# Lecture #21 Force-Balanced MHD Equilibria

## I. Review of MHD Equilibria

A. 1. Momentum Eq:

$$\text{Force/Vol} = \rho\frac{\partial \underline{U}}{\partial t} + \rho \underline{U}\cdot\nabla\underline{U} = \underbrace{-\nabla p + \underline{j}\times\underline{B} - \rho\nabla\Phi_G = 0}_{\text{Equilibrium has zero net force}}$$

2. Neglecting gravity (generally valid for laboratory plasmas)

$$\nabla p = \underline{j}\times\underline{B} \quad \text{required for MHD Equilibrium}$$

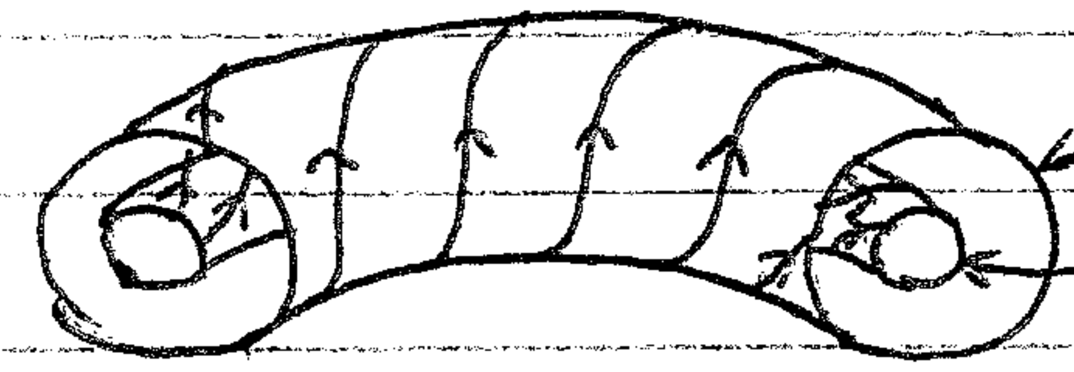
a. If $\underline{j}\times\underline{B} \simeq 0$ $\quad (|\nabla p| \ll |\underline{j}\times\underline{B}|)$ Force-Free

b. If $\underline{j}\times\underline{B} \neq 0$ Force-Balanced

3. Hopf's Theorem:

a. A torus is simplest topological surface satisfying $\nabla\cdot\underline{B}=0$ & $\underline{B}\cdot\nabla\alpha=0$

Thus, to contain a hot plasma, ~~[illegible]~~ magnetic field lines lie on closed surfaces of constant pressure:



Nested Toroidal surfaces of constant pressure.

4. Force-free Equilibrium Solutions

a. Flux Ropes

b. Reverse Field Pinch (RFP)

## II. Force-Balanced Equilibria:

A. 1. Using Ampere's Law, we may write force balance as

$$\boxed{\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{(\underline{B}\cdot\nabla)\underline{B}}{\mu_0}}$$

2. For $p = p(\underline{x})$, $\underline{B} = \underline{B}(\underline{x})$, there exist an infinite number of solutions.

$\Rightarrow$ Let's focus on cylindrical and toroidal geometries

# III. Force-Balanced Equilibria in Cylindrical Geometries

## A. General Case:

1. We'll focus on the family of solutions with a magnetic field with no radial component and only radial dependence,

$$\boxed{\underset{\sim}{B} = B_\phi(r)\,\hat{\phi} + B_z(r)\,\hat{z}} \qquad (B_r = 0)$$

2. The radial component of the force balance (using NRL p. 6-7) gives

$$\boxed{\frac{d}{dr}\left(p(r) + \frac{[B_\phi(r)]^2}{2\mu_0} + \frac{[B_z(r)]^2}{2\mu_0}\right) = \underbrace{-\frac{[B_\phi(r)]^2}{\mu_0 r}}}$$

NOTE: Magnetic tension term depends only on $B_\phi$.

3. Ampere's Law $\underset{\sim}{j} = \frac{1}{\mu_0}(\nabla \times \underset{\sim}{B})$ gives

$$\boxed{\underset{\sim}{j} = -\frac{1}{\mu_0}\frac{dB_z}{dr}\,\hat{\phi} + \frac{1}{\mu_0 r}\frac{d}{dr}(r B_\phi)\,\hat{z}}$$

4. Force balance depends on 3-functions

$$p(r), \quad B_\phi(r), \quad B_z(r)$$

   a. If two are given, the third is determined by force balance (solution to differential equation)

   b. We still have an infinite number of possible solutions
   $\Rightarrow$ Certain limiting cases simplify the force balance equation.

