

Lecture #9: Particle Motion in Slowly Varying \underline{E} -fields Hawes 1

Polarization Drift

I. Polarization Drift

A. Consider an Electric field varying slowly in time $\underline{E}(\tau)$ with a constant Magnetic field $\underline{B} = B_0 \hat{z}$.

1. Note $\nabla \times \underline{B} = \mu_0 \underline{j} + \epsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$, we assume $|\underline{j}| \gg |\epsilon_0 \frac{\partial \underline{E}}{\partial t}|$, so $\frac{d\underline{B}}{dt} \approx 0$.

B. Multiple Time Scale Analysis

1.
$$\frac{d\underline{v}}{dt} = \frac{q}{m} (\underline{E} + \underline{v} \times \underline{B})$$

2. a. Take $\underline{E}(\tau)$ varies only on slow timescale $\tau = \epsilon t$

b.
$$\underline{v} = \underline{v}_1(t) + \epsilon \underline{v}_2(\tau) + \epsilon^2 \underline{v}_3(\tau) + \dots$$

c. Also assume $\nabla \underline{E} = \underline{B} = 0$

3. As before,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$$

4. We'll take a small electric field such that the $\underline{E} \times \underline{B}$ drift velocity $v_E \ll v$, where v is Larmor orbit velocity.

Thus
$$\frac{d\underline{v}}{dt} = \frac{q}{m} (\epsilon \underline{E} + \underline{v} \times \underline{B})$$

5. Substitute expanded solution:

$$\frac{\partial}{\partial t} (\underline{v}_1 + \epsilon \underline{v}_2 + \epsilon^2 \underline{v}_3) + \epsilon \frac{\partial}{\partial \tau} (\underline{v}_1 + \epsilon \underline{v}_2 + \epsilon^2 \underline{v}_3) = \epsilon \frac{q}{m} \underline{E}(\tau)$$

$$+ \frac{q}{m} (\underline{v}_1 + \epsilon \underline{v}_2 + \epsilon^2 \underline{v}_3) \times \underline{B}$$

a. Taking $\underline{B} = B_0 \hat{z}$,

$$\frac{\partial \underline{v}_1}{\partial t} + \epsilon \frac{\partial \underline{v}_2}{\partial \tau} + \epsilon^2 \frac{\partial \underline{v}_3}{\partial \tau} = \epsilon \frac{q \underline{E}(\tau)}{m} + \omega_c \underline{v}_1 \times \hat{z} + \epsilon \omega_c \underline{v}_2 \times \hat{z} + \epsilon^2 \omega_c \underline{v}_3 \times \hat{z}$$

Lecture #9: (Continued)

Pages 2

I, B, (Continued)

$$6. \mathcal{O}(1): \frac{\partial \underline{v}_1}{\partial t} = \omega_c \underline{v}_1 \times \hat{b}$$

a. This is just the usual, fast timescale Larmor gyration about the magnetic field.

b. The general solution for this motion can be written

$$\underline{v}_1 = v_{\perp} \cos(\omega_c t + \phi) \hat{e}_1 - v_{\perp} \sin(\omega_c t + \phi) \hat{e}_2 + v_{\parallel} \hat{b}$$

For a right-handed coordinate system s.t. $\hat{e}_1 \times \hat{e}_2 = \hat{b}$

$$7. \mathcal{O}(\epsilon): 0 = \frac{q}{m} \underline{E}(\tau) + \frac{q B_0}{m} \underline{v}_2 \times \hat{b}$$

a. This is just the slow timescale $\underline{E} \times B$ drift.

b. Operating $\hat{b} \times$ on equation gives:

$$\hat{b} \times \underline{E}(\tau) = \frac{q B_0}{m} \hat{b} \times (\underline{v}_2 \times \hat{b}) = \frac{q B_0}{m} (v_{\perp} (\hat{b} \cdot \hat{b}) - v_{\parallel} \hat{b})$$

or

$$\underline{v}_2 = v_{\parallel} \hat{b} + \frac{\underline{E}(\tau) \times \hat{b}}{B_0}$$

$$8. \mathcal{O}(\epsilon^2): \epsilon^2 \frac{\partial \underline{v}_2}{\partial \tau} = \epsilon^2 \omega_c \underline{v}_3 \times \hat{b}$$

a. At this order, the solution \underline{v}_2 is considered to be known.

