matrix elements.

In what follows I discuss the structure of a rotationally covariant two-body pair current. I use a generic frame - but in order to define a model it is necessary to pick a pair of frames that will be used to define all of the matrix elements.

Formally the covariance condition is

$$\langle (m_f, s_f) p_f, \nu_f' | J^{\mu'}(0) | (m_i, s_i) p_i, \nu_i' \rangle =$$

$$\Lambda^\mu{}_{\mu'} \langle (m_f, s_f) p_{ref-f}, \nu_f' | J^\mu(0) | (m_i, s_i) p_{ref-i}, \nu_i' \rangle \times$$

$$D^{s_i}_{\nu_i'\nu_i}[B^{-1}(p_{ref-i})\Lambda^{-1}B(p_i)]D^{s_f}_{\nu_f'\nu_f}[B^{-1}(p_{ref-f})\Lambda^{-1}B(p_f)]\sqrt{\frac{e_i(p_{ref-i})e_f(p_{ref-f})}{e_i(p_i)e_f(p_f)}}$$

where $p = \Lambda p_{ref}$. These equations define general matrix elements in terms of reference matrix elements. Covariance is compatible with current conservation so if the reference matrix elements satisfy current conservation the transformed matrix elements will also satisfy current conservation.

To construct dynamical two-body currents it is sufficient to construct a rotationally covariant set of matrix elements of the dynamical current in a reference pair of frames which can be used to generate consistent matrix elements is any pair of frames related by boosts.

Below I construct a "pair current" which was an important contribution to elastic-electron deuteron scattering using light front dynamics.

To motivate the structure of two-body current start with the two-potential formalism where $H = H_1 + V$ and V is small. The scattering operator has the form

$$S = \Omega^\dagger_{s+}[I + \sum_{n=1}^\infty \frac{(-i)^n}{n!} T(V(t_1)\cdots V(t_n))]\Omega_{s-} =$$

$$I - 2\pi i \delta(E_f - E_i)T_f \tag{391}$$

Here we assume that $\Omega_{s\pm}$ has only nucleon and electron creation and annihilation operators. It is assumed to be the product of free electron states and eigenstates of a relativistic nuclear Hamiltonian with a realistic nucleon-nucleon interaction, having the operator structure $a^\dagger_N a^\dagger_N a_N a_N a^\dagger_e a_e$ (no nucleon antiparticle creation or annihilation operators). The perturbation V includes the current operators and the part of the pion-nucleon vertex that involves at least one antiparticle creation or annihilation operator. We write this as $V_1 + V_2$ where V_1 is the sum of the electron and nucleon current operators

$$V_1(t) = e\int d^3x e(:\bar\Psi_n(x)\Gamma^\mu\Psi_n(x): + :\bar\Psi_e(x)\gamma^\mu\Psi_e(x):)A_\mu(x) \tag{392}$$

and V_2 is the part of the pseudoscalar pion-nucleon interaction involving antiparticle creation and annihilation operators

$$V_2(t) = -i\frac{f_\pi}{m_\pi}\int d^3x[:\bar\Psi(x)\gamma^5\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)\Psi(x):]_b \tag{393}$$

(the part involving only nucleon creation and annihilation operators is assumed to be included in the initial and final states). To be consistent with the dynamical model the one-pion exchange potential that appears in the current should be replaced by the one-pion exchange part of the Argonne V18 potential.

Note that these calculations are for the S operator which has an energy conserving delta function. The transition operator and potential do not conserve energy on their own - that comes from the time limit that is used to compute the scattering operator. When there is an energy dependence the single-nucleon momenta will be put on shell. This is a prescription rather than a theoretical consequence.

The starting point is to use the pseudoscalar pi-nucleon vertex.

The fields that appear in the vertex and the expressions that will be derived have the following representations in terms of delta function normalized creation and annihilation operators:

$$A_\mu(x) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} \epsilon_\mu(k,\lambda) \left(e^{-ikx} a_\gamma(k,\lambda)\epsilon_\mu(k,\lambda) + e^{ikx} a^\dagger_\gamma(k,\lambda)\epsilon^*_\mu(k,\lambda)\right) \tag{394}$$

$$\Psi(x) = \int \frac{d\mathbf{p}}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega(\mathbf{p})}} \left(a_N(\mathbf{p},\mu)u(\mathbf{p},\mu)e^{-ip\cdot x} + b^\dagger_N(\mathbf{p},\mu)v(\mathbf{p},\mu)e^{ip\cdot x}\right) \tag{395}$$

and

$$\boldsymbol{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{1}{2\omega(\mathbf{p})}} \left(e^{-ip\cdot x}\mathbf{a}_\pi(\mathbf{p}) + e^{ip\cdot x}\mathbf{a}^\dagger_\pi(\mathbf{p})\right). \tag{396}$$

The "pair contribution" to the scattering operator due to the two-body current appears at fourth order in this series (391). The relevant contractions have the following structure

$$\frac{1}{4!}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 [e : \bar{\Psi}_n(x_1)\Gamma^\mu\Psi_n(x_1)A_\mu(x_1) :][e : \bar{\Psi}_e(x_2)\gamma^\nu\Psi_e(x_2)A_\nu(x_2) :]\times$$

$$[-i\frac{f_\pi}{m_\pi} : \bar{\Psi}_n(x_3)\gamma^5\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x_3)\Psi_n(x_3) :][-i\frac{f_\pi}{m_\pi} : \bar{\Psi}_n(x_4)\gamma^5\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x_4)\Psi_n(x_4) :]+\text{permutations}. \tag{397}$$

Here there is one nucleon current operator, one electron current operator and two pi-nucleon current vertices. There are terms corresponding to each of the 4! permutations of the coordinates x_i which lead to 4! identical integrals, so it is enough to evaluate one of the 4! terms and eliminate the factor 1/4!. This results in the following pairings where the unpaired fields will couple to the external electron or nucleon states:

$$-e^2(\frac{f_\pi}{m_\pi})^2 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 : \bar{\Psi}_e(x_2)\gamma^\nu\Psi_e(x_2) : \times$$

$$\langle 0|T(A_\mu(x_1)A_\nu(x_2))|0\rangle\langle 0|T(\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\phi}(x_3)\boldsymbol{\tau}_{cd}\cdot\boldsymbol{\phi}(x_4))|0\rangle\times$$

$$[(\bar{\Psi}_{nc}(x_4)\gamma_5\Psi_{nd}(x_4))\bar{\Psi}(x_1)\Gamma^\mu\langle 0|T(\Psi_n(x_1)\bar{\Psi}_{na}(x_3))|0\rangle_{pair}\gamma_5\Psi_{nb}(x_3)+$$

$$(\bar{\Psi}_{nc}(x_4)\gamma_5\Psi_{nd}(x_4))\bar{\Psi}_{na}(x_3)\gamma_5\langle 0|T(\Psi_{nb}(x_3)\bar{\Psi}_n(x_1))|0\rangle_{pair}\Gamma^\mu\Psi(x_1)+$$

$$(\bar{\Psi}_{na}(x_3)\gamma_5\Psi_{nb}(x_3))\bar{\Psi}(x_1)\Gamma^\mu\langle 0|T(\Psi(x_1)\bar{\Psi}_{nc}(x_4))|0\rangle_{pair}\gamma_5\Psi_{nd}(x_4)+$$

$$(\bar{\Psi}_{na}(x_3)\gamma_5\Psi_{nb}(x_3))\bar{\Psi}_{nc}(x_4)\gamma_5\langle 0|T(\Psi_{nd}(x_4)\bar{\Psi}_n(x_1))|0\rangle_{pair}\Gamma^\mu\Psi_n(x_1)]. \tag{398}$$

The subscript "pair" indicates the part of the propagator that involves two v spinors - discarding the u spinor part. The integrals can all be done using the Fourier representations of the fields and propagators, resulting in 4-momentum conserving delta functions at each vertex. The above expressions for the fields are used to calculate the expressions for the propagators in order to be consistent with the conventions used in these notes.