## B. The Z-Pinch:

1. Here, we take $B_z = 0$. Therefore $j_\phi = 0$.

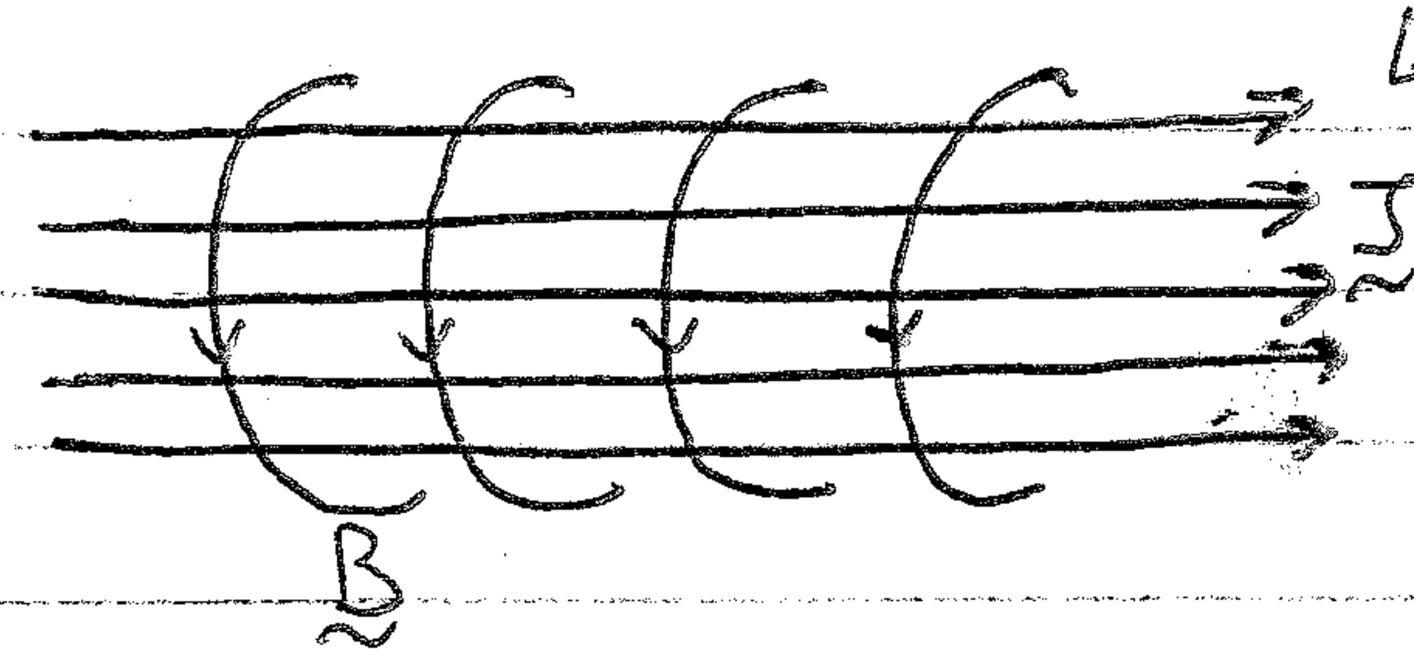
2. $\boxed{\frac{d}{dr}\left(p + \frac{B_\phi^2}{2\mu_0}\right) = -\frac{B_\phi^2}{\mu_0 r}}$

   a. The radial pressure gradient is entirely confined by $B_\phi$.

   b. $B_\phi$ is entirely generated by an axial current $j_z$.

III. $B_0$ (Continued) The Z-Pinch

3.



Strong axial current through plasma produces a confining azimuthal $\underset{\sim}{B}$-field.

4. Consider a plasma of radius $a$ with a constant axial current density inside the plasma, $j_z = j_0$.

a. Total Current: $I_0 = \int_0^{2\pi}\int_0^a r\,dr\,d\phi\, j_0 = \pi a^2 j_0 \Rightarrow j_0 = \frac{I_0}{\pi a^2}$

b. Calculate the resulting $B_\phi$ from Ampere's Law:

i) $j_z = \frac{1}{\mu_0 r}\frac{d}{dr}(r B_\phi)$

ii) $\int_0^r \frac{d}{dr'}(r' B_\phi) = r B_\phi = \int_0^r dr' \mu_0 r' j_z$ where $j_z = \begin{cases} \frac{I_0}{\pi a^2} & r \le a \\ 0 & r > a \end{cases}$

$$= \begin{cases} \int_0^r \frac{\mu_0 I_0}{\pi a^2} r'\,dr' & r \le a \\ \int_0^a \frac{\mu_0 I_0}{\pi a^2} r'\,dr' & r > a \end{cases} = \begin{cases} \frac{\mu_0 I_0 r^2}{2\pi a^2} \\ \frac{\mu_0 I_0 \cancel{a^2}}{2\pi \cancel{a^2}} \end{cases}$$

