Lecture #20 MHD Wave Observations and MHD Equilibria

Howes

I. Observations of MHD Waves:

A. Alfven Waves in the Solar Wind (see Belcher, Davis & Smith, JGR, 74, 2302 (1969).)

1. Consider the eigenmodes for the Alfven Wave with $U_y = U_0$, $U_x = U_z = 0$

a. $(\omega^2 - k_\parallel^2 V_A^2) U_y = 0 \Rightarrow \omega = \pm k_\parallel V_A$

b. NOTE: For $\underset{\sim}{k} = k_\perp \hat{x} + k_\parallel \hat{z}$, $\underset{\sim}{k} \cdot \underset{\sim}{U_1} = 0$ (incompressible)

Thus $\rho_1 = 0$ and $p_1 = 0$

c. $\omega \underset{\sim}{B_1} = \underset{\sim}{B_0} \cancel{(\underset{\sim}{k} \cdot \underset{\sim}{U_1})}^{0} - (\underset{\sim}{B_0} \cdot \underset{\sim}{k}) \underset{\sim}{U_1} \Rightarrow \omega B_y = -B_0 k_\parallel U_y$

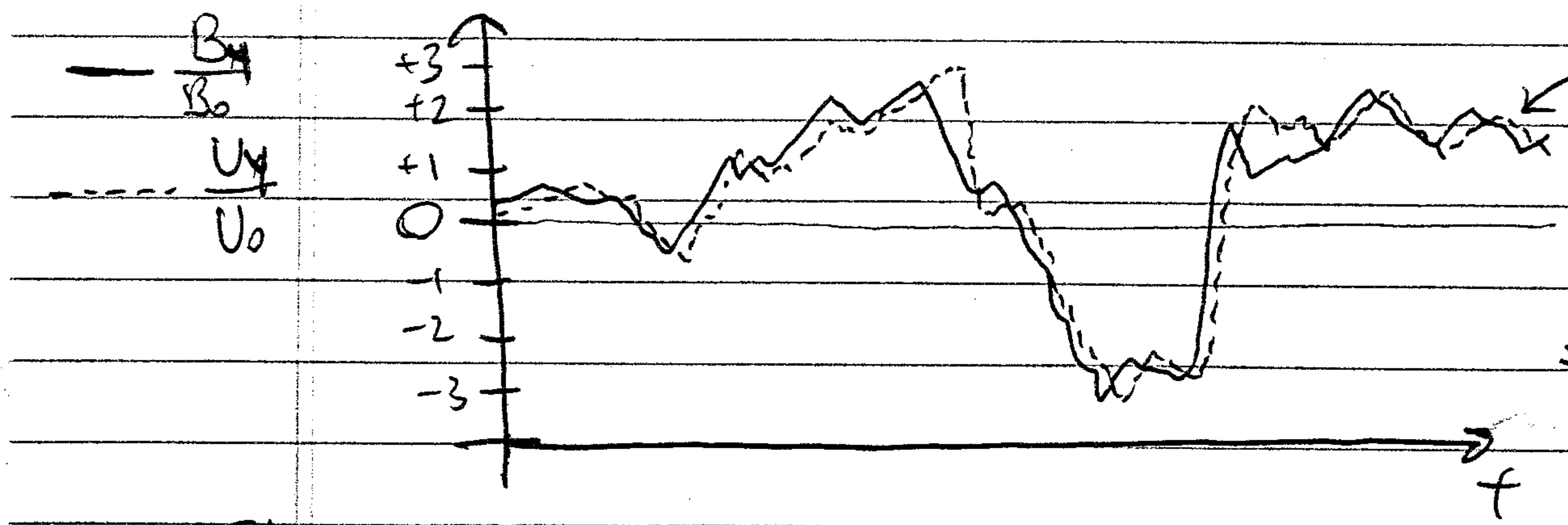
Thus $\dfrac{B_y}{B_0} = -\dfrac{k_\parallel}{\omega} U_y = \dfrac{-k_\parallel}{(\pm k_\parallel V_A)} U_0 = \mp \dfrac{U_0}{V_A}$

d. $\boxed{\dfrac{B_y}{B_0} = \mp \dfrac{U_0}{V_A}}$ $\boxed{U_y = U_0}$ $\boxed{\omega = \pm k_\parallel V_A}$

