Relativistic invariance in Euclidean formulations of quantum mechanics.

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Relativistic invariance in Euclidean formulations of quantum mechanics is discussed. Relativistic treatments of quantum theory are needed to study hadronic systems at sub-hadronic distance scales. Euclidean formulations of relativistic quantum mechanics have some computational advantages. In the Euclidean representation the physical Hilbert space inner product is expressed in terms of Euclidean space-time variables with no need for any analytic continuation. The identification of the complex Euclidean group with the complex Poincaré group relates the infinitesimal generators of both groups. In this work explicit representations of the Poincaré generators in Euclidean space-time variables for all positive-mass positive-energy irreducible representations of the Poincaré group are derived. The commutation relations are checked, both hermiticity and self-adjointness are established, and reflection positivity of the kernels is verified.

I. INTRODUCTION

This paper discusses how relativistic invariance is realized in Euclidean formulations of relativistic quantum theory. In a quantum theory relativistic invariance means that quantum observables, which are probabilities, expectation values and ensemble averages, have the same value for equivalent experiments that are performed in different inertial coordinate systems. This means that experiments performed in an isolated system cannot be used to distinguish inertial coordinate systems. In special relativity different inertial coordinate systems are related by the subgroup of Poincaré group connected to the identity. In 1939 Wigner [1] showed that a necessary and sufficient condition for a quantum system to be relativistically invariant is that vectors representing equivalent quantum states in different inertial coordinate systems are related by a unitary ray representation of this subgroup on the Hilbert space of the quantum theory.

Relativistically invariant quantum theories are needed to study physics on distance scales that are small enough to be sensitive to the internal structure of a nucleon. This is because in order to get wavelengths short enough to resolve the internal structure of a nucleon it is necessary to transfer a momentum to the nucleon that is comparable to or larger than its mass scale.

In quantum theories time evolution is generated by a one-parameter unitary group. The infinitesimal generator of this group is the Hamiltonian, which is a positive self-adjoint operator on the Hilbert space of the quantum theory. Because the spectrum of the Hamiltonian, time can be analytically continued to the lower-half complex time plane. For imaginary times, $t \to -i\tau$, the unitary time evolution group becomes a contractive Hermitian semigroup. For any fixed $\tau > 0$, $e^{-H\tau}$ has the same eigenvectors as the Hamiltonian, and the eigenvalues λ of H are related to the eigenvalues η of $e^{-H\tau}$ by $\lambda = -\ln(\eta)/\tau$. This implies that it is possible to solve dynamical problems directly in a Euclidean representation. For some applications it is enough to replace H by $e^{-\tau H}$. This is a well-behaved bounded operator with a spectrum on the unit interval $[0, 1]$; the parameter $\tau > 0$ can be adjusted to be sensitive to different parts of the spectrum of H. Relativistic invariance normally requires an analytic continuation back to real time. These observations provide the motivation for investigating Euclidean approaches to relativistic quantum field theory and quantum mechanics.

Euclidean approaches were first advocated by Schwinger [2][3] who used the spectral condition in time-ordered Green's functions to establish the existence of an analytic continuation to imaginary times. Independently, axiomatic treatments of quantum field theory [4][5] led to an understanding of the analytic properties of vacuum expectation values of products of fields, also based on the spectral condition. The Euclidean approach to quantum field theory was advocated by Symanzik [6][7], and developed by Nelson [8]. Osterwalder and Schrader [9][10] identified properties of Euclidean covariant distributions that are sufficient to reconstruct a relativistic quantum field theory. Two observations that are contained in the work of Osterwalder and Schrader are (1) that an explicit analytic continuation is

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not necessary to construct a relativistic quantum theory and (2) the reconstruction of a relativistic quantum theory is not limited to local field theories. The discussion that follows is motivated by these two observations.

The Poincaré and four-dimensional Euclidean groups are related because the parameters of both groups can be analytically continued and the covering group of the resulting complex groups are identical, $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. What this means is that the real Poincaré group can be considered to be a complex subgroup of the complex Euclidean group, or conversely, the real Euclidean group can be considered to be a complex subgroup of the complex Poincaré group. These identifications imply formal relations between the infinitesimal generators of the Poincaré group and the real Euclidean group [11][12]. Specifically, if $P_e^0, \mathbf{P}_e, J_e^{ij}, J_e^{0i}$ satisfy the commutation relations of the Euclidean Lie Algebra, then the operators $P_m^0 = -iP_e^0, \mathbf{P}_m = \mathbf{P}_e, J_m^{ij} = J_e^{ij}, J_m^{0i} := -iJ_e^{0i}$, will satisfy the commutation relations of the Poincaré Lie Algebra. However, because of the factors of i, both sets of operators cannot be self-adjoint on the same representation of the Hilbert space.

Osterwalder and Schrader construct a new Hilbert space representation where the Poincaré generators become self adjoint. Osterwalder and Schrader start with a representation of a Hilbert space defined with a Euclidean covariant kernel. On this space the Euclidean transformations are norm preserving which defines a unitary representation of the Euclidean group. Next they choose an arbitrary time direction and multiply the final Euclidean time variables in this kernel by an operator that reverses the sign of all of the final Euclidean times. Introducing this time reflection in the Euclidean kernel breaks the Euclidean invariance and has the effect of making the Poincaré generators constructed from the Euclidean generators Hermitian on this space. The integration variables remain unchanged - they include the Euclidean times. The problem is that the resulting quadratic form cannot be positive for all Euclidean test functions. This is easily seen by taking functions with positive time support and extending them to be even or odd under time reflection. Since the quadratic forms will have opposite signs, they cannot both have positive norm with this new inner product. This flaw is fixed by projecting the test functions on a suitable subspace. The subspace identified by Osterwalder and Schrader is the subspace of functions of Euclidean space-time variables with support for positive absolute and relative Euclidean times. The Euclidean kernels are called *reflection positive* if the norms with respect to the inner product with the Euclidean time reflection is non-negative on this subspace. Reflection positivity is a constraint on the Euclidean distributions [13]. This construction is a specific application of a general construction based on an abstract notion of reflection positivity [11][12].

Because this projection is independent of the form of the Euclidean kernel, cluster properties, which are an important physical requirement, can be expressed entirely in terms of properties of the kernel - the range of the projector does not change. Cluster properties can be a difficult constraint to satisfy in some representations of relativistically quantum mechanics [14][15][16], but it can be easily achieved in the Euclidean approach.

The restriction to positive relative time is because the Euclidean kernels for irreducible representations of the Poincaré group become singular for zero relative times. Since identical particles have an exchange symmetry, this is reflected in the symmetry properties of the Euclidean kernel. As long as the relative time supports are disjoint, the symmetry can be used to reorder the variables so the support satisfies the positive relative-time condition. What separates relativistic quantum theory from local quantum field theory is whether the symmetries involve all of the coordinates in the kernel or just separately involve the initial and final coordinates. This will be discussed in more detail below.

Reflection positivity is a strong constraint, particularly when it is combined with Euclidean covariance and cluster properties. One consequence is that it implies the spectral condition that Schwinger originally used to justify the existence of an analytic continuation. The advantage of the Osterwalder-Schrader reconstruction is that this analytic continuation is never explicitly needed.

In this paper a Euclidean relativistic theory is defined by a finite or infinite collection of Euclidean covariant tempered distributions

$$S_{m:n}(x_m, \cdots, x_1; y_1, \cdots, y_n). \tag{1}$$

These kernels contain the dynamics. The kernels satisfy the permutation symmetry,

$$S_{m:n}(x_m, \cdots, x_1; y_1, \cdots, y_n) = (\pm)^{|\sigma|} S_{m:n}(x_{\sigma(m)} \cdots, x_{\sigma(1)}; y_1, \cdots, y_n)$$

$$= (\pm)^{|\sigma|} S_{m:n}(x_m, \cdots, x_1; y_{\sigma(1)}, \cdots, y_{\sigma(n)}) \tag{2}$$

where $\sigma()$ is a permutation on m or n objects, $|\sigma|$ is 0 if σ is an even permutation and 1 if it is an odd permutation. The $+$ sign is for Bosons and the $(-)$ sign is for Fermions. For local quantum field theories the collection must be infinite, $S_{m:n} = S_{k:l}$ whenever $m + n = k + l$, and the permutation symmetry is with respect to all $n + m$ variables. In (1-2) the x_n can also include spin degrees of freedom.

The symmetry in the local field theory case arises because the domain of analyticity, that comes from the spectral condition, can be extended by complex Lorentz transformations. The extended domain of analyticity includes real space-like separated points (Jost points) that allow the local fields to be reordered [4], relating Green's functions with permuted arguments. This symmetry is not assumed in this work. One consequence of relaxing this condition is that it is possible to have different N-point Green's functions for different numbers of initial and final coordinates.

The setting for a quantum theory is a Hilbert space. A dense set of vectors in the Euclidean representation of the Hilbert space are sequences of Schwartz test functions of Euclidean space-time variables

$$\{\psi_n(x_1 \cdots x_n)\}_{n=0}^N \tag{3}$$

that vanish unless the Euclidean times satisfy $0 < x_1^0 < x_1^0 < \cdots < x_n^0$.

The Hilbert space inner product is

$$\langle\psi|\phi\rangle = \sum_{mn} \int d^4x_m \cdots d^4x_1 d^4y_1 \cdots d^4y_n \psi_m^*(\theta x_m, \cdots, \theta x_1) \times$$

$$S_{m:n}(x_m, \cdots, x_1; y_1, \cdots, y_n)\phi_n(y_1, \cdots, y_n) \tag{4}$$

where θ represents Euclidean time reflection, $\theta(x_i^0, \mathbf{x}_i) := (-x_i^0, \mathbf{x}_i)$. Because of the assumed symmetry properties of the Euclidean distributions, as long as the Euclidean time supports in the functions are ordered for one set of Euclidean times, the permutation symmetry can be used to replace them for one that is ordered as above.

One property of this representation of the Hilbert space, where the inner products has a non-trivial kernel, is that distributions like delta functions represent normalizable vectors.

For this to be a Hilbert space scalar product, this quantity must be non-negative whenever $\{\psi_m\} = \{\phi_n\}$. This condition is called reflection positivity. In general there can be 0-norm vectors. The Hilbert space vectors are Cauchy sequences of equivalence classes of vectors, where two vectors are in the same class if the norm of their difference vanishes. This distinction will be ignored in what follows. For free particles, reflection positivity restricts the form of the allowed distributions [17][18][19]. They are singular when the relative Euclidean coordinates vanish. The restriction picks a domain where the scalar products are finite.

Because the Euclidean time reflection breaks the Euclidean invariance, both Euclidean time translation and rotations in Euclidean space-time planes are no longer unitary on this space. These transformations are nevertheless defined on this space with restricted domains; they represent translations in imaginary time and boosts with imaginary rapidity. The infinitesimal forms of these elementary Euclidean transformations can be used to construct both the Hamiltonian and Lorentz boost generators.