$$\text{Thus, } \frac{\partial \underline{v}_2}{\partial \tau} = \frac{\partial v_{\parallel}}{\partial \tau} \hat{b} + \frac{1}{B_0} \frac{\partial \underline{E}}{\partial \tau} \times \hat{b} = \frac{\frac{\partial \underline{E}}{\partial \tau} \times \hat{b}}{B_0}$$

$$b. \frac{1}{B_0} \frac{\partial \underline{E}}{\partial \tau} \times \hat{b} = \omega_c \underline{v}_3 \times \hat{b}$$

c. Take $\hat{b} \times$ this equation

$$\frac{1}{B_0} \hat{b} \times \left(\frac{\partial \underline{E}}{\partial \tau} \times \hat{b} \right) = \frac{1}{B_0} \left[\frac{\partial \underline{E}}{\partial \tau} (\hat{b} \cdot \hat{b}) - \hat{b} \left(\hat{b} \cdot \frac{\partial \underline{E}}{\partial \tau} \right) \right] = \frac{1}{B_0} \frac{\partial \underline{E}}{\partial \tau}$$

$$\omega_c \hat{b} \times (\underline{v}_3 \times \hat{b}) = \omega_c [v_{\parallel} (\hat{b} \cdot \hat{b}) - \hat{b} (v_{\parallel} \cdot \hat{b})] = \omega_c (v_{\parallel} - v_{\parallel} \hat{b})$$

$$d. \text{ Thus } \underline{v}_3 = v_{\parallel} \hat{b} + \frac{1}{\omega_c B_0} \frac{d \underline{E}(\tau)}{d \tau}$$

Lecture #9 (Continued)

Howes ③

I.B. (Continued)

9. Putting the full solution together: (Taking $\vec{v}_{21} = \vec{v}_{31} = 0$)

$$\vec{v} = v_{\perp} (\cos(\omega t + \phi) \hat{e}_1 - \sin(\omega t + \phi) \hat{e}_2) + \frac{\vec{E}(t) + \vec{B}}{B^2} + \frac{1}{\omega_c B_0} \frac{d\vec{E}}{dt}$$