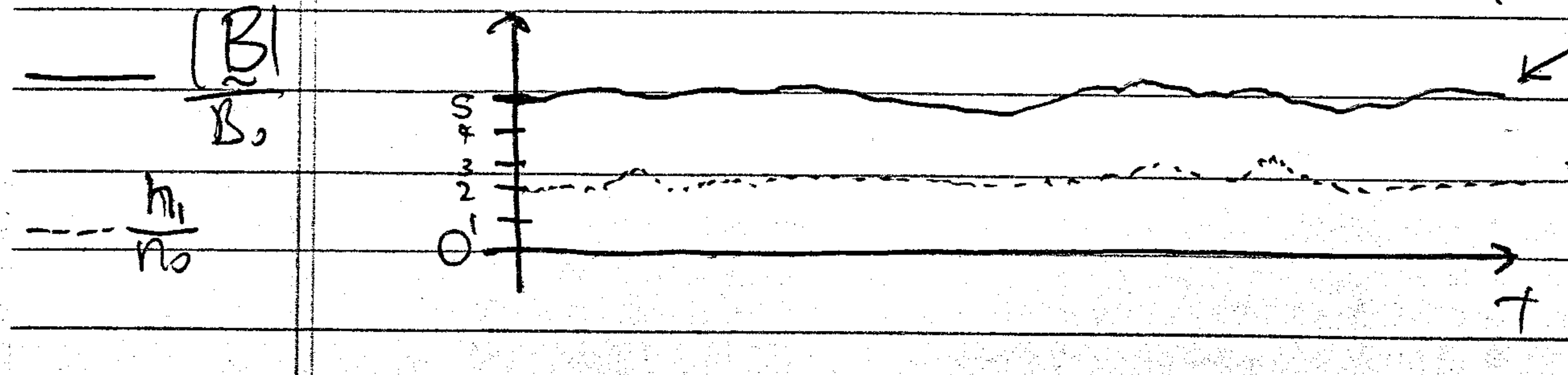
2. For the Alfven wave, the fluctuation $\dfrac{\left(\frac{B_y}{B_0}\right)}{\left(\frac{U_y}{V_A}\right)} = \mp 1$

a. $\boxed{\text{$\underset{\sim}{B_1}$ \& $\underset{\sim}{U_1}$ are either in phase or π out of phase}}$

3. Measurements of magnetic field fluctuations in solar wind



$\underset{\sim}{B_1}$ and U_1 fluctuations are seen to change in phase $\Rightarrow$ Alfven waves