The purpose of this work is to give a detailed discussion of how relativistic invariance is realized in these theories. Rather than consider a general set of Euclidean covariant kernels, this work is limited to Euclidean representations of irreducible representation of the Poincaré group [20]. There are two motivations for this. The first is that the kernels for these representations are known, so it is possible to understand domain issues related to the properties of the kernel and give explicit representations for the Poincaré generators. The second motivation is that any unitary representation of the Poincaré group can be decomposed into a direct integral of irreducible representations. In a relativistic quantum theory these can be identified with the complete set of one-body states plus multi-particle in or out scattering states. These states either transform irreducibly or as tensor products of irreducible representations. The kernel of a general interacting model should be related to the direct integral of irreducible kernels by a unitary transformation. The construction of this direct integral from a general set of Euclidean covariant distributions is the relativistic analog of diagonalizing the Hamiltonian in non-relativistic quantum mechanics. This will not be considered in this work.

In the next section the Poincaré group and its relation to the Euclidean group is discussed. In section three unitary representations of the Poincaré group are discussed, along with structure of positive mass irreducible representations. Section 4 contains explicit forms of Euclidean covariant kernels of irreducible representations of the Poincaré group for any mass and spin. They are shown to be reflection positive. Explicit forms for all the Poincaré generators are constructed, commutation relations are verified, and the generators are shown to be symmetric with respect the inner product with the Euclidean time reflection. Section 5 discuss the self-adjointness of the Hamiltonian and rotationless boost generators. Section 6 has a brief discussion of finite Poincaré transformations. The results are summarized in section 7.

II. BACKGROUND

The Poincaré group is the group of space-time transformations that relate different inertial reference frames in the theory of special relativity. It is the symmetry group that preserves the proper time τ_{ab}, or proper distance, d_{ab}, between any two events with space-time coordinates x_a^μ, x_b^μ

$$-\tau_{ab}^2 = d_{ab}^2 = \eta_{\mu\nu}(x_a - x_b)^\mu (x_a - x_b)^\nu, \tag{5}$$

where $\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = 1$, $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$ is the Minkowski metric tensor. Repeated indices are assumed to be summed. The most general point transformation, $x'^\mu = f^\mu(x)$ satisfying (5) has the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{6}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation satisfying

$$\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \eta_{\alpha\beta} \Lambda^\beta{}_\nu$$

or in matrix form

$$\eta = \Lambda^t \eta \Lambda. \tag{7}$$

Equations (6) and (7) are relativistic generalizations of the fundamental theorem of rigid body motion, which asserts that any motion that preserves the distance between points in a rigid-body in a composition of an orthogonal transformation and a translation.

The full Poincaré group contains discrete transformations that are not associated with special relativity. Equation (7) implies that

$$\det(\Lambda)^2 = 1 \qquad \text{and} \qquad (\Lambda_0^0)^2 = 1 + \sum_i (\Lambda_i^0)^2. \tag{8}$$

This means that the Lorentz group can be decomposed into four topologically disconnected components

- $\det(\Lambda) = 1, \qquad (\Lambda_0^0) \geq 1$; includes identity
- $\det(\Lambda) = -1, \qquad (\Lambda_0^0) \geq 1$; includes space reflection
- $\det(\Lambda) = -1, \qquad (\Lambda_0^0) \leq -1$; includes time reversal
- $\det(\Lambda) = 1, \qquad (\Lambda_0^0) \leq -1$; includes space-time reversal .

Since the discrete symmetries of space reflection and time reversal are not symmetries of the weak interaction, the symmetry group associated with special relativity is normally considered to be the subgroup of Poincaré transformations that is continuously connected to the identity. This subgroup contains the active transformations that can be experimentally realized.

The relation between the Lorentz group and the four-dimensional orthogonal group can be understood by expressing Minkowski, x^μ, and Euclidean, x_e^μ, four vectors as 2×2 matrices:

$$X_m = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(X\sigma_\mu) \tag{9}$$

$$X_e = x_e^\mu \sigma_{e\mu} = \begin{pmatrix} ix_e^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & ix_e^0 - x^3 \end{pmatrix} \qquad x_e^\mu = \frac{1}{2}\mathrm{Tr}(X_e \sigma_{e\mu}^\dagger). \tag{10}$$

where $\sigma_i = \sigma_{ei}$ are the Pauli matrices, σ_0 is the identity and $\sigma_{e0} = i\sigma_0$. The determinants of these matrices are related to the Minkowski and Euclidean line elements respectively:

$$\det(X_m) = (x^0)^2 - \mathbf{x} \cdot \mathbf{x} \qquad \det(X_e) = -\left((x_e^0)^2 + \mathbf{x} \cdot \mathbf{x}\right). \tag{11}$$

The linear transformations that preserve the determinant and hermiticity of X_m have the form

$$X_m \to X'_m = \pm A X_m A^\dagger \qquad \det(A) = 1. \tag{12}$$

The (-) sign represents a space-time reflection, which is not considered part of the symmetry group of special relativity. The group of complex 2×2 matrices with $\det(A) = 1$ is $SL(2, \mathbb{C})$. Similarly linear transformations corresponding to real four-dimensional orthogonal transformations have the general form

$$X_e \to X'_e = A X_e B^t \qquad A, B \in SU(2). \tag{13}$$

Transformations of the form

$$X_e \to X'_e = A X_e B^t \qquad X_m \to X'_m = A X_m B^t \tag{14}$$

with both A and B in $SL(2,\mathbb{C})$ preserve both the Minkowski and Euclidean line elements. However they do not preserve the reality of the four vectors. They represent complex Lorentz or orthogonal transformations.

This shows that the covering group of both the complex Lorentz and complex orthogonal group is $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. This means that the real Lorentz group can be considered to be a subgroup of the complex orthogonal group; similarly the real orthogonal group can be considered to be a complex subgroup of the Poincaré group. The relevant relation that will be exploited in this work is that Euclidean rotations that involve a space and the Euclidean time coordinate can be identified with Lorentz boosts with complex rapidity.

For the full Poincaré group it is necessary to include translations. Euclidean time translations by τ are identified with Minkowski time translations with $t = -i\tau$.

III. UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP

In this section Poincaré group elements are labeled by (Λ, A) where Λ is a $SL(2,\mathbb{C})$ matrix and A is a 2×2 Hermitian matrix representing a translation. In this representation Poincaré transformations have the form

$$X' = \Lambda X \Lambda^\dagger + A \tag{15}$$

where the group multiplication law is

$$(\Lambda_2, A_2)(\Lambda_1, A_1) = (\Lambda_2\Lambda_1, \Lambda_2 A_1 \Lambda_2^\dagger + A_2). \tag{16}$$

Four vector representations of these equations are

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{17}$$

$$(\Lambda^\mu{}_\nu, a^\mu) = (\Lambda^\mu_{2\,\alpha}\Lambda^\alpha_{1\,\nu}, \Lambda^\mu_{2\,\alpha} a_1^\alpha + a_2^\mu), \tag{18}$$

where the four vector and 2×2 representations are related by

$$a^\mu := \frac{1}{2}\mathrm{Tr}(\sigma_\mu A) \qquad \Lambda^\mu{}_\nu := \frac{1}{2}\mathrm{Tr}(\sigma_\mu \Lambda \sigma_\nu \Lambda^\dagger). \tag{19}$$

$SL(2,\mathbb{C})$ is a six parameter group. It has six independent one-parameter subgroups

$$\Lambda_r(\boldsymbol{\theta}) = e^{\frac{i}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\theta}} \qquad \Lambda_b(\boldsymbol{\rho}) = e^{\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\rho}} \tag{20}$$

corresponding to rotations about three different axes and rotationless Lorentz boosts in three different directions. In these expressions $\boldsymbol{\theta}$ represents the angle and axis of a rotation while $\boldsymbol{\rho}$ represents the rapidity and direction of a rotationless boost. The polar decomposition expresses a general $SL(2,\mathbb{C})$ matrix Λ as a product of a rotation (Λ_r unitary) followed by rotationless boost (Λ_b positive Hermitian):

$$\Lambda = \Lambda_b \Lambda_r \tag{21}$$

where

$$\Lambda_b := (\Lambda\Lambda^\dagger)^{1/2} = \Lambda_b(\boldsymbol{\rho}) \qquad \Lambda_r := (\Lambda\Lambda^\dagger)^{-1/2}\Lambda = \Lambda_r(\boldsymbol{\theta}). \tag{22}$$

A unitary representation of the Poincaré group (inhomogeneous $SL(2,\mathbb{C})$) is a set of unitary operators $U(\Lambda, A)$, labeled by elements of $SL(2,\mathbb{C})$ satisfying

$$U(\Lambda_2, A_2)U(\Lambda_1, A_1) = U(\Lambda_2\Lambda_1, \Lambda_2 A_1 \Lambda_2^\dagger + A_2) \tag{23}$$

$$U(I, 0) = I \tag{24}$$

$$U^\dagger(\Lambda, A) = U^{-1}(\Lambda, A) = U(\Lambda^{-1}, -\Lambda^{-1}A(\Lambda^\dagger)^{-1}). \tag{25}$$

The Poincaré group is a 10 parameter group. Infinitesimal generators are the 10 self-adjoint operators defined by

$$H = i\frac{d}{da^0}U(I, a^0\sigma_0)_{|a_0=0} \tag{26}$$

$$P^i = -i\frac{d}{da^j}U(I, a^j\sigma_j)_{|a_i=0} \tag{27}$$

$$J^i = -i\frac{d}{d\theta}U(e^{i\frac{\theta}{2}\sigma_j}, 0)_{|\theta=0} \tag{28}$$

$$K^i = -i\frac{d}{d\rho}U(e^{\frac{\rho}{2}\sigma_j}, 0)_{|\rho=0} \tag{29}$$

where there is no sum in (27) over the repeated j, and $j \in \{1, 2, 3\}$ in (27-29). The group representation property (23) implies that these generators satisfy commutation relations

$$[J^i, J^j] = i\epsilon_{ijk}J^k \qquad [J^i, P^j] = i\epsilon_{ijk}P^k \qquad [J^i, K^j] = i\epsilon_{ijk}K^k \tag{30}$$

$$[K^i, K^j] = -i\epsilon_{ijk}J^k \qquad [J^i, H] = 0 \qquad [P^i, H] = 0 \tag{31}$$

$$[K^j, H] = iP^j \qquad [K^i, P^j] = i\delta_{ij}H. \tag{32}$$

These operators are components of a four vector, P^μ, and an anti-symmetric tensor operator, $J^{\mu\nu}$,

$$P^\mu = (H, \mathbf{P}) \qquad J^{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J_1 & 0 \end{pmatrix}. \tag{33}$$

