

# Lecture #2: Characteristic Scales in a Plasma Hawes ①

Disclaimer: Please accept a blanket disclaimer for the inevitable errors that appear in these notes.

## I. Characteristic Length and Time Scales in a Plasma

### A. Debye Length and Shieldings

1. Consider a plasma in a volume  $L^3$  with  $N$  ions (mass  $m_i$ , charge  $q_i = e$ ) and  $N$  electrons (mass  $m_e$ , charge  $q_e = -e$ )

a. This plasma is quasi-neutral

$$n_i = n_e = \frac{N}{L^3} \equiv n_0$$

b. Add a test particle, an ion with mass  $m_i$ , charge  $q_i = e$ . The potential due to this test particle alone is

$$\phi(r) = \frac{e}{4\pi\epsilon_0 r} \quad \text{where } r \text{ is spherical coordinate.}$$

Electrons will be attracted to this positive charge!

c. Heavy ion approximation For protons,  $\frac{m_i}{m_e} = 1836 \gg 1$ ,

so taking  $\frac{m_i}{m_e} \gg 1$ , we simplify the problem by assuming the "infinitely" heavy ions don't move, they only provide a neutralizing background.

d. If we allow ample time to reach thermal equilibrium, statistical mechanics tells us the electron density will be governed by Maxwell-Boltzmann distribution

$$n_e = n_0 e^{-\frac{e\phi}{kT_e}} \quad \text{where } T_e \text{ is electron temperature.}$$

Lect #2 (Continued)

Howes ②

L.A. (Continued)

d. (Continued) Here  $\phi(r)$  is the potential of the test particle plus the disturbed electron distribution.

e. Solve using Poisson's  $\underline{E} \cdot \nabla \underline{E} = \frac{\rho}{\epsilon_0}$

where charge density  $\rho = q_i n_i + q_e n_e = e n_0 - e n_e$

i) The electrostatic potential  $\underline{E} = -\nabla\phi$ , so

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} (n_0 - n_0 e^{+\frac{e\phi}{kTe}})$$

Valid for  $\frac{e\phi}{kTe} \ll 1$

ii) We'll <sup>Taylor</sup> expand the exponential,  $e^{+\frac{e\phi}{kTe}} \approx 1 + \frac{e\phi}{kTe} + \dots$  <sup>higher order terms</sup> <sub>H.O.T.</sub>

iii) Substituting in and dropping higher order terms:

$$\nabla^2 \phi = -\frac{e n_0}{\epsilon_0} \left[ 1 - \left( 1 + \frac{e\phi}{kTe} \right) \right] = \frac{e^2 n_0}{\epsilon_0 kTe} \phi$$

iv) Using NRL p.8 for Laplacian  $\nabla^2$  in spherical coordinates, ~~and~~ ~~assuming~~ taking spherical symmetry,

$$\boxed{\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = \frac{e^2 n_0}{\epsilon_0 kTe} \phi}$$

f. To solve for  $\phi(r)$ , we'll use a trick to simplify

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) = \frac{1}{r} \frac{d^2(f)}{dr^2} \quad [\text{Prove this in HW}]$$

g. Using  $y = r\phi$ , ~~and~~ and defining  $\lambda_D^2 = \frac{e^2 n_0}{\epsilon_0 kTe}$ , we find

$$\frac{d^2 y}{dr^2} = \frac{y}{\lambda_D^2}$$

## Lect #3 (Continued)

Hawes ③

### I.A.1. (Continued)

h. This equation has two solutions, so a linear combination is

$$y = A e^{-\frac{r}{\lambda_D}} + B e^{\frac{r}{\lambda_D}}$$

i) On physical grounds, we cannot have  $y \rightarrow \infty$  as  $r \rightarrow \infty$ ,  
so  $B = 0$ .

ii) Thus,  $y = A e^{-\frac{r}{\lambda_D}}$  or  $\boxed{\phi = \frac{A}{r} e^{-\frac{r}{\lambda_D}}}$

i. We must now determine the constant  $A$ .

We do this by matching to the solution as  $r \rightarrow 0$  as given by the potential of the test particle,

$$\phi = \frac{e}{4\pi\epsilon_0 r}$$

For  $r \ll \lambda_D$ ,  $\phi = \frac{A}{r} \Rightarrow A = \frac{e}{4\pi\epsilon_0}$