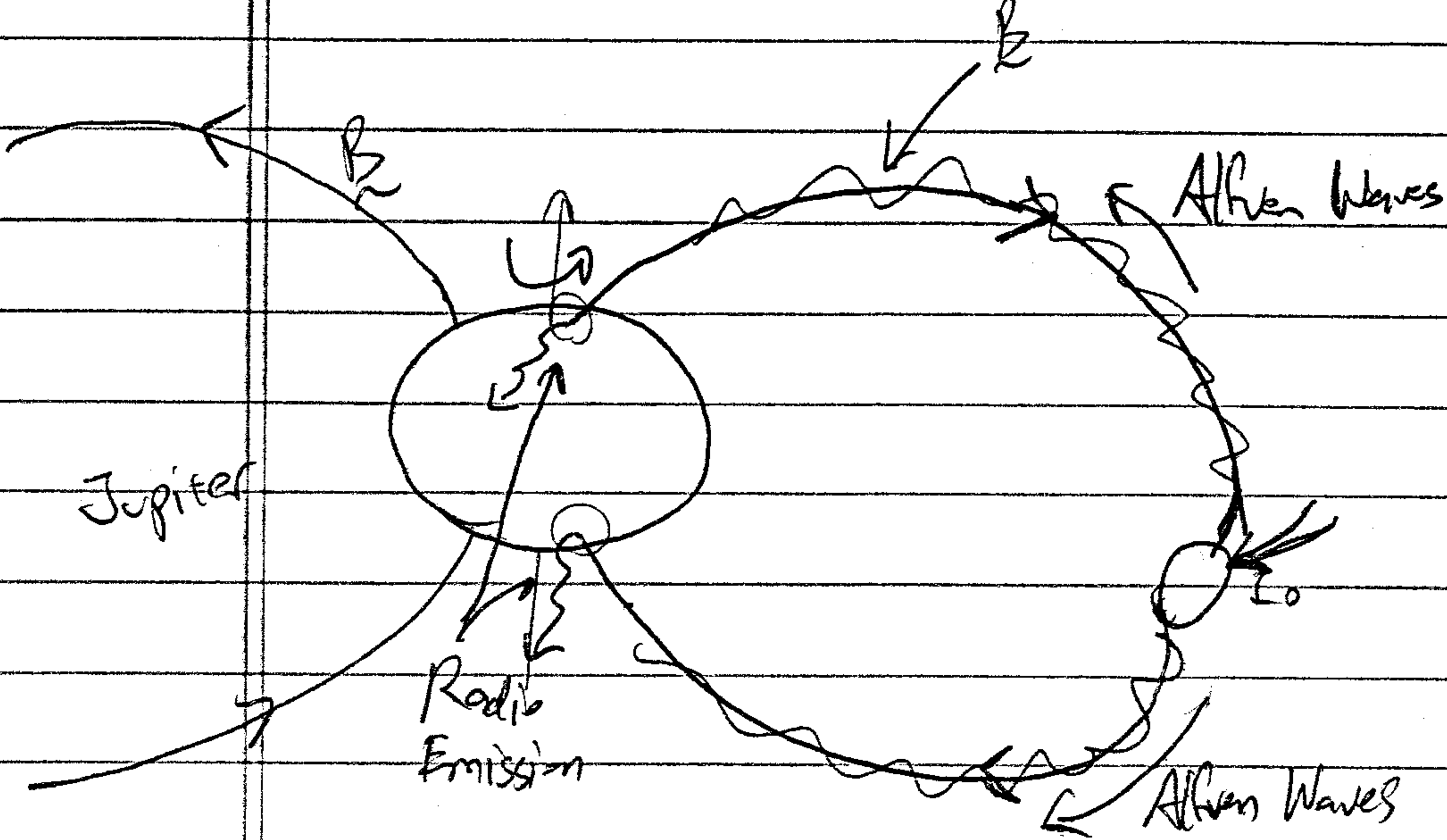


Small fluctuations of $|\underset{\sim}{B_1}|$ and n_1 $\Rightarrow$ Wave is not compressible.

Lecture #20 (Continued) Hawes ②

I. (Continued)

B. Radio Emissions from Jupiter:



Proposed by:
Goldreich & Lynden-Bell
Astrophys. J. 156, 59, (1969).

1. The motion of Jupiter's moon Io through the plasma in Jupiter's magnetic causes Alfven waves.
2. These Alfven waves propagate along the magnetic field to the upper atmosphere of Jupiter
3. When waves strike the atmosphere, the electrons due to the current associated with the Alfven wave (remember $\underset{\sim}{J} = \frac{1}{\mu_0} \nabla \times \underset{\sim}{B}$) cause radio emission.

4. Confirmed during Voyager I flyby of Io
[see Ness, N.F. et.al. Science, 204, 982 (1979).]

Lecture #20 (Continued) Howes (3)

II. MHD Equilibrium

A. Basic concepts about MHD Equilibrium

1. A static equilibrium in MHD exists when $\underset{\sim}{U} = 0$ and the forces on the plasma are in balance.

2. Momentum (or Force) Equation

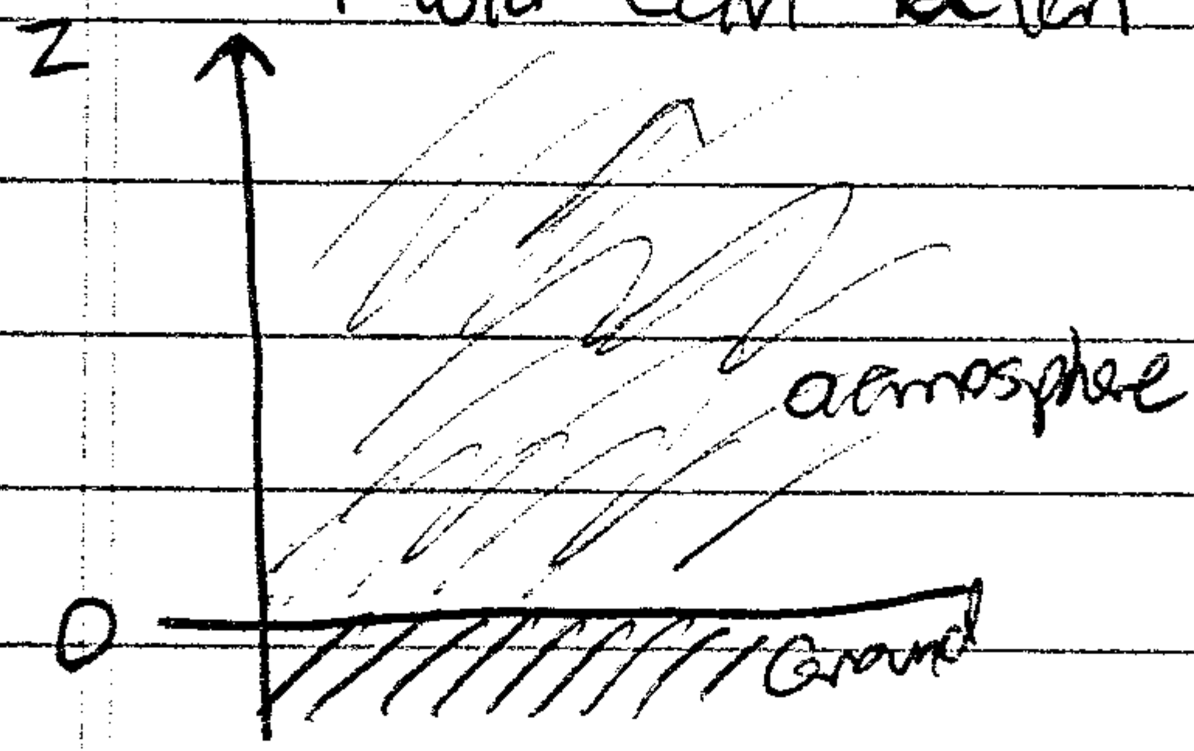
$$\text{Force per unit volume} = \rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla p + \underset{\sim}{j} \times \underset{\sim}{B} - \rho \nabla \underbrace{\Phi_G}_{\text{Gravitational potential}}$$

3. A static equilibrium satisfies $\boxed{0 = -\nabla p + \underset{\sim}{j} \times \underset{\sim}{B} - \rho \nabla \Phi_G}$

NOTE: Equilibrium here refers to force balance, not to thermal equilibrium.