There are two independent polynomial invariants

$$M^2 = (P^0)^2 - \mathbf{P}^2 = -P^\mu P_\mu \tag{34}$$

and

$$W^2 = W^\mu W_\mu \qquad W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}. \tag{35}$$

where W^μ is called the Pauli-Lubanski vector. When $M \neq 0$ the spin is defined by

$$S^2 = W^2/M^2. \tag{36}$$

A spin vector can be defined by an *operator* rotationless (canonical) boost that transforms the angular momentum tensor to the rest frame:

$$s^i = \epsilon_{ijk}\Lambda_c^{-1}(P)^j{}_\mu\Lambda_c^{-1}(P)^k{}_\nu J^{\mu\nu} \tag{37}$$

where

$$\Lambda_c(P)^\mu{}_\nu = \begin{pmatrix} V^0 & \mathbf{V} \\ \mathbf{V} & I + \frac{\mathbf{V}\otimes\mathbf{V}}{1+V^0} \end{pmatrix} \qquad V^\mu = P^\mu/M \tag{38}$$

and P^μ and M are considered operators. Note that $\Lambda_c(p) = \Lambda_b(\boldsymbol{\rho})$ with

$$\mathbf{V} = \mathbf{P}/M = \hat{\boldsymbol{\rho}}\sinh(\rho). \tag{39}$$

This spin vector is called the canonical spin; other types of spin vectors (helicity, light-front spin) are related to the canonical spin by momentum dependent rotations. For the purpose of this work it is sufficient to consider the canonical spin. The canonical spin can also be expressed in terms of the Pauli Lubanski vector:

$$\begin{pmatrix} 0 \\ \mathbf{s}_c \end{pmatrix} = -\frac{1}{2M}\Lambda_c^{-1}(P)^\mu{}_\nu W^\mu. \tag{40}$$

The components of the spin satisfy $SU(2)$ commutations relations:

$$[s_i, s_j] = i\epsilon_{ijk} s^k. \tag{41}$$

With these definitions, for $M > 0$, $M^2, s^2, \mathbf{P}, s_z$ are a maximal set of commuting self-adjoint functions of the Poincaré generators. The spectrum of each component of $\mathbf{P}$ is the real line since each component of $\mathbf{P}$ can be boosted to any value. Similarly the spectrum of spins are restricted to be integral or half integral as a consequence of the $SU(2)$ commutations relations. In a general system these commuting observables are not complete; they can be supplemented by additional Poincaré-invariant degeneracy quantum numbers, which will be denoted by α. A basis for the Hilbert space are the simultaneous eigenstates of $M, S^2, \alpha, \mathbf{P}, s_z$,

$$|(m, s, \alpha)\mathbf{p}, \mu\rangle. \tag{42}$$

Because these vectors are constructed out of eigenvalues of functions of P^μ and $J^{\mu\nu}$, which have well-defined the Poincaré transformation properties, the Poincaré transformation properties of these basis state follow from the definitions

$$U(\Lambda, a)|(m, s, \alpha)p, \mu\rangle = e^{i\Lambda p\cdot a}|(m, s, \alpha)\Lambda p, \nu\rangle D^j_{\nu\mu}[R_{ws}(\Lambda, p)]\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}} \tag{43}$$

where, $R_{cw}(\Lambda, p) := \Lambda_c^{-1}(\Lambda p)\Lambda\Lambda_c(p)$ is the canonical-spin Wigner rotation, $\Lambda_c(p) = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}}$ where $\boldsymbol{\rho}$ is the rapidity of a particle of mass m and momentum $\mathbf{p}$, and $\omega_m(p) := \sqrt{m^2 + \mathbf{p}^2}$ is the energy of the particle.

The Wigner D-function is the finite dimensional unitary representation of the rotation group in the $|s, \mu\rangle$ basis [21]:

$$D^s_{m,m'}[R] = \langle s, \mu|U(R)|s, \mu'\rangle =$$

$$\sum_{k=0}^{s+\mu} \frac{\sqrt{(s+\mu)!(s+\mu')!(s-\mu)!(s-\mu')!}}{k!(s+\mu'-k)!(s+\mu-k)!(k-\mu-\mu')!} R^k_{++} R^{s+\mu'-k}_{+-} R^{s+\mu-k}_{-+} R^{k-\mu-\mu'}_{--}$$

where

$$R = \left(\begin{array}{cc} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{array}\right) = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \sigma_0 \cos(\frac{\theta}{2}) + i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin(\frac{\theta}{2}) \tag{44}$$

is a $SU(2)$ matrix. Because $D^s_{\mu\nu}[R]$ is a degree $2s$ polynomial in the matrix elements of R, and $R = e^{i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}}$ is an entire function of the angles, $\boldsymbol{\theta}$, it follows that $D^s_{\mu,\mu'}[e^{i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}}]$ is an entire function of all three components of $\boldsymbol{\theta}$. This means that the group representation property

$$\sum_{\mu''} D^s_{\mu,\mu''}[R_2]D^s_{\mu'',\mu'}[R_1] - D^s_{\mu,\mu'}[R_2R_1] = 0, \tag{45}$$

and the formulas for adding angular momenta

$$D^s_{\mu,\mu'}[R] - \sum_{\mu_1\mu_2\mu_1'\mu_2'} \langle s, \mu|s_1, \mu_1, s_2, \mu_2\rangle D^{s_1}_{\mu_1,\mu_1'}[R]D^{s_2}_{\mu_2,\mu_2'}[R]\langle s_1, \mu_1', s_2', \mu_2'|s, \mu'\rangle = 0 \tag{46}$$

and

$$D^{s_1}_{\mu_1,\mu_1'}[R]D^{s_2}_{\mu_2,\mu_2'}[R] - \sum_{s\mu\mu'}\langle s_1, \mu_1, s_2, \mu_2|s, \mu\rangle D^s_{\mu,\mu'}[R]\langle s, \mu'|s_1, \mu_1', s_2, \mu_2'\rangle = 0, \tag{47}$$

which hold for real angles, can be analytically continued to complex angles. This means that (45-47) also hold when the $SU(2)$ matrices R are replaced by $SL(2,\mathbb{C})$ matrices. In these expressions, $\langle s, \mu|s_1, \mu_1, s_2, \mu_2\rangle$, are $SU(2)$ Clebsch-Gordan coefficients. While the analytic continuation preserves the group representation and angular momentum addition properties, it does not preserve unitarity.

IV. EUCLIDEAN FORMULATION

The common property of any relativistic quantum theory is that it can be decomposed into a direct integral of irreducible representations. The structure of irreducible representations of the Poincaré group in the Euclidean representation can be understood by starting with Minkowski-space irreducible representations of the Poincaré group. This work considers only positive-mass positive-energy representations. These can be expressed in a basis of simultaneous eigenstates of the mass, spin, linear momentum and z-component of the canonical spin. The action of the unitary representation of the Poincaré group on this basis is given by (43). This is unitary for basis vectors with the normalization:

$$\langle (m',s')p',\mu' \vert (m,s)p,\mu\rangle = \delta_{m'm}\delta_{s's}\delta(\mathbf{p}'-\mathbf{p})\delta_{\mu'\mu}. \tag{48}$$

Because of the unitarity of $R_{wc}(\Lambda,p)$, the $SU(2)$ Wigner rotation can be expressed in two equivalent ways:

$$R_{wc}(\Lambda,p) = \Lambda_c^{-1}(\Lambda p)\Lambda\Lambda_c(p) = \Lambda_c^{\dagger}(\Lambda p)(\Lambda^{\dagger})^{-1}\Lambda_c^{\dagger -1}(p). \tag{49}$$

The $SL(2,\mathbb{C})$ group representation property (45) implies that the unitary representation of the Wigner rotation can be factored into a product of three finite-dimensional representations of $SL(2,\mathbb{C})$ in two different ways:

$$D^s_{\nu\mu}[R_{wc}(\Lambda,p)] = \sum_{\alpha\beta} D^s_{\nu\alpha}[\Lambda_c^{-1}(\Lambda p)]D^s_{\alpha\beta}[\Lambda]D^s_{\beta\mu}[\Lambda_c(p)] \tag{50}$$

or

$$D^s_{\nu\mu}[R_{wc}(\Lambda,p)] = \sum_{\alpha\beta} D^s_{\nu\alpha}[\Lambda_c^{\dagger}(\Lambda p)]D^s_{\alpha\beta}[(\Lambda^{\dagger})^{-1}]D^s_{\beta\mu}[(\Lambda_c^{\dagger})^{-1}(p)]. \tag{51}$$

These relations can be used to rewrite equation (43) in terms of new Lorentz covariant basis states:

$$U(\Lambda,a)\underbrace{\sum_{\alpha} \vert (m,j)p,\alpha\rangle D^s_{\alpha\mu}[\Lambda_b^{-1}(p)]\sqrt{\omega_m(p)}}_{\vert (m,j)p,\mu\rangle_{cov}} =$$

$$e^{i\Lambda p\cdot a}\sum_{\beta}\underbrace{\sum_{\alpha} \vert (m,j)\Lambda p,\alpha\rangle D^s_{\alpha\beta}[\Lambda_c^{-1}(\Lambda p)]\sqrt{\omega_m(\Lambda p)}}_{\vert (m,j)\Lambda p,\beta\rangle_{cov}} D^s_{\beta\mu}[\Lambda] \tag{52}$$

or

$$U(\Lambda,a)\underbrace{\sum_{\alpha} \vert (m,j)p,\alpha\rangle D^s_{\alpha\mu}[\Lambda_c^{-\dagger}(p)]\sqrt{\omega_m(p)}}_{\vert (m,j)p,\mu\rangle_{cov*}} =$$

$$e^{i\Lambda p\cdot a}\sum_{\beta}\underbrace{\sum_{\alpha} \vert (m,j)\Lambda p,\alpha\rangle D^s_{\alpha\beta}[\Lambda_c^{\dagger}(\Lambda p)]\sqrt{\omega_m(\Lambda p)}}_{\vert (m,j)\Lambda p,\beta\rangle_{cov*}} D^s_{\beta\mu}[(\Lambda^{\dagger})^{-1}]. \tag{53}$$

These expressions replace the states (42) that transform covariantly with respect to the Poincaré group with states that transform covariantly with respect to $SL(2,\mathbb{C})$. The transformations relating the Lorentz and Poincaré covariant representations are invertible, however there are two distinct Lorentz covariant representations, because while $R = (R^{\dagger})^{-1}$ for $R\in SU(2)$, the corresponding representations in $SL(2,\mathbb{C})$ are inequivalent. These two representations are called right and left handed representations for reasons that will become apparent.

In the Lorentz covariant representations, (52) and (53), this equivalence can be used to show that the Hilbert space inner product of two $SL(2,\mathbb{C})$ covariant wave functions has a non-trivial kernel

$$\langle\psi\vert\phi\rangle = \sum_{\mu}\int \langle\psi\vert (m,j)p,\mu\rangle d\mathbf{p}\langle (m,j)p,\mu\vert\phi\rangle =$$

$$\int \sum_{\mu\nu} \langle \psi | (m,j)p,\mu\rangle_{cov} D^j_{\mu\nu}[p\cdot\sigma] 2\delta(p^2+m^2)\theta(p^0) d^4p_{cov}\langle (m,j)p,\nu|\phi\rangle \tag{54}$$

$$\langle\psi|\phi\rangle = \int \sum_{\mu} \langle \psi | (m,j)p,\mu\rangle d\mathbf{p} \langle (m,j)p,\mu|\phi\rangle =$$

$$\int \sum_{\mu\nu} \langle \psi | (m,j)p,\mu\rangle_{cov*} D^j_{\mu\nu}[\Pi p\cdot\sigma] 2\delta(p^2+m^2)\theta(p^0) d^4p_{cov*}\langle (m,j)p,\nu|\phi\rangle \tag{55}$$

where $\Lambda_c(p)\Lambda_c^\dagger(p) = \sigma\cdot p$ and $\Lambda_c^{-1}(p)(\Lambda_c^\dagger)^{-1}(p) = \Pi p\cdot\sigma$, was used in these equations. Π is the space reflection operator and $p\cdot\sigma = \omega_m(p)\sigma_0 + \mathbf{p}\cdot\boldsymbol{\sigma}$. These equations explain why (54) and (55) are called right and left handed representations. These kernels are, up to normalization, spin-s two-point Wightman functions [4].