The fermion propagators for the electron and nucleon have the structure

$$\langle 0|T(\Psi_a(x)\bar{\Psi}_b(y))|0\rangle = \int \frac{d\mathbf{p}}{(2\pi)^{3/2}}\frac{d\mathbf{k}}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega(\mathbf{p})}}\sqrt{\frac{m}{\omega(\mathbf{k})}}\times$$

$$\langle 0|\left((a(\mathbf{p},\mu)u_a(\mathbf{p},\mu)e^{-ip\cdot x}a^\dagger(\mathbf{k},\nu)\bar{u}_b(\mathbf{k},\nu)e^{ik\cdot y}\theta(t_x-t_y)\right.$$

$$\left.-b(\mathbf{k},\nu)\bar{v}_b(\mathbf{k},\nu)e^{-ik\cdot y}b^\dagger(\mathbf{p},\mu)v_a(\mathbf{p},\mu)e^{ip\cdot x}\theta(t_y-t_x)\right)|0\rangle =$$

$$\int \frac{d\mathbf{p}}{(2\pi)^3}\frac{m}{\omega(\mathbf{p})}\times$$

$$\left((u_a(\mathbf{p},\mu)\bar{u}_b(\mathbf{p},\mu)e^{-ip\cdot(x-y)}\theta(t_x-t_y) - v_a(\mathbf{p},\mu)\bar{v}_b(\mathbf{p},\mu)e^{-ip\cdot(y-x)}\theta(t_y-t_x)\right). \tag{399}$$

The second term in 399 represents the "pair" contribution. The integral representation of the Heaviside function

$$\theta(x^0-y^0) = \frac{1}{2\pi i}\int ds\frac{e^{is(x^0-y^0)}}{s-i0^+} \tag{400}$$

can be used to express 399 as

$$-i\int \frac{d\mathbf{p}ds}{(2\pi)^4}\frac{m}{\omega(\mathbf{p})}\times$$

$$\left((u_a(\mathbf{p},\mu)\bar{u}_b(\mathbf{p},\mu)\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})-i(\omega(\mathbf{p})-s)(x^0-y^0)}}{s-i0^+} - v_a(\mathbf{p},\mu)\bar{v}_b(\mathbf{p},\mu)\frac{e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})-i(\omega(\mathbf{p})-s)(y^0-x^0)}}{s-i0^+}e^{ip\cdot(y-x)})\right). \tag{401}$$

Next let $p^0=\omega(\mathbf{p})-s$. Then (401) becomes

$$-i\int \frac{d^4p}{(2\pi)^4}\frac{m}{\omega(\mathbf{p})}\times$$

$$\left(u_a(\mathbf{p},\mu)\bar{u}_b(\mathbf{p},\mu)\frac{e^{-ip\cdot(x-y)}}{\omega(\mathbf{p})-p^0-i0^+} - v_a(\mathbf{p},\mu)\bar{v}_b(\mathbf{p},\mu)\frac{e^{-ip\cdot(y-x)}}{\omega(\mathbf{p})-p^0-i0^+}e^{ip\cdot(y-x)}\right) \tag{402}$$

If we let $p \to p' = -p \to p$ in the second term this becomes

$$-i \int \frac{d^4p}{(2\pi)^4} \frac{m}{\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \times$$

$$\left((\frac{u_a(\mathbf{p},\mu)\bar{u}_b(\mathbf{p},\mu)}{\omega(\mathbf{p}) - p^0 - i0^+} - \frac{v_a(-\mathbf{p},\mu)\bar{v}_b(-\mathbf{p},\mu)}{\omega(\mathbf{p}) + p^0 - i0^+} \right). \tag{403}$$

Using (155) and (156) in (403) gives

$$-i \int \frac{d^4p}{(2\pi)^4} \frac{m}{\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \times$$

$$\frac{1}{2m} \left(\frac{(m - \boldsymbol{\gamma} \cdot \mathbf{p} + \omega(\mathbf{p})\gamma^0)_{ab}}{\omega(\mathbf{p}) - p^0 - i0^+} - \frac{(-m + \boldsymbol{\gamma} \cdot \mathbf{p} + \omega(\mathbf{p})\gamma^0)_{ab}}{\omega(\mathbf{p}) + p^0 - i0^+} \right) =$$

$$-i \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \times$$

$$\left(\frac{(m - \boldsymbol{\gamma} \cdot \mathbf{p} + \omega(\mathbf{p})\gamma^0)_{ab}}{\omega(\mathbf{p}) - p^0 - i0^+} + \frac{(m - \boldsymbol{\gamma} \cdot \mathbf{p} - \omega(\mathbf{p})\gamma^0)_{ab}}{\omega(\mathbf{p}) + p^0 - i0^+} \right). \tag{404}$$

This can also be expressed in terms of the projection operators (155) and (156)

$$\sum_\mu u_a(\mathbf{p},\mu)\bar{u}_b(\mathbf{p},\mu) = \Lambda_+(\mathbf{p}) = \frac{m - \mathbf{p}\cdot\boldsymbol{\gamma} + \gamma^0\omega}{2m}$$

$$\sum_\mu v_a(-\mathbf{p},\mu)\bar{v}_b(-\mathbf{p},\mu) = -\Lambda_-(-\mathbf{p}) = \frac{-m + \mathbf{p}\cdot\boldsymbol{\gamma} + \gamma^0\omega}{2m}$$

$$-i \int \frac{d^4p}{(2\pi)^4} \frac{m}{\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \times$$

$$[(\frac{\Lambda_+(\mathbf{p})_{ab}}{\omega(\mathbf{p}) - p^0 - i0^+} + \frac{\Lambda_-(-\mathbf{p})_{ab}}{\omega(\mathbf{p}) + p^0 - i0^+}). \tag{405}$$

This form of the propagator is useful for two-body current calculations when one want to separate out the u and v spinor contributions. The first term is absorbed in initial or final state while the second term involves the antiparticle creation and annihilation operators, and contributes to the two-body currents. As a check note that it follows from (405) that

$$\langle 0|T(\Psi_a(x)\bar{\Psi}_b(y))|0\rangle$$

$$-i \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \times$$

$$\left(\frac{(m - \boldsymbol{\gamma} \cdot \mathbf{p} + \omega(\mathbf{p})\gamma^0)_{ab}}{\omega(\mathbf{p}) - p^0 - i0^+} + \frac{(m - \boldsymbol{\gamma} \cdot \mathbf{p} - \omega(\mathbf{p})\gamma^0)_{ab}}{\omega(\mathbf{p}) + p^0 - i0^+} \right) =$$

$$-i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{\omega(\mathbf{p})} \frac{\omega(\mathbf{p})(m - \boldsymbol{\gamma} \cdot \mathbf{p} + p^0\gamma^0)_{ab}}{\omega(\mathbf{p})^2 - (p^0)^2 - i0^+}$$

$$-i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{(m+p\cdot\gamma)_{ab}}{\omega(\mathbf{p})^2-(p^0)^2-i0^+} =$$

$$-i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{(m+p\cdot\gamma)_{ab}}{\omega(\mathbf{p})^2-(p^0)^2-i0^+} =$$

$$-i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{m-p\cdot\gamma-i0^+} \tag{406}$$

which is that standard form of the time-ordered product. The quantity needed in the "pair" current can be written in several equivalent ways:

$$\langle 0|T(\Psi_a(x)\bar{\Psi}_b(y))|0\rangle_{pair} := -i\int \frac{d^4p}{(2\pi)^4}\frac{1}{2\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \frac{(m-\boldsymbol{\gamma}\cdot\mathbf{p}-\omega(\mathbf{p})\gamma^0)_{ab}}{\omega(\mathbf{p})+p^0-i0^+} = \tag{407a}$$

$$-i\int \frac{d^4p}{(2\pi)^4}\frac{m}{\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \frac{-v_a(-\mathbf{p})\bar{v}_b(-\mathbf{p})}{\omega(\mathbf{p})+p^0-i0^+} = \tag{407b}$$

$$i\int \frac{d^4p}{(2\pi)^4}\frac{m}{\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \frac{\gamma_5 u_a(-\mathbf{p})\bar{u}_b(-\mathbf{p})\gamma_5}{\omega(\mathbf{p})+p^0-i0^+} = \tag{407c}$$

$$i\int \frac{d^4p}{(2\pi)^4}\frac{m}{\omega(\mathbf{p})} e^{-ip\cdot(x-y)} \frac{\gamma_5\gamma^0 u_a(\mathbf{p})\bar{u}_b(\mathbf{p})\gamma^0\gamma_5}{\omega(\mathbf{p})+p^0-i0^+} \tag{407d}$$