Zeroth order Larmor Motion

First-order

ExB drift

Second-order

Polarization Drift

C. Polarization Drift:

1. For slowly varying electric field $\vec{E}(t)$ (slow with respect to the Larmor motion), we define the

Polarization Drift

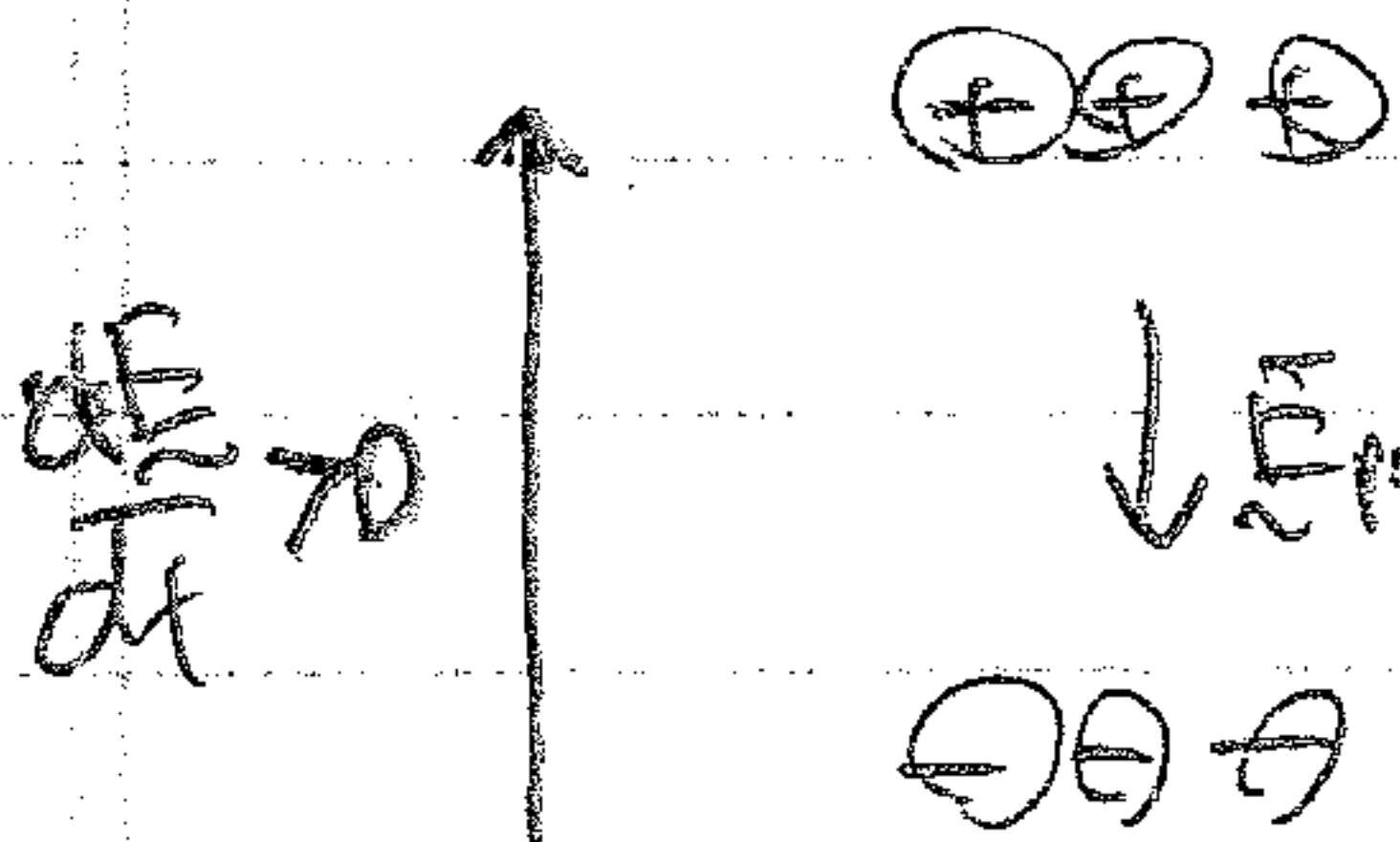
$$\vec{v}_P \equiv \frac{1}{\omega_c B} \frac{d\vec{E}}{dt}$$

2. Using $\omega_c = \frac{qB}{m}$, we have

$$\vec{v}_P = \frac{m}{q B^2} \frac{d\vec{E}}{dt}$$

a. Polarization drift is charge dependent

\Rightarrow ions and electrons drift in opposite directions



b. Resulting polarization of plasma opposes increasing applied Electric field.

c. Because $m_i \gg m_e$, ions dominate the polarization drift.

3. Polarization Current: $\vec{j}_P = \sum_s q_s n_s \vec{v}_P = \sum_s \frac{q_s n_s m_s}{q_s B^2} \frac{d\vec{E}}{dt}$

a. $\vec{j}_P = \sum_s \frac{n_s m_s}{B^2} \frac{d\vec{E}}{dt}$

b. Mass dependence means ion contribute more to polarization current.