iii) Thus

$$\boxed{B_\phi(r) = \begin{cases} \frac{\mu_0 I_0}{2\pi a^2} r & r \le a \\ \frac{\mu_0 I_0}{2\pi r} & r > a \end{cases}}$$

c. Calculate the pressure $p(r)$

i) $$\frac{dp}{dr} = -\frac{d}{dr}\left(\frac{B_\phi^2}{2\mu_0}\right) - \frac{B_\phi^2}{\mu_0 r} = \begin{cases} -\frac{2\mu_0 I_0^2 r}{4\pi^2 a^4} & r \le a \\ 0 & r > a \end{cases}$$

ii) At the edge of the plasma, $r = a$, we take $p(a) = 0$.

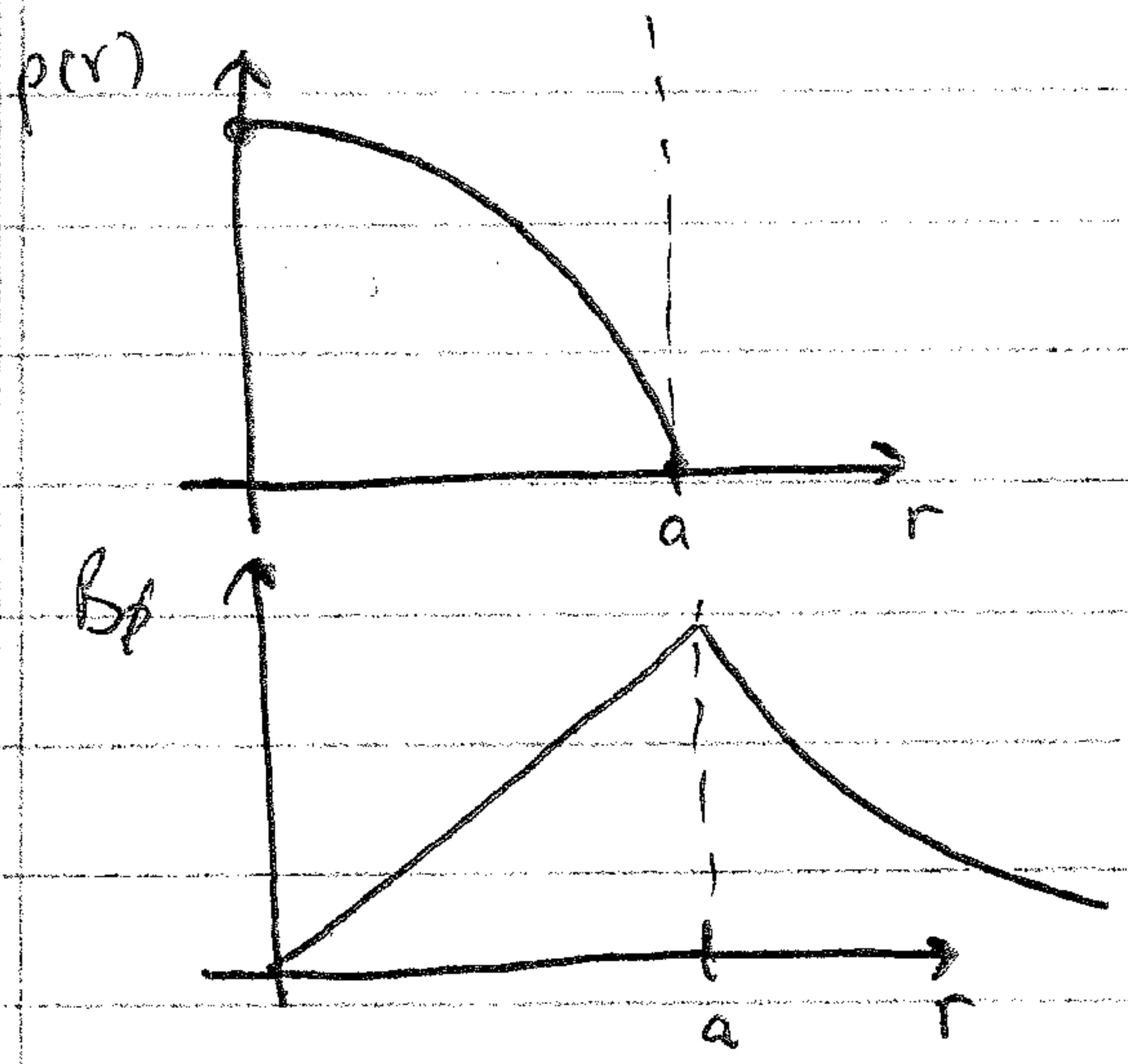
Lecture #21 (Continued) 