To compute the pseudoscalar meson propagator the representation of the heaviside (400) function is used again. The propagator has the form

$$\langle 0|T\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)\boldsymbol{\tau}\cdot\boldsymbol{\phi}(y)\right)|0\rangle =$$

$$\int \frac{d\mathbf{p}}{2\omega(\mathbf{p})(2\pi)^3}\boldsymbol{\tau}\cdot\boldsymbol{\tau}[\int \frac{ds}{(2\pi i)(s-i0^+)} e^{-ip\cdot(x-y)+is(x^0-y^0)} + \int \frac{ds}{(2\pi i)(s-i0^+)} e^{ip\cdot(x-y)-is(x^0-y^0)}] \tag{408}$$

making the variable change $p^0=\omega(\mathbf{p})-s$

$$-i\int \frac{d^4p}{2\omega(\mathbf{p})(2\pi)^4}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\left(\frac{1}{\omega(\mathbf{p})-p^0-i0^+}e^{-ip\cdot(x-y)} + \int \frac{1}{\omega(\mathbf{p})-p^0-i0^+}e^{ip\cdot(x-y)}\right). \tag{409}$$

Next change the sign of p in the second term to get

$$-i\int \frac{d^4p}{2\omega(\mathbf{p})(2\pi)^4}\boldsymbol{\tau}\cdot\boldsymbol{\tau}[\frac{1}{\omega(\mathbf{p})-p^0-i0^+}+\frac{1}{\omega(\mathbf{p})+p^0-i0^+}]e^{-ip\cdot(x-y)} =$$

$$-i\int \frac{d^4p}{(2\pi)^4}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\frac{1}{\omega(\mathbf{p})^2-(p^0)^2-i0^+}e^{-ip\cdot(x-y)} =$$

$$-i\int \frac{d^4p}{(2\pi)^4}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\frac{1}{m^2+p^2-i0^+}e^{-ip\cdot(x-y)} \tag{410}$$

This gives the pion propagator

$$\boxed{\langle 0|T\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)\boldsymbol{\tau}\cdot\boldsymbol{\phi}(y)\right)|0\rangle = -i\int \frac{d^4p}{(2\pi)^4}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\frac{1}{m_\pi^2+p^2-i0^+}e^{-ip\cdot(x-y)}} \tag{411}$$

The terms that we need for (397) are (411),

$$\langle 0|T(\Psi_a(x)\bar{\Psi}_b(y))|0\rangle_{pair} = -i\int \frac{d^4p}{(2\pi)^4}\frac{m}{\omega(\mathbf{p})}e^{-ip\cdot(x-y)}\frac{\Lambda_-(-\mathbf{p})_{ab}}{\omega(\mathbf{p})+p^0-i0^+} =$$

$$i\int \frac{d^4p}{(2\pi)^4}\frac{m}{\omega(\mathbf{p})}e^{-ip\cdot(x-y)}\frac{\gamma_5 u_a(-\mathbf{p})\bar{u}_b(-\mathbf{p})\gamma_5}{\omega(\mathbf{p})+p^0-i0^+} \tag{412}$$

and the photon propagator

$$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = -i\eta_{\mu\nu}\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2+i\epsilon}. \tag{413}$$

Using these in the expression, the terms that contribute to the two nucleon - one electron matrix elements of the time ordered product in the fourth order term in (398)) are

$$\langle 0|a(\mathbf{p},\mu)\bar{\Psi}(x) \to \frac{1}{(2\pi)^{3/2}}\frac{m}{\omega(\mathbf{p})}\bar{u}(\mathbf{p},\mu)$$

$$\Psi(x)a^\dagger(\mathbf{p},\mu)|0\rangle \to= \frac{1}{(2\pi)^{3/2}}\frac{m}{\omega(\mathbf{p})}u(\mathbf{p},\mu)$$

$$-e^2(\frac{f_\pi}{m_\pi})^2\int d^4k d^4q d^4t \frac{m_e m_n^2}{\sqrt{\omega_e(\mathbf{p}_e)\omega_e(\mathbf{p}'_e)\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)\omega_n(\mathbf{p}'_1)\omega_n(\mathbf{p}'_2)\times}}\frac{-i\eta_{\mu\nu}}{q^2+i0^+}\frac{-i\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+k^2-i0^+}\times$$

$$\Big[\frac{(2\pi)^{16}}{(2\pi)^{12}(2\pi)^9}\delta^4(p_1-t-k)\delta^4(k+p_2-p'_2)\delta^4(t+q-p'_1)\delta^4(p_e-q-p'_e)\times$$

$$(a_e^\dagger(\mathbf{p}'_e)\bar{u}_e(\mathbf{p}'_e)\gamma^\nu u_e(\mathbf{p}_e))(a_n^\dagger(\mathbf{p}'_2)\bar{u}_{nc}(\mathbf{p}'_2)\gamma_5 u_{nd}(\mathbf{p}_2)a(\mathbf{p}_2))\times$$

$$a_n^\dagger(\mathbf{p}'_1)\bar{u}_n(\mathbf{p}'_1)\Gamma^\mu\frac{m}{\omega_n(\mathbf{t})}\frac{i\gamma_5 u_n(-\mathbf{t})\bar{u}_{na}(-\mathbf{t})\gamma_5}{\omega_n(\mathbf{t})+t^0-i\epsilon}\gamma_5 u_{nb}(\mathbf{p}_1)a(\mathbf{p}_1)0+$$

$$\frac{(2\pi)^{16}}{(2\pi)^{12}(2\pi)^9}\delta^4(p_1+q-t)\delta^4(t-k-p'_1)\delta^4(p_2+k-p'_2)\delta^4(p_e-q-p'_e)\times$$

$$(a_e^\dagger(\mathbf{p}'_e)\bar{u}_e(\mathbf{p}'_e)\gamma^\nu u_e(\mathbf{p}_e))(a_n^\dagger(\mathbf{p}'_2)\bar{u}_{nc}(\mathbf{p}'_2)\gamma_5 u_{nd}(\mathbf{p}_2)a(\mathbf{p}_2))\times$$

$$a_n^\dagger(\mathbf{p}_1')\bar{u}_{na}(\mathbf{p}_1')\gamma_5\frac{m}{\omega_n(\mathbf{t})}\frac{i\gamma_5 u_{nb}(-\mathbf{t})\bar{u}(-\mathbf{t})\gamma_5}{\omega_n(\mathbf{t})+t^0-i\epsilon}\gamma^\mu u_n(\mathbf{p}_1)a_n(\mathbf{p}_1)+$$

$$\frac{(2\pi)^{16}}{(2\pi)^{12}(2\pi)^9}\delta^4(p_1-k-p_1')\delta^4(p_2+k-t)\delta^4(t+q-p_2')\delta^4(p_e-q-p_e')\times$$

$$(a_e^\dagger(\mathbf{p}_e')\bar{u}_e(\mathbf{p}_e')\gamma^\nu u_e(\mathbf{p}_e))(a_n^\dagger(\mathbf{p}_1')\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)a_n(\mathbf{p}_1))\times$$

$$a_n^\dagger(\mathbf{p}_2')\bar{u}_n(\mathbf{p}_2')\Gamma^\mu\frac{m}{\omega_n(\mathbf{t})}\frac{i\gamma_5 u(-\mathbf{t})\bar{u}_{nc}(-\mathbf{t})\gamma_5}{\omega_n(\mathbf{t})+t^0-i\epsilon}\gamma_5 u_{nd}(\mathbf{p}_2)a(\mathbf{p}_2)+$$