B. Example: Hydrostatic Equilibrium in a plane atmosphere.

1. In the earth's atmosphere, the gravitational force on a fluid can ~~often~~ be well approximated by $-\rho \nabla \Phi_G = -\rho g \hat{\underset{\sim}{z}}$



2. Thus the hydrostatic equilibrium gives

a. $0 = -\nabla p - \rho g \hat{\underset{\sim}{z}}$

b. $\Rightarrow \boxed{\frac{\partial p}{\partial z} = -\rho g}$ (z-component)

3. Consider an Isothermal Atmosphere with $T = T_0$.

a. The ideal gas law gives $p = nkT_0 = \frac{\rho k T_0}{m} \Rightarrow \rho = \frac{m}{kT_0} p$

4. Thus $\frac{\partial p}{\partial z} = -\frac{gm}{kT_0} p \Rightarrow$ Solution $p(z) = p_0 e^{-\frac{z}{H_0}}$

where the scale height $H_0 = \frac{kT_0}{gm}$.

5. $\boxed{\text{The pressure and density in an isothermal atmosphere vary exponentially with scale height } H_0 = \frac{kT_0}{gm}.}$

Lecture #20 (Continued) Howes (4)

II. (Continued)

C. Back to MHD

1. If gravity is negligible (often true in the hot, diffuse plasmas in laboratory devices), then the MHD force balance is

$$\boxed{\nabla p = \mathbf{j} \times \mathbf{B}}$$

a. <u>NOTE:</u> We naturally must also satisfy Maxwell's Eqs: $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, $\nabla \cdot \mathbf{B} = 0$

2. Taking the dot product of the force balance eq. with $\mathbf{j}$ or $\mathbf{B}$ gives

$$\mathbf{j} \cdot \nabla p = 0$$

$$\mathbf{B} \cdot \nabla p = 0$$

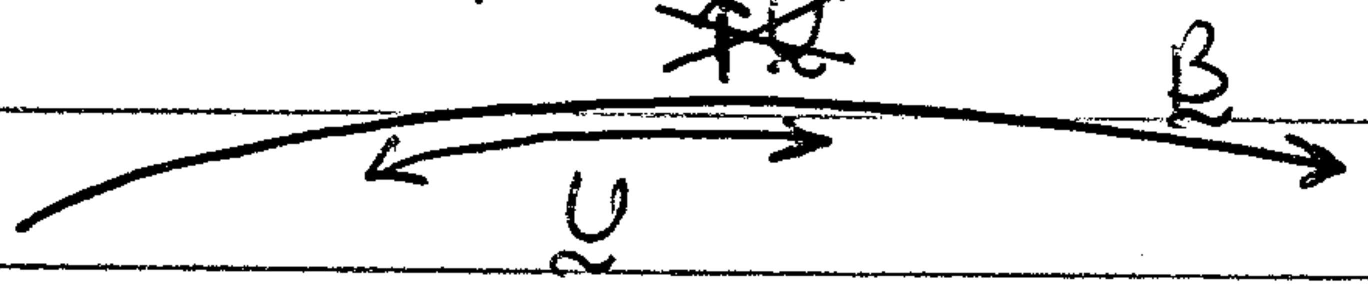
Thus, in force balance $\mathbf{B}$ and $\mathbf{j}$ <u>must</u> lie on surfaces of constant pressure.

3. <u>Magnetic Fusion:</u> The goal of fusion is to contain a hot plasma with a magnetic field.

a. In Ideal MHD, the Frozen-In Flux Theorem tells us that the magnetic field $\mathbf{B}$ is frozen to the plasma.

b. Vice versa, the plasma is stuck to the magnetic field lines.

c. BUT, the plasma can move freely <u>along</u> the $\mathbf{B}$ field line.



d. So, to confine a hot plasma, <u>surfaces of constant pressure must be closed.</u>

4. Static Equilibria can be divided into two classes:

$\mathbf{j} \times \mathbf{B} = 0$	Force-Free
$\mathbf{j} \times \mathbf{B} \neq 0$	Force-Balanced

III. Force-Free MHD Equilibrium ($\mathbf{j} \times \mathbf{B} = 0$)

A.1. This can occur in tenuous or cold plasmas where the pressure can be neglected.