The motivation for considering these $SL(2,\mathbb{C})$ covariant representations is that they are naturally related to the corresponding Euclidean covariant representations.

To show this let $f(x_e,\mu)$ and $g(y_e,\nu)$ be functions of Euclidean space-time variables and spins with positive Euclidean-time support. Consider the following Euclidean covariant kernel:

$$S^s_e(x_e,\mu;y_e,\nu) := \int d^4p \frac{2}{(2\pi)^4} \frac{e^{ip_e\cdot(x_e-y_e)}}{p_e^2+m^2} D^s_{\mu\nu}(p_e\cdot\sigma_e). \tag{56}$$

The physical Hilbert space inner product (4) for this Euclidean Green's function has the form

$$\int \sum_{\mu\nu} d^4x_e d^4y_e f^*(\theta x_e,\mu) S^s_e(x_e,\mu;y_e,\nu) g(y_e,\nu) =$$

$$\int \sum_{\mu\nu} d^4p_e f^*(\theta x_e,\mu) \frac{2}{(2\pi)^4} \frac{e^{ip_e\cdot(x_e-y_e)}}{p_e^2+m^2} D^s_{\mu\nu}(p_e\cdot\sigma_e) g(y_e,\nu) =$$

$$\int \sum_{\mu\nu} \psi^*(\mathbf{p},\mu) \frac{d\mathbf{p}}{\omega_m(\mathbf{p})} D^s_{\mu\nu}(p\cdot\sigma)\phi(\mathbf{p},\nu) \tag{57}$$

where

$$\psi^*(\mathbf{p},\mu) := \frac{1}{(2\pi)^{3/2}} \int d\mathbf{x} d\tau e^{i\mathbf{p}\cdot\mathbf{x}-\omega_m(\mathbf{p})\tau} f^*(\mathbf{x},\tau,\mu) \tag{58}$$

and

$$\phi(\mathbf{p},\nu) := \frac{1}{(2\pi)^{3/2}} \int d\mathbf{x} d\tau e^{-i\mathbf{p}\cdot\mathbf{x}-\omega_m(\mathbf{p})\tau} g(\mathbf{x},\tau,\nu). \tag{59}$$

The Euclidean time-support condition ensures that the Laplace transforms with respect to the Euclidean times in (58) and (59) are well defined. The resulting kernel in (57) is identical to the covariant kernel in (54) after performing the integrals over the p_e^0.

This shows that the "Euclidean" inner product (57) can be identified with the corresponding Lorentz covariant inner product, which itself is identical to the original Poincaré covariant inner product.

This means that

$$S^s_r(x_e,\mu;y_e,\nu) := \int \frac{2d^4p}{(2\pi)^4} \frac{e^{ip_e\cdot(x_e-y_e)}}{p_e^2+m^2} D^s_{\mu\nu}(p_e\cdot\sigma_e) \tag{60}$$

is a Euclidean covariant reflection positive kernel for right handed representations of mass m and spin s respectively. The corresponding kernel for left-handed representations is

$$S^s_l(x_e,\mu;y_e,\nu) := \int \frac{2d^4p}{(2\pi)^4} \frac{e^{ip_e\cdot(x_e-y_e)}}{p_e^2+m^2} D^s_{\mu\nu}(\Pi p_e\cdot\sigma_e). \tag{61}$$

Space reflection interchanges right and left-handed representations. The space reflection operator does not commute with the Euclidean covariant kernel. This implies that space reflected states will not transform correctly under Lorentz transformations in these Lorentz covariant representations. Kernels for systems that allow a linear representation of space reflection can be constructed by taking direct sums of right and left handed kernels.

The kernels (60-61) can be evaluated analytically using the methods in [22]. The results are

$$S^s_r(z_e,\mu,\nu) := \frac{2}{(2\pi)^4}\int \frac{d^4p}{p_e^2+m^2} D^s_{\mu\nu}(p_e\cdot\sigma_e)e^{ip\cdot z_e} =$$

$$D^s_{\mu\nu}(-i\nabla_{ze}\cdot\sigma_e)\frac{2m^2}{(2\pi)^2}\frac{K_1(m\sqrt{z_0^2+\mathbf{z}^2})}{m\sqrt{z_0^2+\mathbf{z}^2}} \tag{62}$$

$$S^s_l(z_e,\mu,\nu) := \frac{2}{(2\pi)^2}\int \frac{d^4p}{p_e^2+m^2} D^s_{\mu\nu}(\Pi p_e\cdot\sigma_e)e^{ip_e\cdot z_e} =$$

$$D^s_{\mu\nu}(-i\Pi\nabla_{ze}\cdot\sigma_e)\frac{2m^2}{(2\pi)^2}\frac{K_1(m\sqrt{z_0^2+\mathbf{z}^2})}{m\sqrt{z_0^2+\mathbf{z}^2}} \tag{63}$$

where $z_e = x_e - y_e$. Note that $\frac{K_1(\eta)}{\eta}$ behaves like $1/\eta^2$ near the origin. Since $D^s_{\mu\nu}(-i\nabla_{ze}\cdot\sigma_e)$ is a degree $2s$ polynomial in $-i\nabla_{ze}$, these kernels have power law singularities at the origin, but fall off exponentially for large values of z_e^2, The restriction of the support of the vectors to positive Euclidean time ensures that $z_e^2 > 0$, so the singularity at $z_e = 0$ never causes a problem. These Green's functions are reflection positive on this space. This is because $D^s_{\mu\nu}(p\cdot\sigma)$ factors into a product of a matrix and its adjoint:

$$D^s_{\mu\nu}(p\cdot\sigma) = \sum_\alpha D^s_{\mu\alpha}(\Lambda_c(p))D^s_{\alpha\nu}(\Lambda_c(p))^\dagger. \tag{64}$$

The treatment of relativity follows from the relation between the four dimensional Euclidean group and the associated complex subgroup of the Lorentz group discussed in section 2. Consider the two matrices

$$p\cdot\sigma := \begin{pmatrix} p^0+p^2 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix} \qquad p_e\cdot\sigma_e := \begin{pmatrix} ip_e^0+p_e^2 & p_e^1-ip_e^2 \\ p_e^1+ip_e^2 & ip_e^0-p_e^3 \end{pmatrix}. \tag{65}$$

The $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ transformation properties of these matrices (denoted by P) are

$$P \to P' = APB^t. \tag{66}$$

The associated complex 4×4 Lorentz and four-dimensional orthogonal transformation matrices are

$$\Lambda(A,B)^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu A\sigma_\nu B^t) \qquad \mathcal{O}(A,B)^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma^\dagger_{e\mu}A\sigma_{e\nu}B^t). \tag{67}$$

For ordinary rotations $A = B^* = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}}$. For rotations about the $\hat{\mathbf{z}}$ axis

$$\mathcal{O}(A,A^*)(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\lambda) & \sin(\lambda) & 0 \\ 0 & -\sin(\lambda) & \cos(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{68}$$

and

$$\Theta\mathcal{O}(A,A^*)(\lambda)\Theta = \mathcal{O}(A,A^*)(\lambda). \tag{69}$$

For real rotations in Euclidean space-time planes $A = B^t = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$. For the case of the $x_e^0 - \hat{\mathbf{z}}$ plane

$$\mathcal{O}(A,A^t)(\lambda)x = \begin{pmatrix} \cos(\lambda) & 0 & 0 & \sin(\lambda) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(\lambda) & 0 & 0 & \cos(\lambda) \end{pmatrix} \tag{70}$$

$$\Theta\mathcal{O}^t(A,A^t)(\lambda)\Theta = \mathcal{O}(A,A^t)(\lambda). \tag{71}$$

While ordinary 3-dimensional rotations are the same for $p\cdot\sigma$ or $p_e\cdot\sigma_e$, real rotations in Euclidean space time planes are interpreted as rotationless Lorentz boosts with imaginary rapidity.

These identifications imply the following algebraic relations between the infinitesimal generators of the four dimensional orthogonal group and the Lorentz group:

$$\mathbf{P}_m = \mathbf{P}_e \qquad J^{ij}_m = J^{ij}_e \tag{72}$$

$$H_m = iH_e \qquad K^i_m = -iJ^{0i}_e \tag{73}$$

Because of the factor of i, if the Euclidean generators are self-adjoint operators on a representation of the Hilbert space, the constructed Poincaré generators cannot be self-adjoint on that representation of the Hilbert space.

In the spinless case ($s=0$) the identifications (68-71) result in the following expressions for the infinitesimal generators of the Poincaré group on the Hilbert space with the time reflection:

$$H\Psi(x_e) = \frac{\partial}{\partial x^0_e}\Psi(x_e) \qquad \mathbf{P}\Psi(x_e) = -i\frac{\partial}{\partial \mathbf{x}_e}\Psi(x_e) \tag{74}$$

$$\mathbf{J}\Psi(x_e) = -i\mathbf{x}\times\boldsymbol{\nabla}_x\Psi(x_e) \qquad K^j\Psi(x_e) = (x^j\frac{\partial}{\partial x^0_e} - x^0_e\frac{\partial}{\partial x^j})\Psi(x_e). \tag{75}$$

It is straightforward to demonstrate that these operators satisfy the Poincaré commutations relations (30-32). For example

$$[K^i,H] = [x^i\frac{\partial}{\partial x^0_e} - x^0_e\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^0_e}] = i(-i\frac{\partial}{\partial x^i}) = iP^i \tag{76}$$

which agrees with (32). The other commutators can be checked similarly.

The Euclidean time reversal of the final state makes both the Hamiltonian H and the boost generators $\mathbf{K}$ formally Hermitian with respect to the scalar product (54). The non-trivial observation is that even an infinitesimal rotation in a Euclidean space time plane can map functions with positive Euclidean time support to functions that violate the support condition. This maps Hilbert space vectors out of the Hilbert space. The resolution of this problem will be discussed in section 6.