III. B. 4. c. (Continued)

iii) Thus $\int_r^a \frac{dp}{dr} dr = \underbrace{p(a)}_{=0 \text{ By Boundary Conditions}} - p(r) = \int_r^a \frac{-2\mu_0 I_0^2 r}{4\pi^2 a^4} dr = -\frac{\mu_0 I_0^2}{4\pi^2 a^2}\left(1 - \frac{r^2}{a^2}\right)$

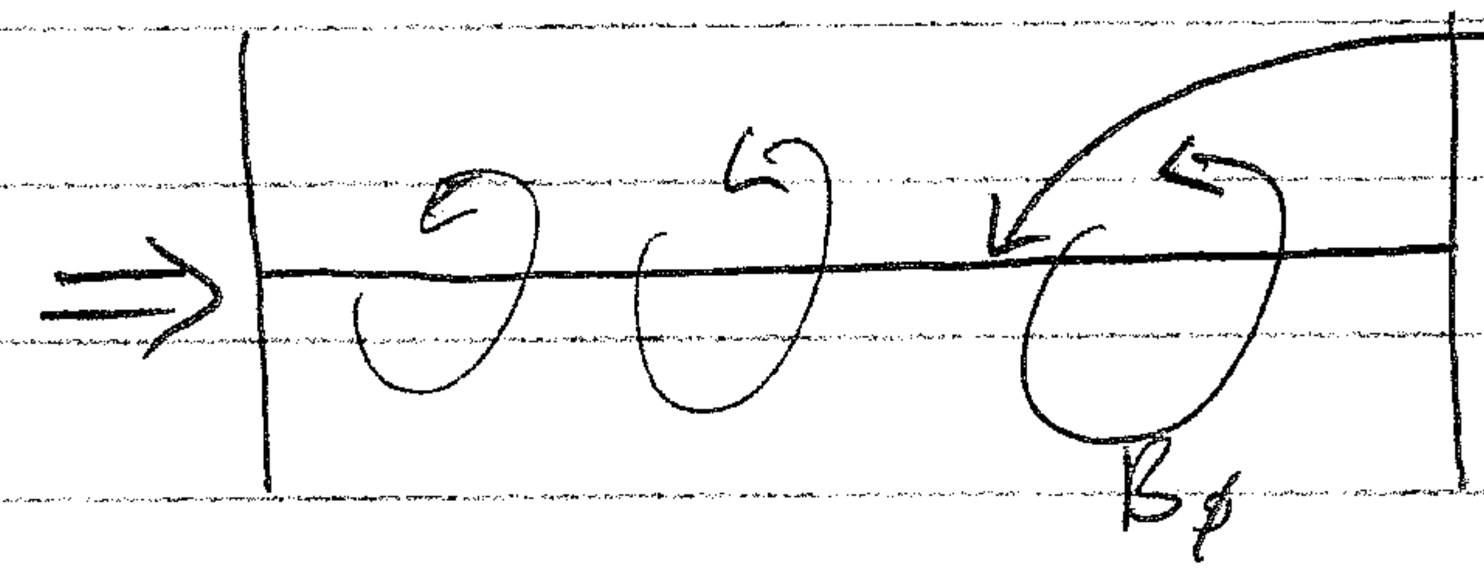
iv) Finally $\boxed{p(r) = \mu_0 \left(\frac{I_0}{2\pi a}\right)^2 \left(1 - \frac{r^2}{a^2}\right)}$ $r \le a$

5. Profiles of $p$ & $B_\phi$



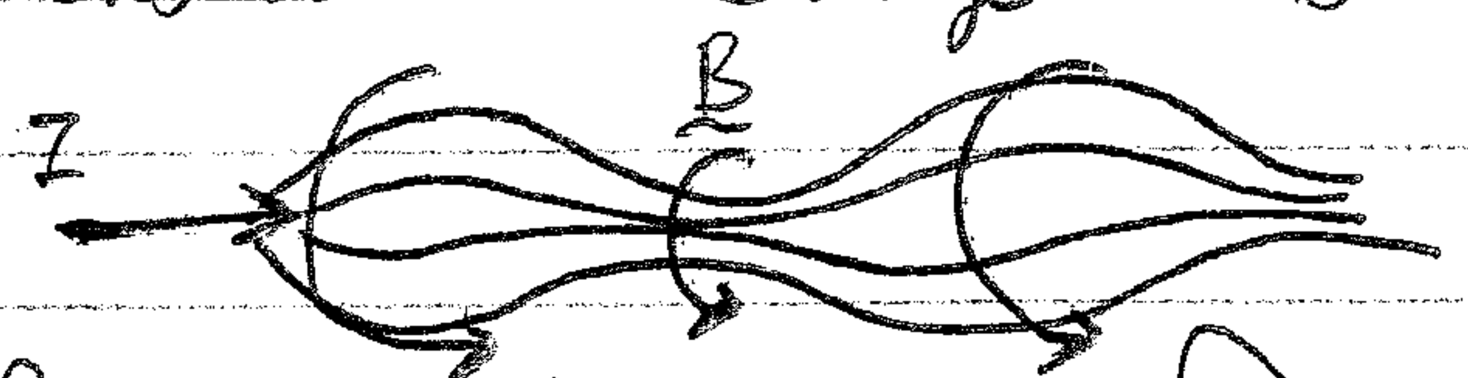

Magnetic pressure and magnetic tension confine thermal pressure.