2. The condition $\mathbf{j} \times \mathbf{B} = 0$ is always satisfied when
$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B} = \alpha \mathbf{B} \quad \text{where } \alpha \text{ is a scalar function.}$$

3. What does the force free condition satisfied by $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ tell us

a. Note: Taking the ~~curl~~ divergence
$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\alpha \mathbf{B})$$
$$0 = \alpha \cancel{(\nabla \cdot \mathbf{B})}^{0} + \mathbf{B} \cdot \nabla \alpha \quad \Rightarrow \boxed{\mathbf{B} \cdot \nabla \alpha = 0}$$

b. Again, $\mathbf{B}$ field lines must lie on surfaces of constant α.

4. In this case, we can show

$\boxed{\text{The constant } \alpha \text{ surface cannot be a simple, closed surface.}}$

Proof: a. Let C be a closed curve along a magnetic field line $\mathbf{B}$.

Thus $\int_C \mathbf{B} \cdot d\boldsymbol{\ell} \neq 0$

b. $$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} \underset{\substack{\text{Stoke's Thm} \\ \text{NRL p.6 (34)}}}{=} \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{A} \underset{\nabla \times \mathbf{B} = \alpha \mathbf{B}}{=} \int_S \alpha \mathbf{B} \cdot d\mathbf{A}$$

S: Surface bounded by curve C.

c. Distort surface S until it coincides with a surface of const α.
This can only be done if the surface is simply closed (Otherwise Stoke's theorem is invalid)

d. Since α = const, we have $= \alpha \int_S \mathbf{B} \cdot d\mathbf{A}$

Lecture #20 (Continued) Howes ⑥

II (Continued)

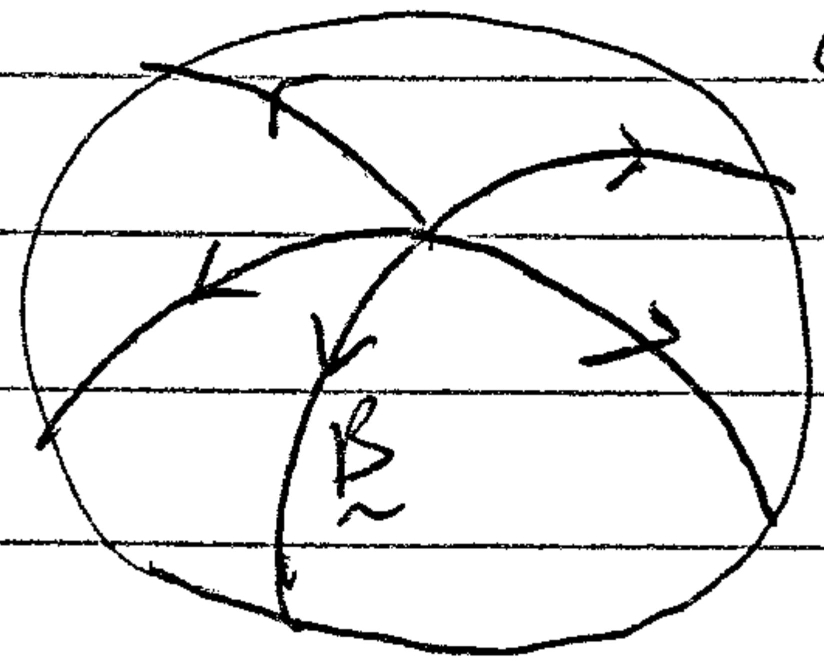
A.4. (Continued)

e. But, $\underline{B}$ lines on the surface of constant α, so
$$\int_S \underline{B}\cdot d\underline{A} = 0.$$

f. We have reached a contradiction!

g. Thus, the surface of constant α cannot be a simply closed surface! QED!

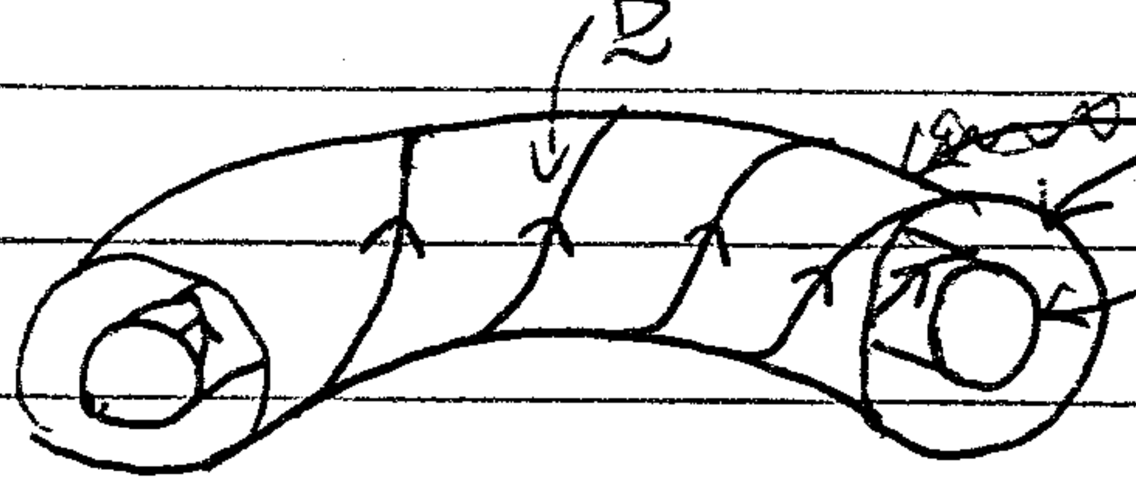
5. A spherical surface of constant α will not work!