$$\frac{(2\pi)^{16}}{(2\pi)^{12}(2\pi)^9}\delta^4(p_1-k-p_1')\delta^4(k+t-p_2')\delta^4(p_2-t+q)\delta^4(p_e-q-p_e')\times$$

$$(a_e^\dagger(\mathbf{p}_e')\bar{u}_e(\mathbf{p}_e')\gamma^\nu u_e(\mathbf{p}_e)a_e(\mathbf{p}_e)(a_n^\dagger(\mathbf{p}_1')\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)a_n(\mathbf{p}_1))\times$$

$$a_n^\dagger(\mathbf{p}_2')\bar{u}_{nc}(\mathbf{p}_2')\gamma_5\frac{m}{\omega_n(\mathbf{t})}\frac{i\gamma_5 u_{nd}(-\mathbf{t})\bar{u}_n(-\mathbf{t})\gamma_5}{\omega_n(\mathbf{t})+t^0-i\epsilon}\Gamma^\mu u_n(\mathbf{p}_2)a(\mathbf{p}_2)\Big] \tag{414}$$

integrating over the momenta, q, t and k gives an overall 4-momentum conserving delta function along with constraints in each of the four terms

$$I \qquad t=p_1+p_2-p_2' \qquad k=p_2'-p_2 \qquad q=p_1'+p_2'-p_1-p_2$$

$$II \qquad t=p_1'+p_2'-p_2 \qquad k=p_2'-p_2 \qquad q=p_1'+p_2'-p_1-p_2$$

$$III \qquad t=p_1+p_2-p_1'; \qquad k=p_1-p_1; \qquad q=p_1'+p_2'-p_1-p_2$$

$$IV \qquad t=p_1'+p_2'-p_1 \qquad k=p_1-p_1' \qquad q=p_1'+p_2'-p_1-p_2$$

in all four terms:

$$(2\pi)^4 ie^2\delta^4(p_e+p_1+p_2-p_e'-p_1'-p_2')\frac{m_e}{(2\pi)^3\sqrt{\omega_e(\mathbf{p}_e)\omega_e(\mathbf{p}_e')}}(a_e^\dagger(\mathbf{p}_e')\bar{u}_e(\mathbf{p}_e')\gamma^\nu u_e(\mathbf{p}_e)a_e(\mathbf{p}_e)]\frac{\eta_{\mu\nu}}{q^2+i0^+}\times$$

$$(\frac{f_\pi}{m_\pi})^2\frac{m_n^2}{(2\pi)^6\sqrt{\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')}}\times$$

$$\Bigg(a_n^\dagger(\mathbf{p}_2')\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{nd}(\mathbf{p}_2)a(\mathbf{p}_2)\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_2'-p_2)^2-i0^+}\times$$

$$a_n^\dagger(\mathbf{p}_1')\bar{u}_n(\mathbf{p}_1')\Gamma^\mu\frac{m}{\omega_n(p_1+p_2-p_2')}\frac{\gamma_5 u_{nd}(-(p_1+p_2-p_2'))\bar{u}_{na}(-(p_1+p_2-p_2'))\gamma_5}{(\omega_n(p_1+p_2-p_2')+\omega_n(p_1)+\omega_n(p_2)-\omega_n(p_2')-i\epsilon)}\gamma_5 u_{nb}(\mathbf{p}_1)a(\mathbf{p}_1)+$$

$$a_n^\dagger(\mathbf{p}_2')\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{nd}(\mathbf{p}_2)a(\mathbf{p}_2)\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_2'-p_2)^2-i0^+}\times$$

$$a_n^\dagger(\mathbf{p}_1')\bar{u}_{na}(\mathbf{p}_1')\gamma_5\frac{m}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)}\frac{\gamma_5 u_{nb}(-\mathbf{p}_1'-\mathbf{p}_2'+\mathbf{p}_2)\bar{u}_n(-\mathbf{p}_1'-\mathbf{p}_2'+\mathbf{p}_2)\gamma_5}{(\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)+\omega(\mathbf{p}_1')+\omega(\mathbf{p}_2')-\omega(\mathbf{p}_2)-i\epsilon)}\Gamma^\mu u_n(\mathbf{p}_1)a_n(\mathbf{p}_1)+$$

$$a_n^\dagger(\mathbf{p}_1')\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)a_n(p_1)\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_1-p_1')^2-i0^+}\times$$

$$a_n^\dagger(\mathbf{p}_2')\bar{u}_n(\mathbf{p}_2')\Gamma^\mu \frac{m}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')} \frac{\gamma_5 u_n(-\mathbf{p}_1-\mathbf{p}_2+\mathbf{p}_1')\bar{u}_{nc}(-\mathbf{p}_1-\mathbf{p}_2+\mathbf{p}_1')\gamma_5}{(\omega_n(-\mathbf{p}_1-\mathbf{p}_2+\mathbf{p}_1')+\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)-\omega(\mathbf{p}_1')-i\epsilon)}\gamma_5 u_{nd}(\mathbf{p}_2)a(\mathbf{p}_2)+$$

$$a_n^\dagger(\mathbf{p}_1')\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)a_n(p_1)\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_1-p_1')^2-i0^+}\times$$

$$a_n^\dagger(\mathbf{p}_2')\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 \frac{m}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)} \frac{\gamma_5 u_{nd}(-\mathbf{p}_1'-\mathbf{p}_2'+\mathbf{p}_1)\bar{u}_{nb}(-\mathbf{p}_1'-\mathbf{p}_2'+\mathbf{p}_1)\gamma_5}{(\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)+(\omega_n(\mathbf{p}_1')+\omega(\mathbf{p}_2')-\omega(\mathbf{p}_1))-i\epsilon)}\Gamma^\mu u_n(\mathbf{p}_2)a(\mathbf{p}_2)\Big) \tag{415}$$

It is also useful to use

$$\bar{u}(-\mathbf{p}) = \bar{u}(\mathbf{p})\gamma^0 \qquad u(-\mathbf{p}) = \gamma^0 u(\mathbf{p})$$

Comparing with (330) gives the two-body current matrix elements

$$\langle p_1', p_2'|J^\mu_{ex}(q)|p_1,p_2\rangle =$$
$$(\frac{f_\pi}{m_\pi})^2 \frac{m_n^2}{(2\pi)^6\sqrt{\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')}}\times$$
$$\Big(\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{nd}(\mathbf{p}_2)\times\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_2'-p_2)^2-i0^+}$$
$$\bar{u}_n(\mathbf{p}_1')\Gamma^\mu \frac{m}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')}\frac{\gamma_5\gamma^0 u_{nd}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')\bar{u}_{na}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2'))\gamma^0\gamma_5}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')++\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)-\omega(\mathbf{p}_2')-i\epsilon}\gamma_5 u_{nb}(\mathbf{p}_1)\times +$$
$$\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{dn}(\mathbf{p}_2)\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_2'-p_2)^2-i0^+}\times$$
$$\bar{u}_n(\mathbf{p}_1')\gamma_5 \frac{m}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)}\frac{\gamma_5\gamma^0 u_{nb}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)\bar{u}_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)\gamma^0\gamma_5}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)+\omega(\mathbf{p}_1')+\omega(\mathbf{p}_2')-\omega(\mathbf{p}_2)-i\epsilon}\Gamma^\mu u_n(\mathbf{p}_1)\times +$$
$$\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)\times\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_1-p_1')^2-i0^+}$$
$$\bar{u}_n(\mathbf{p}_2')\Gamma^\mu \frac{m}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')}\frac{\gamma_5\gamma^0 u_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')\bar{u}_{nc}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')\gamma^0\gamma_5}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')+\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)-\omega(\mathbf{p}_1')-i\epsilon}\gamma_5 u_{nd}(\mathbf{p}_2)\times +$$
$$\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)\times\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_1-p_1')^2-i0^+}$$
$$\bar{u}_{nc}(\mathbf{p}_2')\gamma_{5d}\frac{m}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)}\frac{\gamma_5\gamma^0 u_{nd}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)\bar{u}_{nb}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)\gamma^0\gamma_5}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)+\omega(\mathbf{p}_1')+\omega(\mathbf{p}_2')-\omega(\mathbf{p}_1)-i\epsilon}\Gamma^\mu u_n(\mathbf{p}_2)\Big) \tag{416}$$