6. The Z-Pinch at Sandia National Laboratories




a. Enormous magnetic pressure (and tension) due to axial current confines plasma of deuterium in small volume.

b. Unstable to "Sausage" instability




(Next Semester we'll look at MHD Stability)

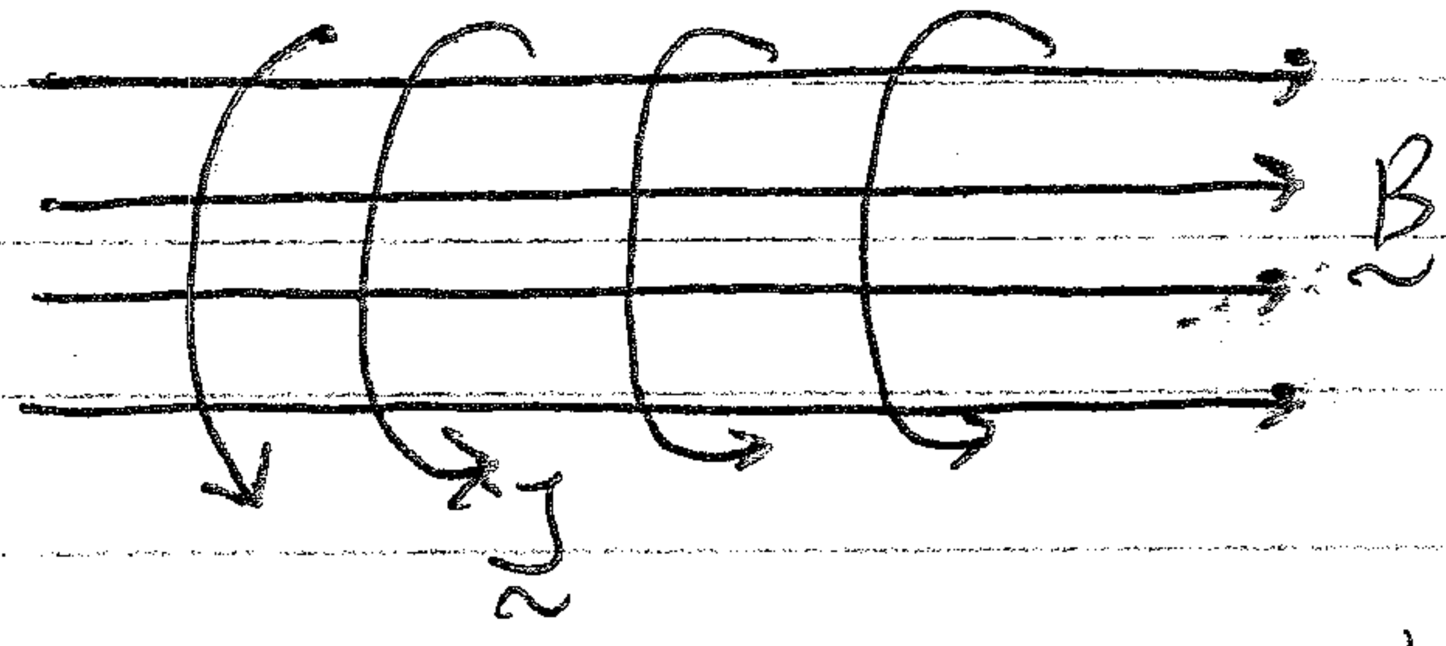
c. Produces copious x-rays from hot plasma. Useful for studies of high-energy-density plasmas!