a. The magnetic field on a simple closed surface cannot be divergence free

"You can't comb the hair everywhere."

6. Hopf's Theorem: The simplest topological surface of constant α that satisfies both conditions $\nabla\cdot\underline{B}=0$ and $\underline{B}\cdot\nabla\alpha=0$ is a torus.



a. This theorem also holds for force-balance equilibria ($\nabla p \neq 0$). Here the surfaces are constant pressure surfaces.

B. Flux Ropes:

Cylindrical (r,ϕ,z)

1. $\nabla\times\underline{B}=\alpha\underline{B}$ can be satisfied by $\underline{B}=B_\phi\hat{\underline{\phi}}+B_z\hat{\underline{z}}$ (cylindrical geometry) with
$$B_\phi=\frac{B_0 kr}{1+k^2r^2}$$
$$B_z=\frac{B_0}{1+k^2r^2}$$

III. B (Continued)

2. The z-component of the current ($\underline{j} = \frac{1}{\mu_0}\nabla\times\underline{B}$) is then

$$j_z = \frac{2kB_0}{\mu_0(1+k^2r^2)^2}$$

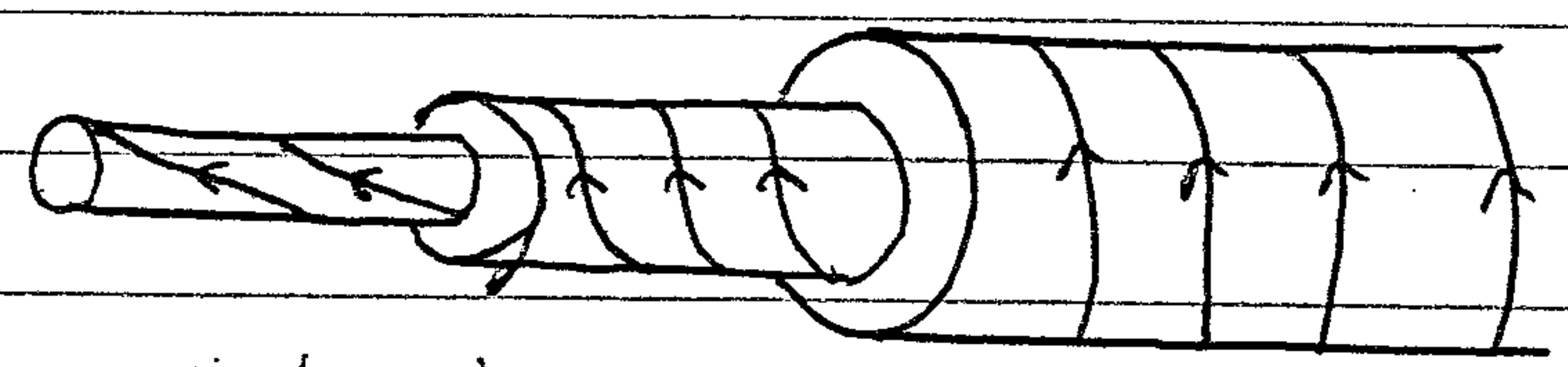
Strong current dies off $\propto \frac{1}{r^4}$

and $\alpha = \frac{\mu_0 j_z}{B_z} = \frac{2k}{1+k^2r^2}$

3. Geometry of Flux Ropes:

a. $\underline{B}$ consists of nested helices.