j. Our solution is therefore

$$\boxed{\phi(r) = \frac{e}{4\pi\epsilon_0 r} e^{-\frac{r}{\lambda_D}}$$

where the Debye Length

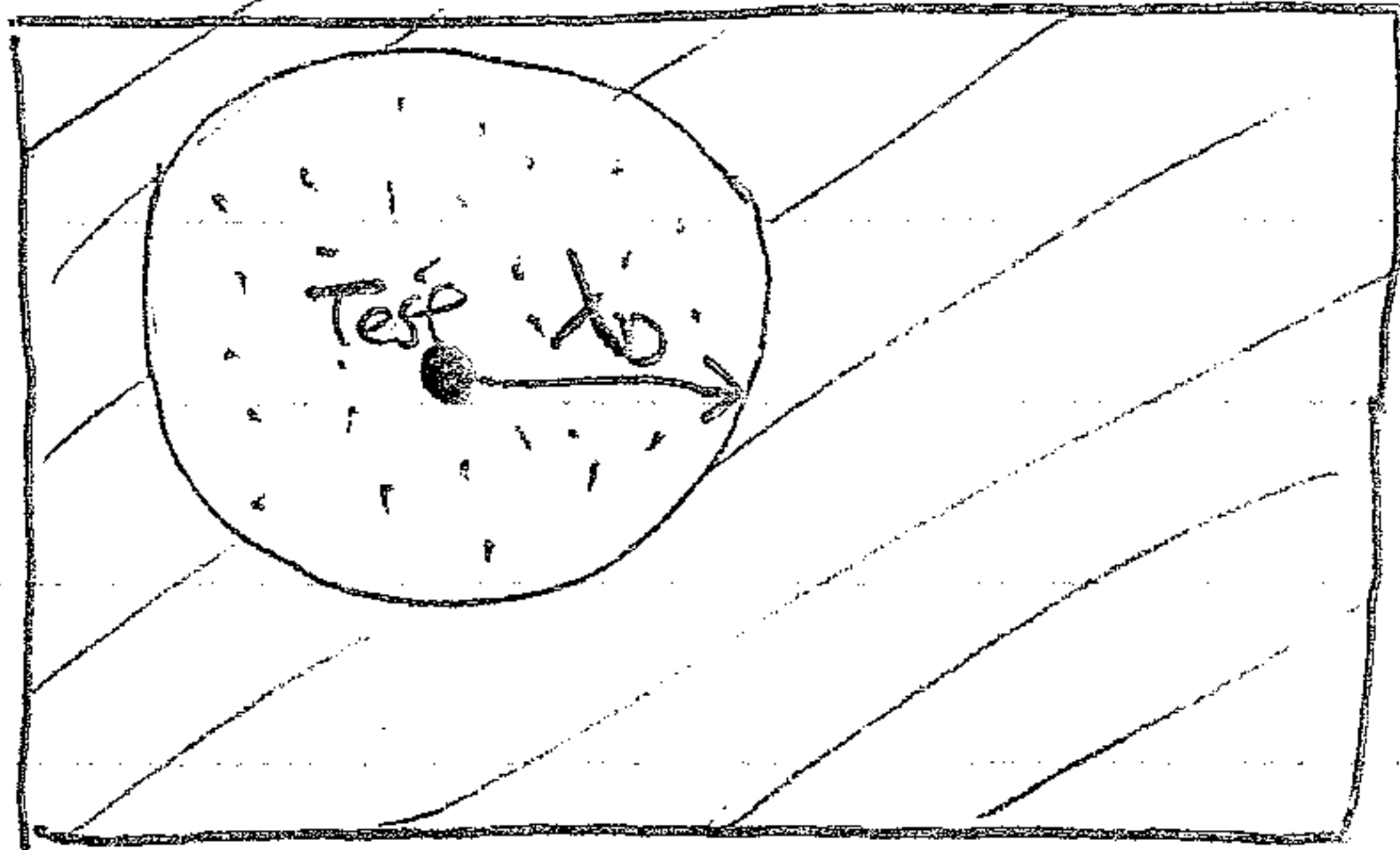
$$\boxed{\lambda_D \equiv \left( \frac{\epsilon_0 k T e}{e^2 n_0} \right)^{\frac{1}{2}}}$$

## 2. Summary:

a. Here we have used Asymptotic matching to connect the solution  $\phi(r)$  at large  $r \gg \lambda_D$  to the potential dominated by the test particle at  $r \ll \lambda_D$ .

- Lect # 2 (Continued)  
 2.  $\lambda_D$  (Continued)  
 3. Interpretation

Hours ④



Debye  
Shielding

- a. Electrons act to shield out the Coulomb field of the test ion  
 b. The net charge inside sphere of radius  $\lambda_D$  is zero.

#### 4. Plasma Parameter, $N_D$ :

- a. For the above picture to be correct, we need many particles in a Debye sphere.

$$N_D \equiv \frac{4\pi}{3} \lambda_D^3 n$$

- b.  $N_D \gg 1$  "Weak coupling", usual requirement for collective behavior

$N_D \ll 1$  "Strong coupling" stellar interiors, very dense plasmas, etc.

c. NOTE that

$$N_D = \left( \frac{4\pi}{3} \frac{\epsilon_0^{3/2} K^{3/2}}{e^3} \right) \frac{T_e^{3/2}}{n_e^{1/2}} \propto \frac{T^{3/2}}{n^{1/2}}$$

$\Rightarrow$  High temperature or low density gives  $N_D \gg 1$ .