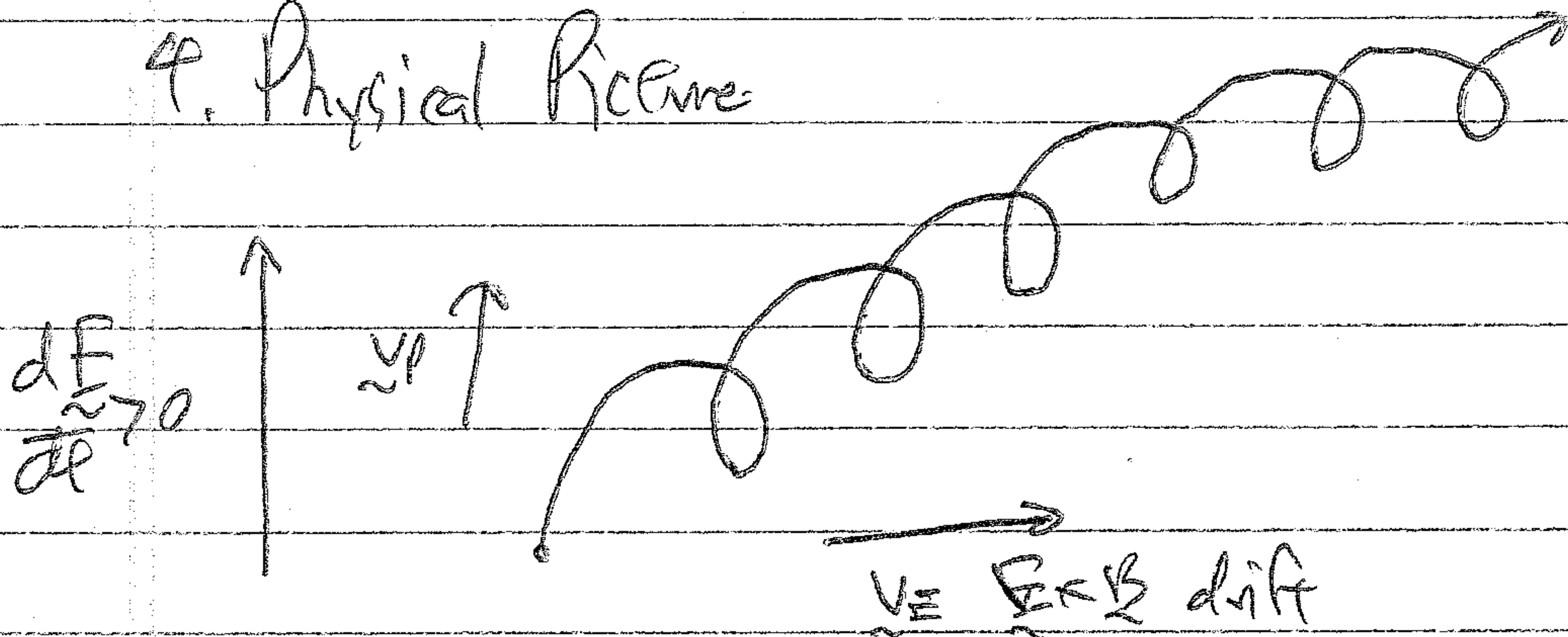
Lecture #9 (Continued)
I.C.3. (Continued)

Hanes ④

b. NOTE: $\underline{E} \times \underline{B}$ velocity is the same for both species, so it cancels, producing no net current.

$$\underline{j}_E = \sum_s q_s n_s \underline{v}_E = \sum_s q_s n_s \frac{\underline{E} \times \underline{B}}{B^2} = \frac{\underline{E} \times \underline{B}}{B^2} \sum_s q_s n_s \stackrel{\rightarrow 0 \text{ by quasineutrality}}{=} 0$$

4. Physical Picture



5. The Polarization Drift can lead to an increase in energy.

a. $\frac{d\varepsilon}{dt} = \underline{v} \cdot \underline{f} = \underline{v} \cdot q(\underline{E} + \underline{v} \times \underline{B}) = q \underline{v} \cdot \underline{E}$

b. $= q \left[v_{\perp} \cos(\omega_c t + \phi) \underline{E} \cdot \hat{e}_1 + v_{\perp} \sin(\omega_c t + \phi) \underline{E} \cdot \hat{e}_2 \right]$

$+ \frac{\underline{E}(\underline{E} \times \underline{B}) \cdot \underline{v}}{B^2} + \frac{1}{\omega_c B} \frac{dE}{dt} \cdot \underline{E}$

Average over Larmor orbit $\Rightarrow 0$.

c. Thus $\frac{d\varepsilon}{dt} = \frac{q}{\omega_c B} \frac{d}{dt} \left(\frac{E^2}{2} \right) = \frac{d}{dt} \left[\frac{1}{2} m \left(\frac{E^2}{B^2} \right) \right]$

d. NOTE: $|\underline{v}_E|^2 = \frac{E^2}{B^2}$, so this can be written $\frac{d}{dt} \left(\frac{1}{2} m v_E^2 \right) = \frac{d\varepsilon}{dt}$

e. The Polarization Drift leads to the acceleration of particles to achieve the $\underline{E} \times \underline{B}$ drift velocity.