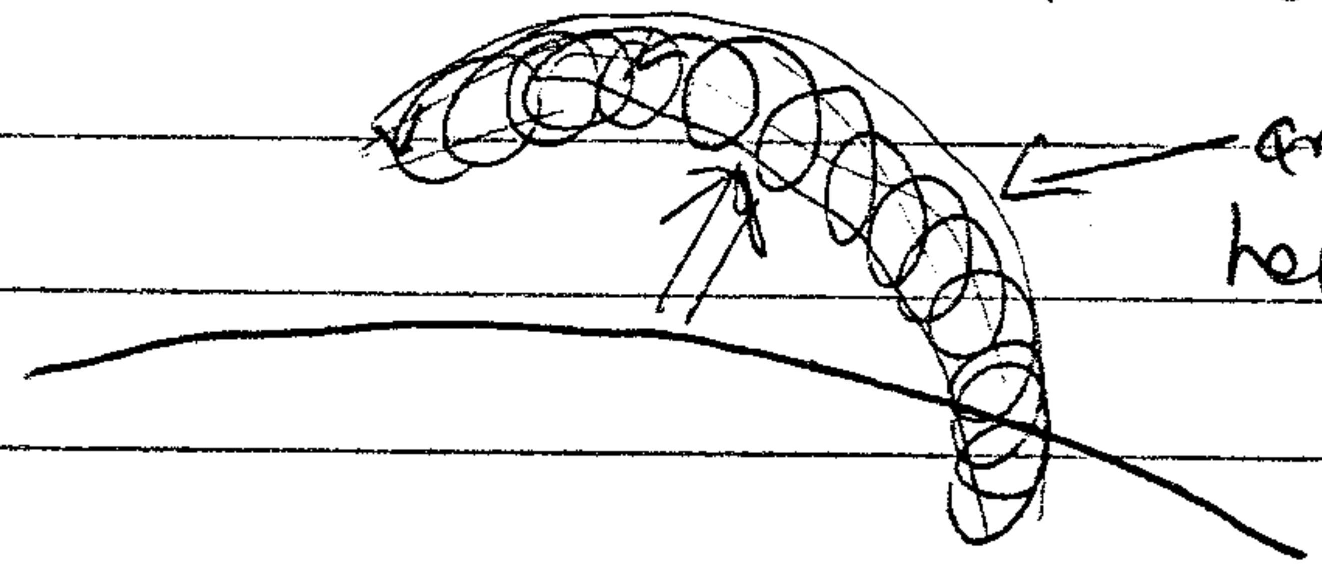
b. Pitch angles (w.r.t. z) from 0 at axis ($r=0$) to $\frac{\pi}{2}$ at large r.



pitch angle increases with r $\rightarrow$ $\lim_{r\to\infty} \frac{B_\phi}{B_z} \propto r$

4. Observations:

a. Prominences in the solar corona can be seen to have such a helical structure.