Eliminating γ_5^2 gives

$$\langle p_1', p_2'|J^\mu_{ex}(q)|p_1,p_2\rangle =$$
$$(\frac{f_\pi}{m_\pi})^2 \frac{m_n^2}{(2\pi)^6\sqrt{\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')}}\times$$
$$\Big(\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{nd}(\mathbf{p}_2)\times\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_2'-p_2)^2-i0^+}$$

$$\bar{u}_n(\mathbf{p}_1')\Gamma^\mu \frac{m}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')} \frac{\gamma_5\gamma^0 u_{nd}((\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2'))\bar{u}_{na}((\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2'))\gamma^0}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')+\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)-\omega(\mathbf{p}_2')-i\epsilon} u_{nb}(\mathbf{p}_1)\times +$$

$$\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{dn}(\mathbf{p}_2)\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_2'-p_2)^2-i0^+}\times$$

$$\bar{u}_n(\mathbf{p}_1')\frac{m}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)} \frac{\gamma^0 u_{nb}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)\bar{u}_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)\gamma^0\gamma_5}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)+\omega(\mathbf{p}_1')+\omega(\mathbf{p}_2')-\omega(\mathbf{p}_2)-i\epsilon}\Gamma^\mu u_n(\mathbf{p}_1)\times +$$

$$\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)\times\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_1-p_1')^2-i0^+}$$

$$\bar{u}_n(\mathbf{p}_2')\Gamma^\mu \frac{m}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')} \frac{\gamma_5\gamma^0 u_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')\bar{u}_{nc}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')\gamma_0}{\omega_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')+\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)-\omega(\mathbf{p}_1')-i\epsilon} u_{nd}(\mathbf{p}_2)\times +$$

$$\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)\times\frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2+(p_1-p_1')^2-i0^+}$$

$$\bar{u}_{nc}(\mathbf{p}_2')\frac{m}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)} \frac{\gamma^0 u_{nd}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)\bar{u}_{nb}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)\gamma^0\gamma_5}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)+\omega(\mathbf{p}_1')+\omega(\mathbf{p}_2')-\omega(\mathbf{p}_1)-i\epsilon}\Gamma^\mu u_n(\mathbf{p}_2)\Big) \tag{417}$$

In (yunfei) the quantities below are approximated by

$$\frac{m_N}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)} \frac{1}{\omega_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)+\omega(\mathbf{p}_1')+\omega(\mathbf{p}_2')-\omega(\mathbf{p}_1)-i\epsilon} \approx 1/2m_N$$

which simplifies this expression to

$$\langle p_1',p_2'|J^\mu_{ex}(q)|p_1,p_2\rangle =$$

$$(\frac{f_\pi}{m_\pi})^2\frac{1}{2m_N}\frac{m_n^2}{(2\pi)^6\sqrt{\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')}}\times$$

$$\Big(\frac{\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{nd}(\mathbf{p}_2)\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}\bar{u}_{na}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')\gamma^0 u_{nb}(\mathbf{p}_1)\bar{u}_n(\mathbf{p}_1')\Gamma^\mu\gamma_5\gamma^0 u_{nd}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_2')+}{m_\pi^2+(p_2'-p_2)^2-i0^+}$$

$$\frac{\bar{u}_{nc}(\mathbf{p}_2')\gamma_5 u_{dn}(\mathbf{p}_2)\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}\bar{u}_n(\mathbf{p}_1')\gamma^0 u_{nb}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)\bar{u}_n(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_2)\gamma^0\gamma_5\Gamma^\mu u_n(\mathbf{p}_1)}{m_\pi^2+(p_2'-p_2)^2-i0^+}+$$

$$\frac{\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}\bar{u}_{nc}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')\gamma_0 u_{nd}(\mathbf{p}_2)\bar{u}_n(\mathbf{p}_2')\Gamma^\mu\gamma_5\gamma^0 u_n(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1')}{m_\pi^2+(p_1-p_1')^2-i0^+}$$

$$\frac{\bar{u}_{na}(\mathbf{p}_1')\gamma_5 u_{nb}(\mathbf{p}_1)\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}\bar{u}_{nc}(\mathbf{p}_2')\gamma^0 u_{nd}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)\bar{u}_{nb}(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1)\gamma^0\gamma_5\Gamma^\mu\bar{u}_{nc}(\mathbf{p}_2)}{m_\pi^2+(p_1-p_1')^2-i0^+}\Big) \tag{418}$$

In these expressions the quantities in the red can be absorbed into the initial or final states, so mathematically this still looks like the calculation of an impulse matrix element. These matrix elements are rotationally covariant. They do not transform correctly with respect dynamical boosts and they do not satisfy current conservation. This matrix element, along with the impulse current, can be defined as the current in a given pair of frames - matrix elements in any other pair of frame can be determined by covariance - these matrix elements will not be the same as the corresponding matrix elements in the other pair of frames. The sensitivity of calculations to this difference provides a measure of the violations of current conservation of current covariance at the operator level.

Note that this current is not necessarily covariant or conserved; however if current matrix elements are is evaluated in a given frame, and defined in all other frames by covariance and current conservation, the resulting matrix elements are will be matrix elements of a covariant current. The result assumes that the current agrees with matrix elements of the above current in the given frame and is related to matrix elements in any other frame by covariance and current conservation.

The spinor quantities that enter the two-body current are

$$\bar{u}(p')\gamma^5 u(p) = \frac{1}{2}(\tilde{\Lambda}_c(p')\Lambda_c(p) - \Lambda_c(p')\tilde{\Lambda}_c(p))$$

$$\bar{u}(p')\gamma^0 u(p) = u^\dagger(p')u(p) = \frac{1}{2}(\tilde{\Lambda}_c(p')\tilde{\Lambda}_c(p) - \Lambda_c(p')\Lambda_c(p)))$$

The other terms that enter are

$$\bar{u}(p')\Gamma^\mu\gamma^5\gamma^0 u(p) = \frac{1}{2}(\tilde{\Lambda}_c(p'), \Lambda_c(p')\Gamma^\mu \begin{pmatrix} \tilde{\Lambda}_c(p) \\ -\Lambda_c(p) \end{pmatrix}$$

$$\bar{u}(p')\gamma^0\gamma^5\Gamma^\mu u(p)\frac{1}{2}(\Lambda_c(p'), -\tilde{\Lambda}_c(p))\Gamma^\mu \begin{pmatrix} \Lambda_c(p) \\ \tilde{\Lambda}_c(p) \end{pmatrix}$$

where

$$\Lambda_c(p) = \sigma_0 \cosh(\rho/2) + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\rho}}\sinh(\rho/2) = \frac{1}{\sqrt{2m(m+p^0)}}\big((p^0+m)\sigma_0 + \mathbf{p}\cdot\boldsymbol{\sigma}\big)$$

$$\tilde{\Lambda}_c(p) = \sigma_0 \cosh(\rho/2) - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\rho}}\sinh(\rho/2) = \frac{1}{\sqrt{2m(m+p^0)}}\big((p^0+m)\sigma_0 - \mathbf{p}\cdot\boldsymbol{\sigma}\big)$$

and

$$\Gamma^\mu = F_1\gamma^\mu + i\frac{(p'-p)_\nu\sigma^{\nu\mu}}{2m}F_2$$

I used the representation

$$\gamma^\mu := \begin{pmatrix} 0 & \tilde{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} \qquad -i\sigma^{0i} = \frac{1}{2}[\gamma^0,\gamma^i] = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \qquad \sigma^{ij} = \frac{i}{2}[\gamma^i,\gamma^j] = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

to compute these quantities.