Lecture #21 (Continued)

II. (Continued)

C. The "Theta" Pinch

1. In this case, we take $B_\phi = 0$ and can solve for $p(r)$ in terms of $B_z(r)$.



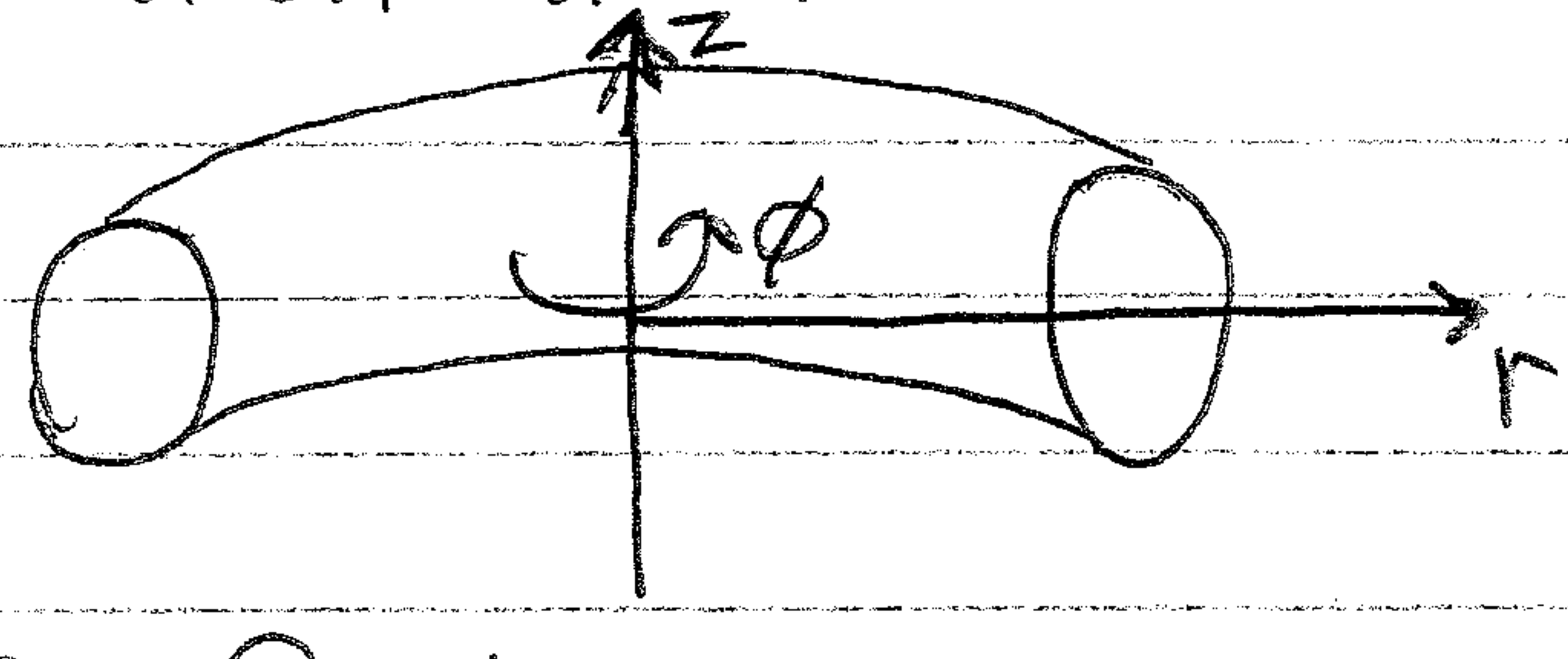

This is a homework problem!

Current in $\phi$ direction produces $B_z$ that confines pressure.

III. Force-Balanced Equilibria in Toroidal Geometries

A. 1. Although the cylindrical cases give us a good intuition of Force-Balanced MHD Equilibrium, it is toroidal geometries that are necessary to have confined in 3-D (Hopf's theorem tells us we must at least have a torus).

2. We now consider toroidal geometries with symmetry in the toroidal direction $\phi$.




B. Magnetic Flux Coordinates:

1. For $\frac{\partial}{\partial \phi} = 0$, $\nabla \cdot \underset{\sim}{B} = 0$ implies $\frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0$.

2. If we use the vector potential $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$, then $\nabla \cdot \underset{\sim}{B} = 0$ is automatically satisfied.

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III. B. (Continued)

3. The toroidal symmetry implies a simplification ($\frac{\partial}{\partial\phi} = 0$)

From NRL p.6

$$B_r = (\nabla \times \underset{\sim}{A})_r = \frac{1}{r}\cancel{\frac{\partial A_z}{\partial \phi}}^{0} - \frac{\partial A_\phi}{\partial z} = -\frac{\partial A_\phi}{\partial z}$$

$$B_z = (\nabla \times \underset{\sim}{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) - \frac{1}{r}\cancel{\frac{\partial A_r}{\partial \phi}}^{0} = \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)$$

a. Both $B_r$ & $B_z$ depend only on $A_\phi$!