To show the hermiticity of the rotationless boost generators (75) note that rotational invariance of the Euclidean Green's function in Euclidean space-time planes means that the Euclidean rotation generators commute with the Euclidean Green's function:

$$(-ix^i\frac{\partial}{\partial x^0_e} + ix^0_e\frac{\partial}{\partial x^i})S^0_e(x-y) = S^0_e(x-y)(-iy^i\frac{\partial}{\partial y^0_e} + iy^0_e\frac{\partial}{\partial y^i}). \tag{77}$$

Multiplying both sides by i gives

$$(x^i\frac{\partial}{\partial x^0_e} - x^0_e\frac{\partial}{\partial x^i})S^0_e(x-y) = S^0_e(x-y)(y^i\frac{\partial}{\partial y^0_e} - y^0_e\frac{\partial}{\partial y^i}). \tag{78}$$

Next consider the inner product

$$\langle f|K^i|g\rangle =$$

$$\int d^4xd^4yf^*(\mathbf{x},-x^0_e)S^0_e(x-y)(y^i\frac{\partial}{\partial y^0_e} - y^0_e\frac{\partial}{\partial y^i})g(\mathbf{y},y^0_e). \tag{79}$$

Using (78) in (79) gives

$$= \int d^4xd^4yf^*(\mathbf{x},-x^0_e)(x^i\frac{\partial}{\partial x^0_e} - x^0_e\frac{\partial}{\partial x^i})S^0_e(x-y)g(\mathbf{y},y^0_e). \tag{80}$$

Integrating by parts again gives

$$= -\int d^4xd^4y(x^i\frac{\partial}{\partial x^0_e} + x^0_e\frac{\partial}{\partial x^i})(\theta f)^*(\mathbf{x},x^0_e)S^0_e(x-y)g(\mathbf{y},y^0_e). \tag{81}$$

Finally factoring the time reversal out of f gives

$$-(x^i \frac{\partial}{\partial x_e^0} + x_e^0 \frac{\partial}{\partial x^i})\theta f^*(\mathbf{x}, x_e^0) = \theta \left((x^i \frac{\partial}{\partial x_e^0} - x_e^0 \frac{\partial}{\partial x^i}) f^*(\mathbf{x}, x_e^0) \right) \tag{82}$$

which when used in (81) gives

$$\langle f|K^i|g\rangle = \int d^4xd^4y f^*(\mathbf{x}, -x_e^0) S_e^0(x-y)(y^i \frac{\partial}{\partial y_e^0} - y_e^0 \frac{\partial}{\partial y^i}) g(\mathbf{y}, y_e^0) =$$

$$\int d^4xd^4y \theta((x^i \frac{\partial}{\partial x_e^0} - x_e^0 \frac{\partial}{\partial x^i}) f(\mathbf{x}, x_e^0))^* S_e^0(x-y) g(\mathbf{y}, y_e^0) = \langle K^i f|g\rangle. \tag{83}$$

This shows that K^i is a Hermitian operator on this representation of the Hilbert space.

The other non-trivial operator is the Hamiltonian (74). In this case

$$\langle f|H|g\rangle = \int d^4xd^4y f^*(\mathbf{x}, -x_e^0) S_e^0(x-y) \frac{\partial}{\partial y_e^0} g(\mathbf{y}, y_e^0) =$$

$$-\int d^4xd^4y f^*(\mathbf{x}, -x_e^0) \frac{\partial}{\partial y_e^0} S_e^0(x-y) g(\mathbf{y}, y_e^0) = \int d^4xd^4y f^*(\mathbf{x}, -x_e^0) \frac{\partial}{\partial x_e^0} S_e^0(x-y) g(\mathbf{y}, y_e^0) =$$

$$-\int d^4xd^4y \frac{\partial}{\partial x_e^0} f^*(\mathbf{x}, -x_e^0) S_e^0(x-y) g(\mathbf{y}, y_e^0) = \int d^4xd^4y \frac{\partial f^*}{\partial x^0}(\mathbf{x}, -x_e^0) S_e^0(x-y) g(\mathbf{y}, y_e^0) = \langle Hf|g\rangle. \tag{84}$$

The Euclidean time reversal does not change the linear or angular momentum operators. These methods can be used to demonstrate that all of the $s = 0$ generators (74-75) are Hermitian in this representation of the Hilbert space and satisfy the Poincaré Lie algebra.

V. SPIN

In this section explicit formulas for generators for particles with arbitrary spin are derived, generalizing the method used in the previous section for scalar particles.

In the original Poincaré covariant theory the spin is associated with the observable that is the $\hat{\mathbf{z}}$-component of the spin that would be measured in the particle's rest frame if it was transformed to the rest frame with a rotationless Lorentz transformation. The spin in the covariant wave function is related to this spin by multiplying by one of the $SL(2,\mathbb{C})$ matrices, $D^s_{\mu\nu}(\Lambda_c(p)^{-1})$ or $D^s_{\mu\nu}(\Lambda_c(p)^\dagger)$. These transformations lead to distinct right or left handed spinors. In discussing spin it is important to understand that the Poincaré covariant spinors and the Lorentz covariant spinors are related, but different. Representations of the Poincaré generators for each type of covariant spin must be considered separately. In addition, for each type of covariant spinor there are invariant linear functionals that define dual spinors. The dual spinors are spinor analogs of covariant and contravariant vectors. In conventional treatments [4][23] [24] the right-handed spinors are denoted by ξ^a, left handed spinors are denoted by $\xi^{\dot{a}}$ and their duals are denoted by ξ_a and $\xi_{\dot{a}}$ respectively.

The first step is to consider the $SL(2,\mathbb{C})$ transformation properties of the Euclidean kernels for right and left handed covariant spinors and their duals.

Euclidean four vectors can be represented by any of the four matrices:

$$p_e \cdot \sigma_e = p_e^\mu \sigma_{e\mu} \qquad p_e \cdot (\sigma_2 \sigma_e \sigma_2) = p_e^\mu \sigma_2 \sigma_{e\mu} \sigma_2 \qquad p_e \cdot \sigma_e^t = p_e^\mu \sigma_{e\mu}^t \qquad p_e \cdot (\sigma_2 \sigma_e^t \sigma_2) = p_e^\mu \sigma_2 \sigma_{e\mu}^t \sigma_2. \tag{85}$$

The determinant of each of these matrices is (-) the square of the Euclidean length of p_e, which is preserved under linear transformations of the form

$$P' = APB^t \tag{86}$$

where P represents any of the matrices in (85), and $A, B \in SL(2,\mathbb{C})$. Real four-dimensional orthogonal transformations are obtained by restricting A and B to be elements of $SU(2)$.

The 4×4 orthogonal matrix $\mathbb{O}(A,B)^\mu{}_\nu$ is related to the pair (A,B) by

$$\mathbb{O}(A,B)^\mu{}_\nu := \frac{1}{2} Tr(\sigma^\dagger_{e\mu} A \sigma_{e\nu} B^t). \tag{87}$$

It follows that

$$A p_e^\mu \sigma_{e\mu} B^t = \sigma_{e\mu} \mathbb{O}(A,B)^\mu{}_\nu p_e^\nu = \sigma_{e\mu} (\mathbb{O}(A,B) p_e)^\mu. \tag{88}$$

Multiplying (88) by σ_2 on both sides gives

$$A^*(p_e \cdot \sigma_2 \sigma_e \sigma_2) B^\dagger = (\mathbb{O}(A,B)p)_e \cdot \sigma_2 \sigma_e \sigma_2. \tag{89}$$

Taking transposes of the 2×2 matrices (88) and (89) give

$$B(p_e \sigma_e^t) A^t = \sigma_e^t \cdot (\mathbb{O}(A,B) p_e) \tag{90}$$

and

$$B^*(p_e \cdot \sigma_2 \sigma_e^t \sigma_2) A^\dagger = \sigma_2 \sigma_e^t \sigma_2 \cdot (\mathbb{O}(A,B) p_e) \tag{91}$$

where $\sigma_2 A \sigma_2 = A^*$ for $A \in SU(2)$ was used in (89-91). In all four of these expressions A, B and the orthogonal matrix $\mathbb{O}(A,B)$ are unchanged. All four of the matrices (85) become positive when p_e is replaced by the on-shell Minkowski four momentum, $p_m^\mu = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$ and σ_e^μ is replaced by σ^μ.

These matrices appear in the Euclidean covariant kernels for the right and left-handed representations and their duals. The spin s Euclidean covariant kernels for each type of covariant spinor are:

$$S_e^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}[p_e \cdot \sigma_e]}{p_e^2 + m^2} e^{ip_e \cdot x_e} d^4 p_e \tag{92}$$

$$S_{ed}^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}[p_e \cdot (\sigma_2 \sigma_e \sigma_2)]}{p_e^2 + m^2} e^{ip_e \cdot x_e} d^4 p_e \tag{93}$$

$$S_{e*}^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}[p_e \cdot \sigma_e^t]}{p_e^2 + m^2} e^{ip_e \cdot x_e} d^4 p_e \tag{94}$$

$$S_{ed*}^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}[p_e \cdot (\sigma_2 \sigma_e^t \sigma_2)]}{p_e^2 + m^2} e^{ip_e \cdot x_e} d^4 p_e. \tag{95}$$

The physical Hilbert space inner product associated with each of these kernels is

$$\langle \psi_e | \phi_e \rangle = \int \sum_{\mu\nu} \psi_e^*(\theta x, \mu) S_e^s(x_e - y_e; \mu, \nu) \phi_e(y, \nu) d^4x d^4y \tag{96}$$

$$\langle \psi_{ed} | \phi_{ed} \rangle = \int \sum_{\mu\nu} \psi_{ed}^*(\theta x, \mu) S_{ed}^s(x_e - y_e; \mu, \nu) \phi_{ed}(y, \nu) d^4x d^4y \tag{97}$$

$$\langle \psi_{e*} | \phi_{e*} \rangle = \int \sum_{\mu\nu} \psi_{e*}^*(\theta x, \mu) S_{e*}^s(x_e - y_e; \mu, \nu) \phi_{e*}(y, \nu) d^4x d^4y \tag{98}$$

$$\langle \psi_{ed*} | \phi_{ed*} \rangle = \int \sum_{\mu\nu} \psi_{ed*}^*(\theta x, \mu) S_{ed*}^s(x_e - y_e; \mu, \nu) \phi_{ed*}(y, \nu) d^4x d^4y. \tag{99}$$

For wave functions with positive Euclidean time support, the p_e^0 integral can be evaluated by the residue theorem, closing the contour in the lower half plane. This replaces p_e^0 by $-i\omega_m(\mathbf{p})$. The kernels become the two-point Minkowski

Wightman functions [4] for mass m spin s irreducible representations of the Lorentz group. Equations (97) and (99) are dual representations of the right-handed kernel, while (96) and (98) are dual representations of the left-handed kernel. σ_2 behaves like a metric tensor for the Lorentz covariant spinors, relating the representations (96) and (97) or (98) and (99). Contraction of the two types of right or left handed spinors are Lorentz invariant. The results of performing the p_e^0 integral are

$$\langle\psi_e|\phi_e\rangle = \int\sum_{\mu\nu} f_m^*(\mathbf{p},\mu)\frac{d\mathbf{p}D^s_{\mu\nu}[p_m\cdot\sigma]}{\omega_m(\mathbf{p})}g_m(\mathbf{p},\nu) \tag{100}$$

$$\langle\psi_{ed}|\phi_{ed}\rangle = \int\sum_{\mu\nu} f_m^*(\mathbf{p},\mu)\frac{d\mathbf{p}D^s_{\mu\nu}[p_m\cdot\sigma_2\sigma\sigma_2]}{\omega_m(\mathbf{p})}g_m(\mathbf{p},\nu) \tag{101}$$

$$\langle\psi_{e*}|\phi_{e*}\rangle = \int\sum_{\mu\nu} f_m^*(\mathbf{p},\mu)\frac{d\mathbf{p}D^s_{\mu\nu}[p_m\cdot\sigma^*]}{\omega_m(\mathbf{p})}g_m(\mathbf{p},\nu) \tag{102}$$

$$\langle\psi_{ed*}|\phi_{ed*}\rangle = \int\sum_{\mu\nu} f_m^*(\mathbf{p},\mu)\frac{d\mathbf{p}D^s_{\mu\nu}[p_m\cdot\sigma_2\sigma^*\sigma_2]}{\omega_m(\mathbf{p})}g_m(\mathbf{p},\nu) \tag{103}$$

where

$$f_m^*(\mathbf{p},\mu) := \int\frac{d^4x}{(2\pi)^{3/2}}\psi^*(x,\mu)e^{i\mathbf{p}\cdot\mathbf{x}-\omega_m(\mathbf{p})x^0} \tag{104}$$

$$g_m(\mathbf{p},\nu) := \frac{d^4y}{(2\pi)^{3/2}}\psi(y,\nu)e^{-i\mathbf{p}\cdot\mathbf{y}-\omega_m(\mathbf{p})y^0} \tag{105}$$

for each type of spinor wave function.