C. Reverse Field Pinch (RFP)

1. Another class of force-free solutions with cylindrical symmetry has

$$B_\phi = B_0 J_1(\alpha r)$$

$$B_z = B_0 J_0(\alpha r)$$

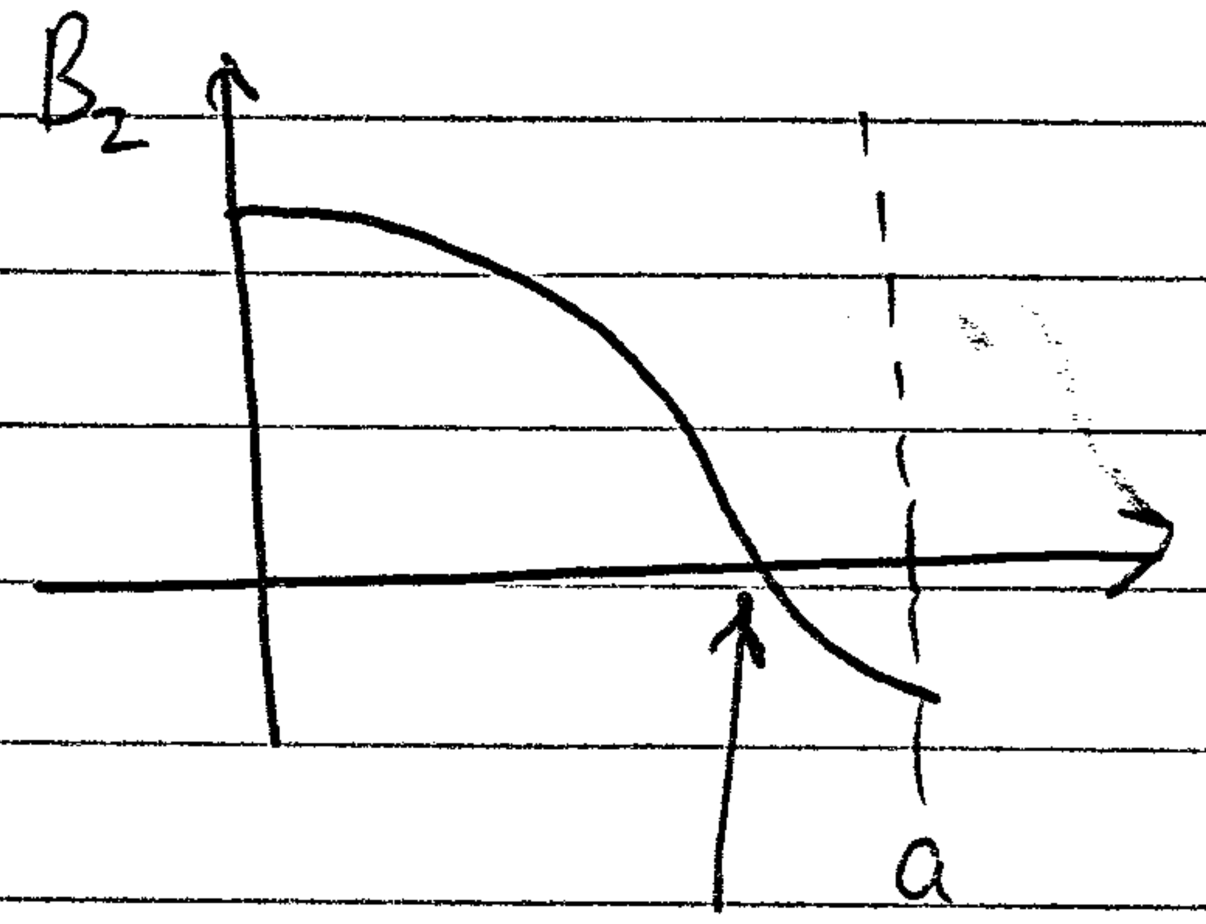
a. α, B_0 are positive constants

b. J_0 & J_1 are zeroth & first order Bessel functions.

III. C. (Continued)

2. ~~[illegible]~~ Magnetic Fields again are nested helices with varying pitch angle.

3. For a plasma of radius $a > \frac{2.4}{\alpha}$, the axial field reverses outside the first zero of the Bessel Function J_0.

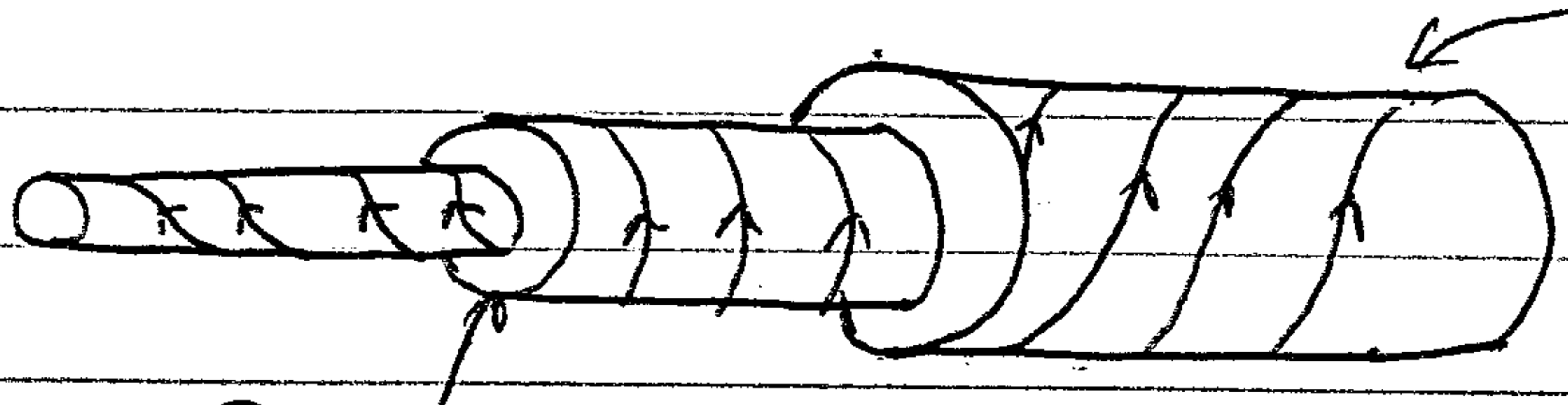


$\alpha r \simeq 2.4 \Rightarrow J_0(\alpha r) = 0$

Such a configuration is known as a

$\boxed{\text{Reverse Field Pinch (RFP)}}$

These have good confinement properties for fusion.



Surface with $B_z = B_0 J_0(\alpha r) = 0$

D. Taylor Relaxation [Taylor, J.B. Phys. Rev. Lett. 33, 1139 (1974)]

1. Taylor proposed RFPs spontaneously form when a turbulent toroidally confined plasma is allowed to relax

a. Magnetic Helicity $H_m \equiv \int d^3x \, \underset{\sim}{A} \cdot \underset{\sim}{B}$ remains constant (it is a conserved quantity in IDEAL MHD)

while

b. Plasma relaxes to state of minimum potential energy

$$E = \underbrace{\frac{1}{2}\rho U^2}_{\text{Kinetic Energy}} + \underbrace{\frac{P}{\gamma - 1} + \frac{B^2}{2\mu_0}}_{\text{Potential Energy}}, \quad U^2 \to 0$$

$\boxed{\text{Taylor Relaxation}}$