The Breit frame is useful frame; we also use rotational covariance to choose the 3 direction as the direction of momentum transfer, and finally have to ensure rotational covariance about the z axis. In the Breit frame

$$p_1 + p_2 = (2\omega, 0, 0, -\frac{q}{2})$$

$$p_1' + p_2' = (2\omega, 0, 0, \frac{q}{2})$$

when

$$p_1 = p_1' \text{ then } p_2' - p_2 = (0, 0, 0, q)$$

$$p_2 = p_2' \text{ then } p_1' - p_2 = (0, 0, 0, q)$$

The Breit frame quantities are related to the lab quantities by a Lorentz boost that depends on the final momentum.

Next we need to identify the part of this expression that corresponds to a one pion exchange potential

To do this first note that the Born approximation to the scattering operator is

$$S = I - 2\pi i\delta(E_f - E_i)T_{fi} \to I - 2\pi i\delta(E_f - E_i)V_{fi}$$

This suggest calculating the S operator to leading order, and removing the energy conservation constraint to get an expression of the interaction. The resulting expression will have a non-trival dependence on the initial and final energies. Handling these issues involves choosing a perscription. The first step is to calculate the Born approximation using the interaction picture. For scattering with a vertex interaction this is second order in the interaction:

$$S = I + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty} (-i)^2 (\frac{f_\pi}{m_\pi})^2 d^3x d^3y \bar{\Psi}(y)\gamma_5\Psi(y)\langle 0|T(\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)\boldsymbol{\tau}\cdot\boldsymbol{\phi}(y))|0\rangle\bar{\Psi}(x)\gamma_5\Psi(x)$$

I extract the parts of the fields involving two annihilation operators followed by two creation operators

$$= I + (\frac{f_\pi}{m_\pi})^2 \frac{1}{(2\pi)^6} \frac{m^2}{\sqrt{\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)}} e^{i(p_1'-p_1)\cdot x}\bar{u}(\mathbf{p}_1')\gamma_5 u(\mathbf{p}_1) e^{i(p_2'-p_2)\cdot y}\bar{u}(\mathbf{p}_2')\gamma_5 u(\mathbf{p}_2)\times$$

$$(-\frac{i}{(2\pi)^4}\int \frac{d^4k e^{\frac{-ik(x-y)}{(2\pi)^4}} \boldsymbol{\tau}\cdot\boldsymbol{\tau}}{m_\pi^2 + k^2 - i0^+})$$

$$= I - i(\frac{f_\pi}{m_\pi})^2 \frac{(2\pi)^4}{(2\pi)^6} \frac{m^2}{\sqrt{\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)}} \delta^4(p_1' - p_1 - k)\delta^4(p_2' - p_2 + k)\times$$

$$\bar{u}_a(\mathbf{p}_1')\gamma_5 u_b(\mathbf{p}_1) \frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2 + (p_1' - p_1)^2 - i0^+} \bar{u}_c(\mathbf{p}_2')\gamma_5 u_d(\mathbf{p}_2)$$

The potential can be extracted as

$$\frac{i}{2\pi}(-i)(\frac{f_\pi}{m_\pi})^2 \frac{(2\pi)^4}{(2\pi)^6} \frac{m^2}{\sqrt{\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)}} \delta^4(p_1' - p_1 - k)\delta^4(p_2' - p_2 + k)\times$$

$$\bar{u}_a(\mathbf{p}_1')\gamma_5 u_b(\mathbf{p}_1) \frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2 + (p_1' - p_1)^2 - i0^+} \bar{u}_c(\mathbf{p}_2')\gamma_5 u_d(\mathbf{p}_2)$$

removing the $-(2\pi)i\delta(E_f - E_i)$ gives

$$\langle \mathbf{p}_1', \mathbf{p}_2' | V | \mathbf{p}_1, \mathbf{p}_2 \rangle =$$

$$\frac{1}{(2\pi)^3}(\frac{f_\pi}{m_\pi})^2 \frac{m^2}{\sqrt{\omega_n(\mathbf{p}_1')\omega_n(\mathbf{p}_2')\omega_n(\mathbf{p}_1)\omega_n(\mathbf{p}_2)}} \delta^3(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)\times$$

$$\bar{u}_a(\mathbf{p}_1')\gamma_5 u_b(\mathbf{p}_1) \frac{\boldsymbol{\tau}_{ab}\cdot\boldsymbol{\tau}_{cd}}{m_\pi^2 + (p_1' - p_1)^2 - i0^+} \bar{u}_c(\mathbf{p}_2')\gamma_5 u_d(\mathbf{p}_2)$$

XXII. ELECTROWEAK CURRENTS

The structure of the interaction term follows the particle data book p180 equation (10.2). This (with a $-$ sign), evaluated at time 0, is the weak interaction that appears in the Hamiltonian. For nuclear physics applications the interaction with up and down quarks should be replaced by interactions with protons and neutrons with appropriate form factors.

The theory has $U(2)$ and $SU(2)$ gauge fields, $B^\mu(x)$ and $W_a^\mu(x)$. The covariant derivative is

$$D^\mu = \partial^\mu - gW_a^\mu T_a - ig' B^\mu S \tag{419}$$

with the kinetic term is

$$\mathcal{L}_K = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W_{a\mu\nu} \tag{420}$$

The electroweak fields are

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \tag{421}$$

$$W_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu - g\epsilon_{abc}W_b^\mu W_c^\nu \tag{422}$$

where

$$[T^a, T^b] = i\epsilon_{abc}T^c \qquad \mathbf{T} = \frac{1}{2}\boldsymbol{\sigma} \tag{423}$$

the coupling constants are related by the Weinberg angle

$$g = g' \tan(\theta_W). \tag{424}$$

$$S = Q - T^3 \tag{425}$$

where Q is the electric charge operator.

The familiar fields are related to the Gauge fields by

$$W^{\mu+}(x), W^{\mu-}, W(x)^{\mu}_3 \qquad SU(2) \text{ flavor} \qquad B_\mu(x) \qquad U(1) \tag{426}$$

These are related to the electromagnetic vector potential and neutral current by

$$A_\mu(x) = B_\mu(x)\cos(\theta_W) + W^3_\mu(x)\sin(\theta_W) \tag{427}$$

$$Z_\mu(x) = W^3_\mu(x)\cos(\theta_W) - B_\mu(x)\sin(\theta_W) + \tag{428}$$

$$W^{\pm}_\mu(x) = \frac{1}{\sqrt{2}}(W^1_\mu(x) \mp iW^2_\mu(x)) \tag{429}$$

Fermion famlies transform as SU(2) doublets

$$\Psi_i(x) = \begin{pmatrix} \nu_i(x) \\ l_{i-}(x) \end{pmatrix} \qquad \Psi_i(x) = \begin{pmatrix} u_i(x) \\ d'_i(x) \end{pmatrix} \tag{430}$$

where

$$d'_i = \sum_j V_{ij} d_j \qquad V_{ij} = \text{ CKM matrix} \tag{431}$$

The interaction terms with the fermions is

$$-\mathcal{H}_I(x) = \mathcal{L}_I(x) - e\sum_i q_i \bar{\Psi}_i(x)\gamma^\mu \Psi_i(x) A_\mu(x)$$

$$-\frac{g}{2\cos(\theta_M)}\sum_i \bar{\Psi}_i(x)(\gamma^\mu(g^i_V - \gamma^5 g^i_A)\Psi_i(x) Z_\mu(x)$$

$$-\frac{g}{2\sqrt{2}}\sum_i \bar{\Psi}_i(x)(\gamma^\mu(1-\gamma^5)(T^+_i W^+_\mu + T^-_i W^-_\mu)\psi_i \tag{432}$$