Each of the spin matrices, $D^s_{\mu\nu}[p_m\cdot\sigma]$, $D^s_{\mu\nu}[p_m\cdot\sigma_2\sigma\sigma_2]$, $D^s_{\mu\nu}[p_m\cdot\sigma^*]$ and $D^s_{\mu\nu}[p_m\cdot\sigma_2\sigma^*\sigma_2]$ are positive Hermitian matrices, so the Euclidean Green's functions (92-95) are all reflection positive.

The first step to find the spinor parts of the Poincaré generators in the Euclidean representation is to use the identities (88-91) which lead to

$$\int\sum_{\mu\nu}\psi_e^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[\mathbb{O}p\cdot\sigma_e]\phi_e(y,\nu)d^4xd^4yd^4p =$$

$$\int\sum_{\mu\nu}\psi_e^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[p\cdot A\sigma_e B^t]\phi_e(y,\nu)d^4xd^4yd^4p \tag{106}$$

$$\int\sum_{\mu\nu}\psi_{ed}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[\mathbb{O}p\cdot\sigma_2\sigma_e\sigma_2]\phi_{ed}(y,\nu)d^4xd^4yd^4p =$$

$$\int\sum_{\mu\nu}\psi_{ed}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[p\cdot A^*\sigma_2\sigma_e\sigma_2 B^\dagger]\phi_{ed}(y,\nu)d^4xd^4yd^4p \tag{107}$$

$$\int\sum_{\mu\nu}\psi_{e*}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[\mathbb{O}p\cdot\sigma_e^t]\phi_{e*}(y,\nu)d^4xd^4yd^4p =$$

$$\int\sum_{\mu\nu}\psi_{e*}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[p\cdot B\sigma_e^t A^t]\phi_{e*}(y,\nu)d^4xd^4yd^4p \tag{108}$$

$$\int \sum_{\mu\nu} \psi^*_{ed*}(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2} D^s_{\mu\nu}[\mathbb{O}p\cdot \sigma_2\sigma^t_e\sigma_2]\phi_{ed*}(y,\nu)d^4xd^4yd^4p =$$

$$\int \sum_{\mu\nu} \psi^*_{ed*}(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot B^*\sigma_2\sigma^t_e\sigma_2 A^\dagger]\phi_{ed*}(y,\nu)d^4xd^4yd^4p. \tag{109}$$

The next step is to move the transformations from the kernels to the wave functions. The Euclidean invariance of the measures and scalar products, the group representation properties of the Wigner functions, and re-definitions of the wave functions can be used to show that (106-109) are equivalent to

$$\int \sum (D^s_{\mu\alpha}[A^\dagger]^{-1}\psi_e(\theta\mathbb{O}^t\theta x,\alpha))^*\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot\sigma_e]\phi_e(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum \psi^*_e(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^j_{\mu\alpha}[p\cdot\sigma_e]D^s_{\alpha\nu}[B^t]\phi_e(\mathbb{O}y,\nu)d^4xd^4yd^4p \tag{110}$$

$$\int \sum (D^s_{\mu\alpha}[A^t]^{-1}\psi_{ed}(\theta\mathbb{O}^t\theta x,\alpha))^*\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot\sigma_2\sigma_e\sigma_2]\phi_{ed}(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum \psi^*_{ed}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^j_{\mu\alpha}[p\cdot\sigma_2\sigma_e\sigma_2]D^s_{\alpha\nu}[B^\dagger]\phi_{ed}(\mathbb{O}y,\nu)d^4xd^4yd^4p \tag{111}$$

$$\int \sum (D^s_{\mu\alpha}[B^\dagger]^{-1}\psi_{e*}(\theta\mathbb{O}^t\theta x,\alpha))^*\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot\sigma^t_e]\phi_{e*}(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum \psi^*_{e*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^j_{\mu\alpha}[p\cdot\sigma^t_e]D^s_{\alpha\nu}[A^t]\tilde{\phi}_{e*}(\mathbb{O}y,\nu)d^4xd^4yd^4p \tag{112}$$

$$\int \sum (D^s_{\mu\alpha}[B^t]^{-1}\psi_{ed*}(\theta\mathbb{O}^t\theta x,\alpha))^*\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot\sigma_2\sigma^t_e\sigma_2]\phi_{ed*}(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum \psi^*_{ed*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}[p\cdot\sigma_2\sigma^t_e\sigma_2]D^s_{\alpha\nu}[A^\dagger]\phi_{ed*}(\mathbb{O}y,\nu)d^4xd^4yd^4p. \tag{113}$$

To derive expressions for the generators for each type of spinor, check the hermiticity and verify the commutation relations the first step is to replace A and B with the pairs of $SU(2)$ matrices representing one-parameter groups for both ordinary rotations about a fixed axis and rotations in a Euclidean space time plane.

For ordinary rotations about the $\hat{\mathbf{n}}$ axis, the one-parameter group is

$$A(\lambda) = B^*(\lambda) = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} \tag{114}$$

and $(\theta\mathbb{O}^t(\lambda)\theta) = \mathbb{O}^t(\lambda)$, while for rotations in Euclidean $\hat{\mathbf{n}}$-x^0 space-time planes the one-parameter group is

$$A(\lambda) = B^t(\lambda) = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} \tag{115}$$

and $(\theta\mathbb{O}^t(\lambda)\theta) = \mathbb{O}(\lambda)$. The 4×4 orthogonal transformations, $\mathbb{O}(\lambda)$ associated with each type of transformation are shown explicitly for rotations about the $\hat{\mathbf{z}}$ axis and for rotations in the $\hat{\mathbf{z}}$-x^0 plane: For rotations about the $\hat{\mathbf{z}}$ axis

$$\mathbb{O}(A,A^*)(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\lambda) & \sin(\lambda) & 0 \\ 0 & -\sin(\lambda) & \cos(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{116}$$

and

$$\Theta \mathbb{O}(A, A^*)(\lambda)\Theta = \mathbb{O}(A, A^*)(\lambda). \tag{117}$$

For rotations in the $\hat{\mathbf{z}}$-x^0 plane

$$\mathbb{O}(A, A^t)(\lambda) = \begin{pmatrix} \cos(\lambda) & 0 & 0 & \sin(\lambda) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(\lambda) & 0 & 0 & \cos(\lambda) \end{pmatrix} \tag{118}$$

and

$$\theta \mathbb{O}^t(A, A^t)(\lambda)\theta = \mathbb{O}(A, A^t)(\lambda) \tag{119}$$

For the case of ordinary rotations $A = B^*$ and equations (110-113) become

$$\int \sum (D^s_{\mu\alpha}[A]\psi_e(\mathbb{O}^t(\lambda)x, \alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^s_{\mu\nu}[p \cdot \sigma_e]\phi_e(y, \nu) d^4x d^4y d^4p$$

$$= \int \sum \psi^*_e(x, \mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^j_{\mu\alpha}[p \cdot \sigma_e] D^s_{\alpha\nu}[A^\dagger]\phi_e(\mathbb{O}(\lambda)y, \nu) d^4x d^4y d^4p \tag{120}$$

$$\int \sum (D^s_{\mu\alpha}[A^*]\psi_{ed}(\mathbb{O}^t(\lambda)x, \alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^s_{\mu\nu}[p \cdot \sigma_2\sigma_e\sigma_2]\phi_{ed}(y, \nu) d^4x d^4y d^4p$$

$$= \int \sum \psi^*_{ed}(x, \mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^j_{\mu\alpha}[p \cdot \sigma_2\sigma_e\sigma_2] D^s_{\alpha\nu}[A^t]\phi_{ed}(\mathbb{O}(\lambda)y, \nu) d^4x d^4y d^4p \tag{121}$$

$$\int \sum (D^s_{\mu\alpha}[A^*]\psi_{e*}(\mathbb{O}^t(\lambda)x, \alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^s_{\mu\nu}[p \cdot \sigma^t_e]\phi_{e*}(y, \nu) d^4x d^4y d^4p$$

$$= \int \sum \psi^*_{e*}(x, \mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^j_{\mu\alpha}[p \cdot \sigma^t_e] D^s_{\alpha\nu}[A^t]\phi_{e*}(\mathbb{O}(\lambda)y, \nu) d^4x d^4y d^4p \tag{122}$$

$$\int \sum (D^s_{\mu\alpha}[A]\psi_{ed*}(\mathbb{O}^t(\lambda)x, \alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^s_{\mu\nu}[p \cdot \sigma_2\sigma^t_e\sigma_2]\phi_{ed*}(y, \nu) d^4x d^4y d^4p$$

$$= \int \sum \psi^*_{ed*}(x, \mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^s_{\mu\alpha}[p \cdot \sigma_2\sigma^t_e\sigma_2] D^s_{\alpha\nu}[A^\dagger]\phi_{ed*}(\mathbb{O}(\lambda)y, \nu) d^4x d^4y d^4p \tag{123}$$

For the case of rotations in Euclidean space-time planes for $A = B^t$ equations (110-113) become

$$\int \sum (D^s_{\mu\alpha}[A]\psi_e(\mathbb{O}(\lambda)x, \alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^s_{\mu\nu}[p \cdot \sigma_e]\phi_e(y, \nu) d^4x d^4y d^4p$$

$$= \int \sum {\psi_e}^*(x, \mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^j_{\mu\alpha}[p \cdot \sigma_e] D^s_{\alpha\nu}[A]\phi_e(\mathbb{O}(\lambda)y, \nu) d^4x d^4y d^4p \tag{124}$$

$$\int \sum (D^s_{\mu\alpha}[A^*]\psi_{ed}(\mathbb{O}(\lambda)x, \alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2 + m^2} D^s_{\mu\nu}[p \cdot \sigma_2\sigma_e\sigma_2]\phi_{ed}(y, \nu) d^4x d^4y d^4p$$

$$= \int \sum \psi^*_{ed}(x,\mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2+m^2} D^j_{\mu\alpha}[p\cdot \sigma_2 \sigma_e \sigma_2] D^s_{\alpha\nu}[A^*] \phi_{ed}(\mathbb{O}y,\nu) d^4x d^4y d^4p \tag{125}$$

$$\int \sum (D^s_{\mu\alpha}[A^t]\psi_{e*}(\mathbb{O}(\lambda)x,\alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot \sigma^t_e] \phi_{e*}(y,\nu) d^4x d^4y d^4p$$

$$= \int \sum \psi^*_{e*}(x,\mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2+m^2} D^j_{\mu\alpha}[p\cdot \sigma^t_e] D^s_{\alpha\nu}[A^t] \phi_{e*}(\mathbb{O}(\lambda)y,\nu) d^4x d^4y d^4p \tag{126}$$

$$\int \sum (D^s_{\mu\alpha}[A^\dagger]\psi_{ed*}(\mathbb{O}(\lambda)x,\alpha))^* \frac{e^{ip\cdot(\theta x - y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot \sigma_2 \sigma^t_e \sigma_2] \phi_{ed*}(y,\nu) d^4x d^4y d^4p$$

$$= \int \sum \psi^*_{ed*}(x,\mu) \frac{e^{ip\cdot(\theta x - y)}}{p^2+m^2} D^s_{\mu\alpha}[p\cdot \sigma_2 \sigma^t_e \sigma_2] D^s_{\alpha\nu}[A^\dagger] \phi_{ed*}(\mathbb{O}(\lambda)y,\nu) d^4x d^4y d^4p. \tag{127}$$