4. Defining a flux function $\psi(r,z) = r A_\phi$, we get

$$\boxed{B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}} \qquad \boxed{B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}}$$

5. NOTE: For $\psi(r,z)$, $\underset{\sim}{B}\cdot\nabla\psi = B_r\frac{\partial\psi}{\partial r} + B_z\frac{\partial \psi}{\partial z} = B_r(B_z r) + B_z(-B_r r) = 0$

a. Thus, ~~surfaces of constant~~ $\boxed{\underset{\sim}{B}\cdot\nabla\psi = 0}$

Magnetic fields line must lie on surfaces of $\psi$ = constant.

b. But, we also know ~~[illegible]~~ $\underset{\sim}{B}\cdot\nabla p = 0$, from the force balance, so $\boxed{p = p(\psi)}$

6. Consider the $\phi$-component of $\underset{\sim}{j}\times\underset{\sim}{B} = \nabla p$:

a. $j_z B_r - j_r B_z = \frac{1}{r}\frac{\partial p}{\partial \phi}$ but $\frac{1}{r}\frac{\partial p(\psi)}{\partial\phi} = 0$ since $\psi = \psi(r,z)$.

NRL p.6 b. Ampere's law gives: $j_r = \frac{1}{\mu_0}(\nabla\times\underset{\sim}{B})_r = \frac{1}{\mu_0}\left[\frac{1}{r}\cancel{\frac{\partial B_z}{\partial\phi}}^{0} - \frac{\partial B_\phi}{\partial z}\right]$

$$j_z = \frac{1}{\mu_0}(\nabla\times\underset{\sim}{B})_z = \frac{1}{\mu_0}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) - \frac{1}{r}\cancel{\frac{\partial B_r}{\partial\phi}}^{0}\right]$$

As with $\underset{\sim}{B}$ in terms of $\underset{\sim}{A}$, $\underset{\sim}{j}$ depends only on $B_\phi(r,z)$.

c. Analogous, we define a second flux function $\boxed{F = r B_\phi}$ to satisfy $j_z B_r - j_r B_z = 0$.

III. B. G. (Continued)

d. It follows that $B_r \frac{\partial F}{\partial r} + B_z \frac{\partial F}{\partial z} = 0 \Rightarrow \underset{\sim}{B} \cdot \nabla F = 0$

and so $\boxed{F = F(\psi)}$

7. Thus, we can express the toroidally symmetric magnetic field in terms of two scalar flux functions $\psi(r,z)$, $F(r,z)$.

a. $$\boxed{\underset{\sim}{B} = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z}\right)\hat{\underset{\sim}{r}} + \left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\hat{\underset{\sim}{z}} + \left(\frac{F(\psi)}{r}\right)\hat{\underset{\sim}{\phi}}}$$

Magnetic Field expressed using Magnetic Flux Coordinates

b. This can also be expressed $\underset{\sim}{B} = \frac{\nabla \psi}{r} \times \hat{\underset{\sim}{\phi}} + \frac{F(\psi)}{r}\hat{\underset{\sim}{\phi}}$

C. The Grad-Shafranov Equation:

1. Now, let's consider the r-component of the force balance.

$$j_\phi B_z - j_z B_\phi = \frac{\partial p}{\partial r}$$

$j_\phi$ 2. First use Ampere's Law to calculate $j_\phi$

$$\mu_0 j_\phi = (\nabla \times \underset{\sim}{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \underset{\uparrow}{=} -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) \equiv -\frac{1}{r}\Delta^* \psi$$

$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$, $B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$

$j_z$ 3. Similarly

$$\mu_0 j_z = (\nabla \times \underset{\sim}{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) - \cancelto{0}{\frac{1}{r}\frac{\partial B_r}{\partial \phi}} \underset{\uparrow}{=} \frac{1}{r}\frac{\partial F}{\partial r}$$

$F = rB_\phi$

4. Thus $-\frac{1}{r}\Delta^*\psi \, B_z - \frac{1}{r}\frac{\partial F}{\partial r}B_\phi = \mu_0 \frac{\partial p}{\partial r}$

a. Again, substituting for $B_z$ & $B_\phi$ gives

$$-\frac{1}{r}\Delta^*\psi\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) - \frac{1}{r}\frac{\partial F}{\partial r}\left(\frac{F}{r}\right) = \mu_0 \frac{\partial p}{\partial r}$$

Lecture #21 (Continued)

III. C. (Continued)

5. NOTE: $F = F(\psi)$ and $p = p(\psi)$, so

$$\frac{\partial (F^2)}{\partial r} = \frac{d(F^2)}{d\psi}\frac{\partial \psi}{\partial r} \quad \text{and} \quad \frac{\partial p}{\partial r} = \frac{dp}{d\psi}\frac{\partial \psi}{\partial r}.$$

Thus

$$-\frac{1}{r^2}\Delta^*\psi \left(\cancel{\frac{\partial \psi}{\partial r}}\right) - \frac{1}{2r^2}\frac{d(F^2)}{d\psi}\left(\cancel{\frac{\partial \psi}{\partial r}}\right) = \mu_0 \frac{dp}{d\psi}\left(\cancel{\frac{\partial \psi}{\partial r}}\right)$$