This has the general form

$$\mathcal{H}_I(x) = \sum_i \lambda_i J^\mu_i(x) V_{i\mu}(x) \tag{433}$$

The constants are

$$G_F = 1.16637876 \times 10^{-5}(Gev)^{-2} \tag{434}$$

$$G_F = g^2 \frac{\sqrt{2}}{8M_W^2} \qquad \text{small momenta} \tag{435}$$

$$g_v^i = T_{3l}^i - 2Q_i \sin^2(\theta) \tag{436}$$

$$g_a^i = T_{3l}^i \tag{437}$$

$$\sin^2(\theta_W) = .22337 \tag{438}$$

$$e = g \sin(\theta_W) \tag{439}$$

Structure of the 4 Fermi interaction

$$-\frac{G_F}{\sqrt{2}}([\bar{\Psi}_n \Gamma^i \Psi_n(x)][\bar{\Psi}_e(x)\Gamma_i \Psi_\nu(x)] + hc) \tag{440}$$

where

$$\Gamma = I, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu} \tag{441}$$

Kinematic variables - neutrino nucleon scattering

$$\nu(p) + N \to l^-(p') + X \tag{442}$$

$$s = (p + p + n)^2 - 2ME \tag{443}$$

$$Q^2 = -q^2 = -(p + p')^2 = 4EE' \sin^2(\frac{\theta}{2}) \tag{444}$$

$$\nu = E - E' = \frac{q \cdot p_N}{M} \tag{445}$$

$$W^2 = -Q^1 + 2M\nu + M^2 \tag{446}$$

$$x = \frac{-q^2}{2q \cdot p_N} = \frac{Q^2}{2Mv} \tag{447}$$

$$y = \frac{\nu}{E} = \frac{Q^2}{2MEx} = \frac{q \cdot p_N}{p \cdot p_n} \tag{448}$$

Appendix 1 Derivation of D function - used Schwinger method

Define

$$n_\pm := j \pm \mu$$

$$j = \frac{1}{2}(n_+ + n_-) \qquad \mu = \frac{1}{2}(n_+ - n_-)$$

In this notation

$$|n_+, n_-\rangle := |j, \mu\rangle$$

the raising and lowering operators become

$$J_+|n_+, n_-\rangle = \sqrt{(n_+ + 1)n_-}|n_+ + 1, n_- - 1\rangle$$

$$J_-|n_+, n_-\rangle = \sqrt{n_+(n_- + 1)}|n_+ - 1, n_- + 1\rangle$$

Introduce the operators $a_+, a_+^\dagger, a_-, a_-^\dagger$

$$a_+|n_+, n_-\rangle = \sqrt{n_+}|n_+ - 1, n_-\rangle \qquad a_+^\dagger|n_+, n_-\rangle = \sqrt{n_+ + 1}|n_+ + 1, n_-\rangle$$

$$a_-|n_+, n_-\rangle = \sqrt{n_-}|n_+, n_- - 1\rangle \qquad a_-^\dagger|n_+, n_-\rangle = \sqrt{n_- + 1}|n_+, n_- + 1\rangle$$

Using these operators the angular momentum operators can be expressed as

$$J_+ = a_+^\dagger a_- \qquad J_- = a_-^\dagger a_+$$

$$\mathbf{J} = (a_+^\dagger a_+^\dagger)\frac{1}{2}\boldsymbol{\sigma}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

where

$$\boldsymbol{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

are the Pauli matrices. In what follows we use the short hand notation:

$$a := \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \qquad a^\dagger = (a_+^\dagger, a_-^\dagger) \qquad \mathbf{J} = \frac{1}{2}a^\dagger\boldsymbol{\sigma}a$$

In this representation the unitary rotation operator has the form

$$U(R) = e^{i\theta\hat{\mathbf{n}}\cdot\mathbf{J}}$$

The Wigner D functions are matrix elements of this operator in an angular momentum basis

$$D^j_{\mu\nu}[R] := \langle j, \mu|U(R)|j', \nu'\rangle = \langle n_+, n_-|U(R)|n'_+, n'_-\rangle$$

Using the creation and annihilation operator to express the normalized eigenstates of J^2 and $\mathbf{J}\cdot\hat{\mathbf{z}}$

$$|n_+, n_-\rangle = \frac{(a_+^\dagger)^{n^+}}{\sqrt{n_+!}}\frac{(a_-^\dagger)^{n^-}}{\sqrt{n_-!}}|0, 0\rangle$$

this becomes

$$D^j_{\mu\nu}[R] = \langle 0,0| \frac{a_+^{n^+}}{\sqrt{n_+!}} \frac{a_-^{n^-}}{\sqrt{n_-!}} e^{i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a} \frac{(a_+^\dagger)^{n'_+}}{\sqrt{n'_+!}} \frac{(a_-^\dagger)^{n'_-}}{\sqrt{n'_-!}} |0,0\rangle$$

where

$$j = \frac{1}{2}(n_+ + n_-) = \frac{1}{2}(n'_+ + n'_-)$$

$$\nu = \frac{1}{2}(n'_+ - n'_-)$$

$$\mu = \nu \frac{1}{2}(n_+ - n_-)$$

Since $a_\pm$ is an annihilation operator

$$|0,0\rangle = e^{-i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a}|0,0\rangle$$

the above identity can be expressed in the form

$$D^j_{\mu\nu}[R] = \langle 0,0| \frac{a_+^{n^+}}{\sqrt{n_+!}} \frac{a_-^{n^-}}{\sqrt{n_-!}} |e^{i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a} \frac{(a_+^\dagger)^{n'_+}}{\sqrt{n'_+!}} \frac{(a_-^\dagger)^{n'_-}}{\sqrt{n'_-!}} e^{-i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a}|0,0\rangle =$$

$$\langle 0,0| \frac{a_+^{n^+}}{\sqrt{n_+!}} \frac{a_-^{n^-}}{\sqrt{n_-!}} \frac{(e^{i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a} a_+^\dagger e^{-i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a})^{n'_+}}{\sqrt{n'_+!}} \frac{(e^{i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a} a_-^\dagger e^{-i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a})^{n'_-}}{\sqrt{n_-!}} |0,0\rangle =$$

In order to evaluate this expression use

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] \cdots$$

with $+ = 1, - = 2$ note

$$e^{i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a} a_i^\dagger e^{-i\theta\hat{\mathbf{n}}\cdot a^\dagger \frac{\boldsymbol{\sigma}}{2} a} =$$

$$a_i^\dagger + i\theta \sum_{jk} [a_j^\dagger \hat{\mathbf{n}} \cdot \frac{\boldsymbol{\sigma}_{jk}}{2} a_k, a_i^\dagger] + \cdots =$$

$$a_i^\dagger + i\theta \sum_j a_j^\dagger \hat{\mathbf{n}} \cdot \frac{\boldsymbol{\sigma}_{ji}}{2} + \cdots =$$

$$a_j^\dagger (e^{i\frac{\theta}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}})_{ji} = \sum_j a_j^\dagger (\delta_{ij}\cos(\frac{\theta}{2}) + i\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}_{ji}\sin(\frac{\theta}{2})) \sum_j a_j^\dagger R(\theta\hat{\mathbf{n}})_{ji}$$

where

$$R(\hat{\mathbf{n}})_{ji} = e^{i\frac{\theta}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$$

is the $SU(2)$ representation of the rotation. Using these results in the expression for $D^j_{\mu\nu}$ gives

$$D^j_{\mu\nu}[R] =$$

$$\langle 0,0|\frac{a_+^{n^+}}{\sqrt{n_+!}}\frac{a_-^{n^-}}{\sqrt{n_-!}}\frac{(\sum_{m=1}^2 a_m^\dagger R_{m+})^{n'_+}}{\sqrt{n'_+!}}\frac{(\sum_{l=1}^2 a_l^\dagger R_{l-})^{n'_-}}{\sqrt{n'_-!}}|0,0\rangle =$$

$$\langle 0,0|\frac{a_+^{n_+}}{\sqrt{n_+!}}\frac{a_-^{n^-}}{\sqrt{n_-!}}\frac{(a_+^\dagger R_{++}+a_-^\dagger R_{-+})^{n'_+}}{\sqrt{n'_+!}}\frac{(a_+^\dagger R_{+-}+a_-^\dagger R_{--})^{n'_-}}{\sqrt{n'_-!}}|0,0\rangle =$$