Note that the transformations above represent inverse Lorentz transformations since

$$\langle x,\nu|U(\Lambda,0)|\psi\rangle = \langle \psi|U^\dagger(\Lambda,0)|x,\nu\rangle^* = \langle \mathbb{O}^t x,\nu|\psi\rangle^* = \langle \psi|\mathbb{O}^t x,\nu\rangle. \tag{128}$$

To construct generator of ordinary rotations differentiate the right hand side of (120-123) by λ, set $\lambda = 0$, and multiply the result by i. To construct the generators of Euclidean space-time rotations differentiate the right hand side of (124-127) by λ, set $\lambda = 0$, and multiply the result by i to get expressions for the generators. To get expressions for the Lorentz Boost generators multiply the Euclidean space-time rotation generators by an additional factor of $-i$. The derivatives of the Wigner functions can be computed using

$$\frac{d}{d\lambda} D^s_{\mu\nu}[A(\lambda)]_{|\lambda=0} = \frac{d}{d\lambda}\langle s,\mu|e^{i\lambda\hat{\mathbf{n}}\cdot\mathbf{S}}|s,\nu\rangle_{|\lambda=0} = i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle \tag{129}$$

$$\frac{d}{d\lambda} D^s_{\mu\nu}[A(\lambda)^\dagger]_{|\lambda=0} = \frac{d}{d\lambda}\langle s,\mu|e^{-i\lambda\hat{\mathbf{n}}\cdot\mathbf{S}}|s,\nu\rangle_{|\lambda=0} = -i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle \tag{130}$$

$$\frac{d}{d\lambda} D^s_{\mu\nu}[A^*(\lambda)]_{|\lambda=0} = \frac{d}{d\lambda}(D^s_{\mu\nu}[A(\lambda)])^*_{|\lambda=0} = -i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle^* = -i\langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle \tag{131}$$

$$\frac{d}{d\lambda} D^s_{\mu\nu}[A^t(\lambda)]_{|\lambda=0} = \frac{d}{d\lambda}(D^s_{\mu\nu}([A(\lambda)])^*)^{-1}_{|\lambda=0} = i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle^* = i\langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle. \tag{132}$$

These can be evaluated using S_z and angular momentum raising and lowering operators. The rotation generators for each type of spinor representation can be read off of (120-123):

$$\langle x,s,\nu|\mathbf{J}|\psi_e\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times(-i\frac{\partial}{\partial\mathbf{x}}) + \langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle\right)\langle x,s,\nu|\psi_e\rangle \tag{133}$$

$$\langle x,s,\nu|\mathbf{J}|\psi_{ed}\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times(-i\frac{\partial}{\partial\mathbf{x}}) - \langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle\right)\langle x,s,\nu|\psi_{ed}\rangle \tag{134}$$

$$\langle x,s,\nu|\mathbf{J}|\psi_{e*}\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times(-i\frac{\partial}{\partial\mathbf{x}}) - \langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle\right)\langle x,s,\nu|\psi_{e*}\rangle \tag{135}$$

$$\langle x,s,\nu|\mathbf{J}|\psi_{ed*}\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times(-i\frac{\partial}{\partial\mathbf{x}}) + \langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle\right)\langle x,s,\nu|\psi_{ed*}\rangle \tag{136}$$

The first and fourth term are representations of standard rotation generators . In the second and third terms the spin generator matrix elements are transposed and multiplied by with a (-) sign. To show that these operator satisfy $SU(2)$ commutation relations, consider matrices satisfying $SU(2)$ commutation relations:

$$[M_i, M_j] = i\epsilon_{ijk} M_k. \tag{137}$$

The transposes satisfy

$$[M_j^t, M_i^t] = i\epsilon_{ijk} M_k^t \tag{138}$$

$$[(-M_i^t), (-M_j^t)] = i\epsilon_{ijk}(-M_k^t) \tag{139}$$

which shows that the negative transpose of these matrices also satisfy $SU(2)$ commutation relations. This shows that all of the spin generator satisfy $SU(2)$ commutation relations.

Generators for rotations in Euclidean space-time planes are constructed the same way from

$$\langle x, s, \nu | J^{0\hat{n}} | \psi_e \rangle = \sum_\nu \left(i\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) - \langle s, \mu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \nu \rangle \right) \langle x, s, \nu | \psi_e \rangle \tag{140}$$

$$\langle x, s, \nu | J^{0\hat{n}} | \psi_{ed} \rangle = \sum_\nu \left(i\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) + \langle s, \nu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \mu \rangle \right) \langle x, s, \nu | \psi_{ed} \rangle \tag{141}$$

$$\langle x, s, \nu | J^{0\hat{n}} | \psi_{e*} \rangle = \sum_\nu \left(i\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) - \langle s, \nu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \mu \rangle \right) \langle x, s, \nu | \psi_{e*} \rangle \tag{142}$$

$$\langle x, s, \nu | J^{0\hat{n}} | \psi_{ed*} \rangle = \sum_\nu \left(i\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) + \langle s, \mu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \nu \rangle \right) \langle x, s, \nu | \psi_{ed*} \rangle \tag{143}$$

In order to construct the boost generators it is necessary to multiply these expression by an additional factor of (-i)

$$\langle x, s, \nu | \mathbf{K} | \psi_e \rangle = \sum_\nu \left(\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) + i \langle s, \mu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \nu \rangle \right) \langle x, s, \nu | \psi_e \rangle \tag{144}$$

$$\langle x, s, \nu | \mathbf{K} | \psi_{ed} \rangle = \sum_\nu \left(\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) - i \langle s, \nu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \mu \rangle \right) \langle x, s, \nu | \psi_{ed} \rangle \tag{145}$$

$$\langle x, s, \nu | \mathbf{K} | \psi_{e*} \rangle = \sum_\nu \left(\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) + i \langle s, \nu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \mu \rangle \right) \langle x, s, \nu | \psi_{e*} \rangle \tag{146}$$

$$\langle x, s, \nu | \mathbf{K} | \psi_{ed*} \rangle = \sum_\nu \left(\delta_{\mu\nu} (\mathbf{x} \frac{\partial}{\partial x^0} - x^0 \frac{\partial}{\partial \mathbf{x}}) - i \langle s, \mu | \hat{\mathbf{n}} \cdot \mathbf{S} | s, \nu \rangle \right) \langle x, s, \nu | \psi_{ed*} \rangle \tag{147}$$

The continuous part of these expressions agree with (74-75) for spinless operators. The relevant commutators involving the spin parts of the boost generators in each of the four representations are

$$[K_i, K_j] = [iS_i, iS_j] = -i\epsilon_{ijk} S_k \tag{148}$$

$$[K_i, K_j] = [-iS_i^t, -iS_j^t] = -\epsilon_{ijk}(-S_k^t) \tag{149}$$

$$[K_i, K_j] = [iS_i^t, iS_j^t] = -\epsilon_{ijk}(-S_k^t) \tag{150}$$

$$[K_i, K_j] = [-iS_i, -iS_j] = -i\epsilon_{ijk} S_k \tag{151}$$

$$[K_i, S_j] = [iS_i, S_j] = \epsilon_{ijk} iS_k = \epsilon_{ijk} K_k \tag{152}$$

$$[K_i, S_j] = [-iS_i^t, -S_j^t] = \epsilon_{ijk} - iS_k^t = \epsilon_{ijk} K_k \tag{153}$$

$$[K_i, S_j] = [iS_i^t, -S_j^t] = \epsilon_{ijk} iS_k^t = \epsilon_{ijk} K_k \tag{154}$$

$$[K_i, S_j] = [-iS_i, S_j] = -i\epsilon_{ijk}(-iS_k) = \epsilon_{ijk} K_k \tag{155}$$

where the spin generators in (149,150,153) and (154) are (-) the transposes of the matrices satisfying $SU(2)$ commutation relations, which were shown in (137-139) to satisfy $SU(2)$ commutation relations. It follows that the expressions (133-136) and (144-147) for the Lorentz generators in each of the spinor representations satisfy the Poincaré commutation relations.

The hermiticity of these generators follow from the expressions (120-123) and (124-127). Each of equations (120-123) has the form

$$\langle U^\dagger(\lambda)\psi|\phi\rangle = \langle\psi|U(\lambda)|\phi\rangle \tag{156}$$

so the rotation operators, which are generators of unitary one-parameter groups [25] are self-adjoint in the Hilbert spaces with inner products (96-99).

For the boost generators hermiticity follows from (124-127). In this case all of these equations have the form

$$\langle T(\lambda)\psi|\phi\rangle = \langle\psi|T(\lambda)|\phi\rangle \tag{157}$$

In these cases $T(\lambda)$ is Hermitian, but the generators are constructed by multiplying the λ derivative $1 = (i)(-i)$ rather than i, resulting in Hermitian operators.

In these covariant representations the spin does not enter in the Hamiltonian or the linear momentum operators. These operators all commute with the spin operators and commutators with these operators follow from the scalar case.

The main result of this section the expressions (133-136) and (144-147) for the Poincaré generators. The construction relates the Euclidean spinors to the Lorentz covariant spinors.

VI. SELF ADJOINTNESS

While the self-adjointness of the generators of ordinary rotations follows from the unitarity of the one-parameter group of rotations on the Hilbert spaces (96-99), this argument does not apply to either the Hamiltonian or the boost generators. In both cases the operators were derived from the corresponding Euclidean generators by multiplication by an imaginary constant. The Euclidean generators and corresponding Lorentz generators act on different Hilbert space representations. The problem is that the corresponding finite Euclidean transformations can map functions with positive time support to functions that violate this condition.