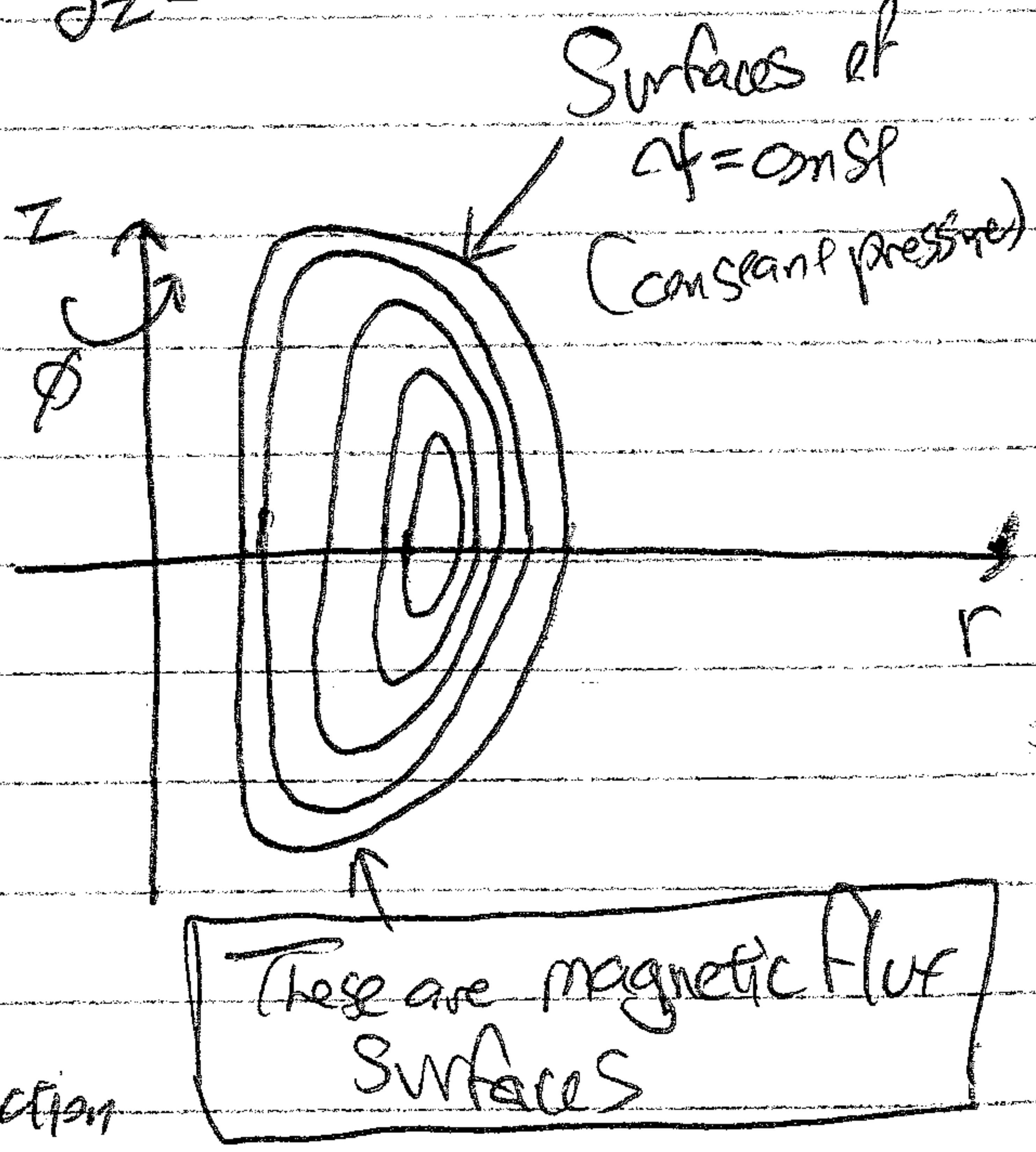
6. Finally, we obtain

$$\boxed{\Delta^*\psi = -\mu_0 r^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}}$$ Grad-Shafranov Equation

where $\Delta^*\psi \equiv r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2}$

D. Application

1. Grad-Shafranov equations is used to calculate <u>Magnetostatic Equilibria in axisymmetric toroidal systems.</u>




2. In practice:

a. Specify $p(\psi)$ and $F(\psi)$
Pressure — Toroidal Field Function

b. Solve Grad-Shafranov Eq (Numerically) with specified boundary conditions for $\psi(r,z)$

c. Pressure profile is then determine $p = p(\psi(r,z))$

3. Exact analytical solutions, known as Solov'ev equilibria, are often used in analysis of toroidal magnetic fusion devices.