Since $[a_+^\dagger a_-^\dagger]=0$ the powers can be expanded in a binomial series

$$(a_+^\dagger R_{++}+a_-^\dagger R_{-+})^{n'_+} = \sum_{k=0}^{n'_+}\frac{n'_+!}{k!(n'_+-k)!}(R_{++}a_+^\dagger)^k(R_{-+}a_-^\dagger)^{n'_+-k}$$

$$(a_+^\dagger R_{+-}+a_-^\dagger R_{--})^{n'_-} = \sum_{l=0}^{n'_-}\frac{n_-!}{l!(n'_--l)!}(R_{+-}a_+^\dagger)^l(R_{--}a_-^\dagger)^{n'_+-l}$$

Next use these expansions in the above and noting that the non-zero terms must have the same number of creation and annihilation operators with normalization

$$\langle 0|(a_i)^n(a_i^\dagger)^n|0\rangle = n!,$$

In the above expression

$$n_+ = k+l \qquad n_- = 2n'_+ - k - l$$

because of these constrains l can be eliminated

$$l = n_+ - k$$

This gives

$$\langle 0,0|\frac{a_+^{n^+}}{\sqrt{n_+!}}\frac{a_-^{n^-}}{\sqrt{n_-!}}\frac{(\sum_{r=1}^2 a_r^\dagger R_{r+})^{n'_+}}{\sqrt{n'_+!}}\frac{(\sum_{s=1}^2 a_s^\dagger R_{s-})^{n'_-}}{\sqrt{n'_-!}}|0,0\rangle =$$

$$\langle 0,0|\frac{a_+^{n^+}}{\sqrt{n_+!}}\frac{a_-^{n^-}}{\sqrt{n_-!}}\frac{(a_+^\dagger R_{++}+a_-^\dagger R_{-+})^{n'_+}}{\sqrt{n'_+!}}\frac{(a_+^\dagger R_{+-}+a_-^\dagger R_{--})^{n_-}}{\sqrt{n'_-!}}|0,0\rangle =$$

$$\sum_{l,m}\langle 0,0|\frac{a_+^{n^+}}{\sqrt{n_+!}}\frac{a_-^{n^-}}{\sqrt{n_-!}}\frac{n'_+!n'_-!}{l!k!(n'_+-k)!(n'_--l)!}\frac{(a_+^\dagger R_{++})^k(a_-^\dagger R_{-+})^{n'_+-k}}{\sqrt{n'_+!}}\frac{(a_+^\dagger R_{+-})^l(a_-^\dagger R_{--})^{n'_--l}}{\sqrt{n'_-!}}|0,0\rangle =$$

$$\sum_{k=0}^{n'_+}\sum_{l=0}^{n'_-}\frac{\sqrt{n_+!n_-!n'_+!n'_-!}}{l!k!(n'_+-k)!(n'_--l)!}R_{++}^k R_{-+}^{n'_+-k}R_{+-}^l R_{--}^{n'_-l} =$$

Using $l = n_+ - k$

$$\sum_{k=0}^{n'_+}\frac{\sqrt{n_+!n_-!n'_+!n'_-!}}{(n_+-k)!k!(n'_+-k)!(n'_--n_++k)!}R_{++}^k R_{-+}^{n'_+-k}R_{+-}^{n'_+-k}R_{--}^{n'_--n_++k}$$

the last step is to replace $n_\pm = j \pm \mu$

$$\sum_{k=0}^{j+\nu} \frac{\sqrt{(j+\mu)!(j-\mu)!(j+\nu)!(j-\nu)!}}{(j+\mu-k)!k!(j+\nu-k)!(k-\nu-\mu)!} R_{++}^{k} R_{-+}^{j+\mu-k} R_{+-}^{j+\nu-k} R_{--}^{k-\nu-\mu}$$

which is the expression of the $D^j_{\mu\nu}[R]$ as a function of the $SU(2)$ matrix elements

$$\boxed{D^j_{\mu\nu}[R] = \sum_{k=0}^{j+\nu} \frac{\sqrt{(j+\mu)!(j-\mu)!(j+\nu)!(j-\nu)!}}{(j+\mu-k)!k!(j+\nu-k)!(k-\nu-\mu)!} R_{++}^{k} R_{-+}^{j+\mu-k} R_{+-}^{j+\nu-k} R_{--}^{k-\nu-\mu}}$$

where

$$R = e^{i\frac{\theta}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}$$

1. *Elastic neutrino-deuteron scattering*

$$\langle \mathbf{p}'_D, \mu'_D, D, \mathbf{p}'_\nu, \mu'_\nu | T | \mathbf{p}_D, \mu_D, D, \mathbf{p}_\nu, \mu_\nu \rangle = \\ -\frac{G_F}{\sqrt{2}} (2\pi)^3 \langle \mathbf{p}'_D, \mu'_D, D | J_w^\alpha(0) | \Phi_D | \mathbf{p}_D, \mu_D, D \rangle g_{\alpha\beta} \langle \mathbf{p}'_\nu, \mu'_\nu | J_\nu^\beta(0) | \mathbf{p}_\nu, \mu_\nu \rangle \,, \qquad (449)$$

where G_F is the Fermi constant. The neutrino matrix element is written as

$$\langle \mathbf{p}'_\nu, \mu'_\nu | J_\nu^\beta(0) | \mathbf{p}_\nu, \mu_\beta \rangle \equiv \frac{1}{(2\pi)^3} \frac{1}{\sqrt{4|\mathbf{p}_\nu||\mathbf{p}'_\nu|}} L^\beta \left(\mathbf{p}'_\nu, \mu'_\nu, \mathbf{p}_\nu, \mu_\nu\right) \,, \qquad (450)$$

with

$$L^\beta \left(\mathbf{p}'_\nu, \mu'_\nu, \mathbf{p}_\nu, \mu_\nu\right) = \bar{u}_\nu(\mathbf{p}'_\nu, \mu'_\nu) \gamma^\beta \left(1 - \gamma_5\right) u_\nu(\mathbf{p}_\nu, \mu_\nu) \qquad (451)$$

and the Dirac spinors for massless neutrinos defined as

$$u_\nu(\mathbf{p}_\nu, \mu_\nu) = \sqrt{|\mathbf{p}_\nu|} \begin{pmatrix} \chi_{\mu_\nu} \\ \frac{p_\nu \cdot \boldsymbol{\sigma}}{|\mathbf{p}_\nu|} \chi_{\mu_\nu} \end{pmatrix} . \qquad (452)$$

Also for weak reactions we include in the nuclear matrix elements

$$\langle \mathbf{p}'_D, \mu'_D, D | J_w^\alpha(0) | \mathbf{p}_D, \mu_D, D \rangle \equiv \frac{1}{(2\pi)^3} N^\alpha \left(\mathbf{p}'_D, \mu'_D, \mathbf{p}_D, \mu_D\right) \,,$$

only the single-nucleon contributions in the well known form:

$$p', \mu', \tau' J_k^\mu(0) p, \mu, \tau \\ = \delta_{\tau'\tau} \, \bar{u}_N(p', \mu') \left(F_{1,\tau}^N(Q^2) \gamma^\mu + \frac{i}{2m} \sigma^{\mu\nu} q_\nu F_{2,\tau}^N(Q^2) \\ + F_{A,\tau}^N(Q^2) \gamma^\mu \gamma_5 + \frac{q^\mu}{m} \gamma_5 F_{P,\tau}^N(Q^2) \right) u_N(p, \mu) \,, \qquad (453)$$

where $q^\mu = p'^\mu - p^\mu$ and the weak neutral-current nucleon form factors $F_{i,\tau}^N$ depend on the nucleon isospin. For these quantities we use the parametrizations from Refs. [**? ?**]. Actually the part with $F_{P,\tau}$ gives no contribution in Eq. (453) in the case of massless neutrinos but we keep it, since the single nucleon charged current has the same functional form, despite different isospin dependence.