For the Hamiltonian this can be treated by only considering translations in the positive Euclidean time direction. These translations map functions with positive Euclidean time support into functions with positive Euclidean time support. Reflection positivity can be used to show that translations in the positive Euclidean time direction define a contractive Hermitian semigroup on the Hilbert space with the scalar products (54-57). The argument [26] uses the Schwartz inequality on both the physical and Euclidean Hilbert spaces. One application of the Schwartz inequality on the physical Hilbert space gives

$$\| |e^{-Hx^0}|\phi\rangle\| = \langle e^{-Hx^0}\phi|e^{-Hx^0}|\phi\rangle^{1/2} = \langle\phi|e^{-H2x^0}|\phi\rangle^{1/2} \leq \| |e^{-H2x^0}|\phi\rangle\|^{1/2} \| |\phi\rangle\|^{1/2}. \tag{158}$$

Repeating these steps n-times gives

$$\| |e^{-Hx^0}|\phi\rangle\| \leq \| |e^{-H2^n x^0}|\phi\rangle\|^{1/2^n} \| |\phi\rangle\|^{1-1/2^n}. \tag{159}$$

The quantity

$$\| |e^{-H2^n x^0}|\phi\rangle\| \leq \|\theta U_e(2^n x^0)|\psi\rangle\|_e < \| |\psi\rangle\|_e < \infty \tag{160}$$

is bounded by the Euclidean norm, $\| \cdot \|_e$, since $U_e(2^n x^0)$ is unitary and $\|\theta\|_e = 1$ on that Hilbert space. Since this is finite and independent of n, taking the limit as $n \to \infty$ gives

$$\| |e^{-Hx^0} |\phi\rangle \| \leq \| |\phi\rangle \|. \tag{161}$$

It follows that positive Euclidean time translations define a contractive Hermitian semigroup on the Hilbert spaces (96-99). The generator is a positive self-adjoint operator [25][27].

Boosts present additional complications. Even an infinitesimal rotation in a Euclidean space time plane will map a general function with positive Euclidean time support to one that violates this condition. The self-adjointness of the boost generator cannot be demonstrated by showing that it defines a unitary one-parameter group or contractive semigroup, however it turns out that rotations in Euclidean space time planes, which are interpreted as boosts with complex rapidity, define local symmetric semigroups [28][29] [30] on the Hilbert spaces (96-99). These have self-adjoint generators, which are exactly the boost generators.

The conditions for a local symmetric semigroup [28] are

1. For each $\theta \in [0, \theta_0]$, there is a linear subset $\mathcal{D}_\theta$ such that $\mathcal{D}_{\theta_1} \supset \mathcal{D}_{\theta_2}$ if $\theta_1 < \theta_2$, and $\cup_{0<\theta<\theta_0} \mathcal{D}_{\theta_2}$ is dense.
2. For each $\theta \in [0, \theta_0]$, $E(\theta)$ is a linear operator on the Hilbert space with domain $\mathcal{D}_\theta$
3. $E(0) = I$, $E(\theta_1) : \mathcal{D}_{\theta_2} \to \mathcal{D}_{\theta_2 - \theta_1}$, and $E(\theta_1)E(\theta_2) = E(\theta_1 + \theta_2)$ on $\mathcal{D}_{\theta_1+\theta_2}$ for $\theta_1, \theta_2, \theta_1 + \theta_2 \in [0, \theta_0]$
4. $E(\theta)$ is Hermitian for $\theta \in [0, \theta_0]$
5. $E(\theta)$ is weakly continuous on $[0, \theta_0]$

When these conditions are satisfied there is a unique self-adjoint operator K such that $\mathcal{D}_\theta \subset \mathcal{D}_{e^{-K\theta}}$ and $E(\theta)$ is the restriction of $e^{-K\theta}$ to $\mathcal{D}_\theta$.

In this case $E(\theta)$ represents Euclidean space time rotations considered as operators on the Hilbert space (4) restricted to domains that will be described below.

The domains are Schwartz functions with space Euclidean time support the wedge shaped region defined by

$$\mathbf{x} \cdot \hat{\mathbf{n}} - \frac{x_e^0}{\epsilon} + \epsilon < 0 \tag{162}$$

$$\mathbf{x} \cdot \hat{\mathbf{n}} + \frac{x_e^0}{\epsilon} - \epsilon > 0 \tag{163}$$

The wedge shaped region becomes the positive Euclidean time half plane in the limit that $\epsilon \to 0$. Schwartz functions with support on this half plane are dense. In addition, if this domain is rotated by an angle less than $\theta_\epsilon := \pm \tan^{-1}(\epsilon)$, it will still be contained in the positive Euclidean time half plane. Schwartz functions with support in these wedge shaped regions can be constructed from Schwartz functions that have support or positive Euclidean time by multiplying the function by $g(x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon)$ where

$$g(x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon)) = h(\frac{x_e^0}{\epsilon} - \epsilon + \mathbf{x} \cdot \hat{\mathbf{n}}) h(\frac{x_e^0}{\epsilon} - \epsilon - \mathbf{x} \cdot \hat{\mathbf{n}}). \tag{164}$$

and

$$h(\lambda) = \left\{ \begin{array}{cc} e^{-\frac{1}{(\lambda)^2}} & \lambda > 0 \\ 0 & \lambda \leq 0 \end{array} \right. . \tag{165}$$

is a smoothed Heaviside function. $g(x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon))$ is a Schwartz function with support in the wedge shaped region (162-163) that approaches 1 as $\epsilon(\theta)$ approaches 0.

The domain $\mathcal{D}_\theta$ is taken as the space of Schwartz functions with positive time support multiplied by the function $g(x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon))$ where $\theta = \theta_\epsilon$. The Euclidean space time rotations restricted to these domains have all of the properties of local symmetric semigroup. It follows that the boost generators $\mathbf{K}$ are self-adjoint on the physical Hilbert space.

VII. FINITE TRANSFORMATIONS

Finite Poincaré transformations are needed for most applications. While the generators for space translations and rotations were constructed from the associated unitary one-parameter groups, the construction of the boost generators and Hamiltonian was not as direct. Because self adjointness was established for the Hamiltonian and boost generators, each one of these generators has a dense set of analytic vectors [31] where exponential series for the unitary one parameter groups converge. This ensures that the differential operators (75) that define the boost generators applied to a dense set of functions with positive time support have positive time support.

Directly summing the exponential series is inefficient. On the other hand, the structure of the finite unitary transformations is fixed by (43) when they act on irreducible basis states. The situation is analogous to non-relativistic quantum mechanics - time evolution becomes trivial once the Hamiltonian is diagonalized. In the Euclidean framework, the analogous problem is to diagonalize the mass squared operator. This is a dynamical problem that depends on the choice of reflection positive Euclidean covariant distributions. For the Green functions discussed in sections 4-5 the mass operator is the four dimensional Euclidean Laplacian. For more general Euclidean covariant Green's functions is it second order differential operator. The spectral condition ensures that there are no negative energy states.

The Euclidean Green's functions are manifestly covariant with respect to space translations and rotations. Given a mass eigenstate, the translational and rotational covariance can be used to decompose the mass eigenstate into a linear superposition of simultaneous eigenstates of linear momentum, and spins. On these states the unitary representation of the Poincaré group acts irreducibly.

In the Euclidean formalism, since the dynamics is in the Green function, mass eigenstates are solutions to

$$\langle \phi | (M^2 - m^2) | \psi \rangle = 0 \tag{166}$$

for all ϕ satisfying the support condition. Methods for constructing mass eigenstates satisfying the support condition are discussed in [32].

Mass-momentum eigenstates can be constructed using

$$|m, \mathbf{p}\rangle := \int e^{-i\mathbf{p}\cdot\mathbf{a}} U(\mathbf{a}) |\psi\rangle d\mathbf{a}. \tag{167}$$

Applying the translation operator $U(\mathbf{a}')$ to this vector gives

$$U(\mathbf{a}')|m, \mathbf{p}\rangle = U(\mathbf{a}') \int e^{-i\mathbf{p}\cdot\mathbf{a}} U(\mathbf{a})|\psi\rangle d\mathbf{a} = \int e^{-i\mathbf{p}\cdot\mathbf{a}} U(\mathbf{a}+\mathbf{a}')|\psi\rangle d\mathbf{a} =$$

$$\int e^{-i\mathbf{p}\cdot(\mathbf{a}''-\mathbf{a}')} U(\mathbf{a}'')|\psi\rangle d\mathbf{a}'' = e^{i\mathbf{p}\cdot\mathbf{a}'} \int e^{-i\mathbf{p}\cdot\mathbf{a}''} U(\mathbf{a}'')|\psi\rangle d\mathbf{a}'' = e^{i\mathbf{p}\cdot\mathbf{a}'}|m, \mathbf{p}\rangle \tag{168}$$

which shows that (167) is either 0 or an eigenstate of linear momentum with eigenvalue $\mathbf{p}$.

The mass-momentum eigenstates can be decomposed into spin eigenstates. Right and left handed kernels with the covariance properties (96) or (99) after integration become kernels for the covariant representations (60) and (61). For Green's functions with these rotational covariance properties the covariant basis states, up to normalization, can be constructed as follows,

$$|(m,s)\mathbf{p}, \mu\rangle := \int U(R)|m, R^{-1}\mathbf{p}\rangle D^{s*}_{\mu 0}[R] dR \tag{169}$$

where the integral is over the $SU(2)$ Haar measure. For a fixed rotation R':

$$U(R')|(m,s)R'^{-1}\mathbf{p}, \mu\rangle = U(R') \int U(R)|m, R^{-1}R'^{-1}\mathbf{p}\rangle D^{s*}_{\mu 0}[R] dR = \int U(R'R)|m, (R'R)^{-1}\mathbf{p}\rangle D^{s*}_{\mu 0}[R] dR =$$

$$\int U(R'')|m, R''^{-1}\mathbf{p}\rangle D^{s*}_{\mu\nu}[R^{'-1}] D^{s*}_{\nu 0}[R''] dR'' = \int U(R'')|m, R''^{-1}R'\mathbf{p}\rangle D^{s*}_{\nu 0}[R''] dR'' D^{s}_{\nu\mu}[R'] = |(m,s)\mathbf{p}, \nu\rangle D^{s}_{\nu\mu}[R']. \tag{170}$$

If R' is a rotation about the z axis, it follows that the resulting vector is an eigenstate of s^2 and s_z. This shows how mass eigenstates can be decomposed into a superposition of Lorentz covariant states that transform irreducibly with respect to the Poincaré group.

VIII. SUMMARY AND CONCLUSION

The purpose of this paper is to provide explicit representations for Poincaré generators for systems of particles of any spin in Euclidean representations of relativistic quantum mechanics, demonstrate that these generators satisfy the commutation relations of the Poincaré Lie Algebra and are self-adjoint with respect to a reflection positive scalar product. This was done by starting with irreducible unitary representations of the Poincaré group and expressing them in a manifestly Lorentz covariant form. The inner product in the Lorentz covariant representation necessarily had a non-trivial kernel, which could be expressed in terms of reflection positive Green functions. Expressions for the generators for any spin were derived based on these relations.

While the results are specifically for positive mass irreducible representations, they apply more generally since any unitary representation of the Poincaré group can be decomposed into a direct integral of positive-mass positive-energy irreducible representations.

Two consequences of the Osterwalder-Schrader reconstruction theorem are (1) the locality axiom is logically independent of the other Euclidean axioms and (2) the Hilbert space representation of the quantum theory does not require explicit analytic continuation. These observations suggest the possibility of formulating phenomenological non-local relativistic quantum mechanical models in a purely Euclidean representation [33][34][32]. The new feature is that the dynamics appears in model Euclidean Green's functions rather than in the Hamiltonian, which is a simple differential operator. One of the advantages of the Euclidean formulation is that Euclidean Green's functions are moments of a Euclidean path integral, which provides a formal connection to the dynamics of Lagrangian field theories. Models can be formulated by perturbing products of free Green Euclidean functions with Euclidean covariant interactions that preserve reflection positivity.

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