## Leet #2 (Continued)

HW #5

### I.A. (Continued)

5. Check validity of  $\frac{e\phi}{kT} \ll 1$  for outer solution at  $r \sim \lambda_D$

$$a. \frac{e\phi}{kT} = \frac{(e^2 n_0)}{4\pi \underbrace{(\epsilon_0 kT)}_{\frac{1}{\lambda_D^2}} r n_0} e^{-\frac{r}{\lambda_D}} = \frac{\lambda_D}{\left(\frac{4\pi \lambda_D^3 n_0}{3}\right) 3r} e^{-\frac{r}{\lambda_D}}$$

$$b. \frac{e\phi}{kT} \sim \frac{1}{N_0} \frac{\lambda_D}{3r} e^{-\frac{r}{\lambda_D}} \ll 1 \quad \checkmark$$

### B. Average Particle Spacing:

1. For a plasma of density  $n_0$ , average particle spacing is  $n_0^{-\frac{1}{3}}$

2. How do kinetic and potential energies compare?

$$a. P.E. \sim \frac{e^2}{4\pi \epsilon_0 n_0^{-\frac{1}{3}}}$$

$$b. K.E. \sim \frac{3}{2} kT$$

$$c. \text{ Thus } \frac{K.E.}{P.E.} \sim \frac{\frac{3}{2} kT}{\frac{e^2}{4\pi \epsilon_0 n_0^{-\frac{1}{3}}}} = 4\pi \underbrace{(\epsilon_0 kT)}_{\lambda_D^2} \frac{1}{2} n_0 n_0^{-\frac{1}{3}} = \left(\frac{4\pi \lambda_D^3 n_0}{3}\right) \frac{1}{2} \frac{n_0^{-\frac{1}{3}}}{\lambda_D}$$

$$= N_0 \frac{1}{2} \frac{n_0^{-\frac{1}{3}}}{\lambda_D} \gg 1$$

3. a. The kinetic energy of a particle dominates over the potential energy due to the interaction with one other particle.

b. Most of the potential energy arises from the many particles between  $n_0^{-\frac{1}{3}}$  and  $\lambda_D$ .

c. The smoothed field of many particles dominates over discrete particle effects  $\Rightarrow$  Statistical description appropriate.

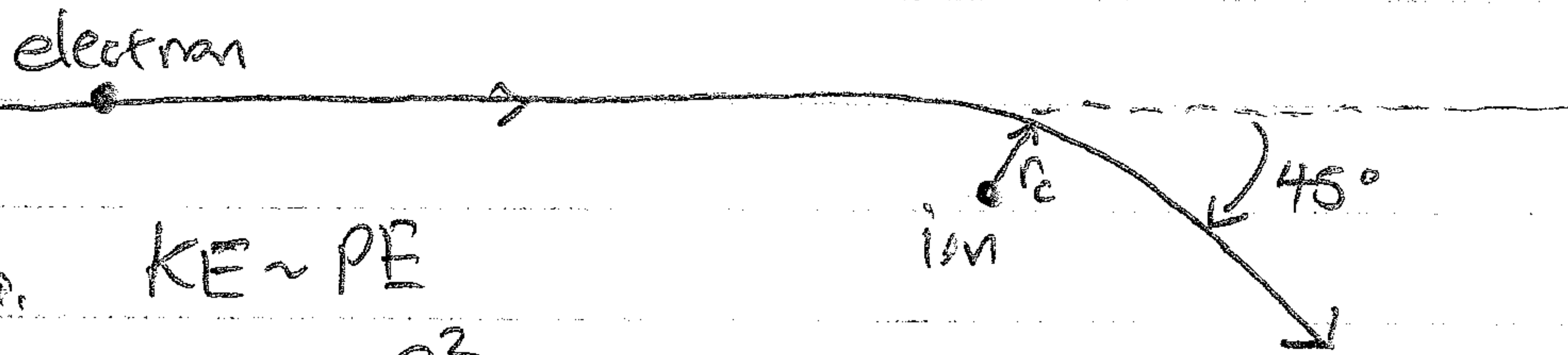
### C. System Size: $L$

1. Another important <sup>length</sup> scale is the scale over which we view the system, or the system size  $L$ .

2. a. To observe the "collective behavior" characteristic of plasmas, we require  $\lambda_D \ll L$
- b. Otherwise, for  $L \ll \lambda_D$ , particles interact little and follow relatively straight-line trajectories.

D. Mean Free Path and Collisions:

1. An electron is deflected by  $\sim 45^\circ$  when it passes close enough to an ion such that  $KE \sim PE$ .



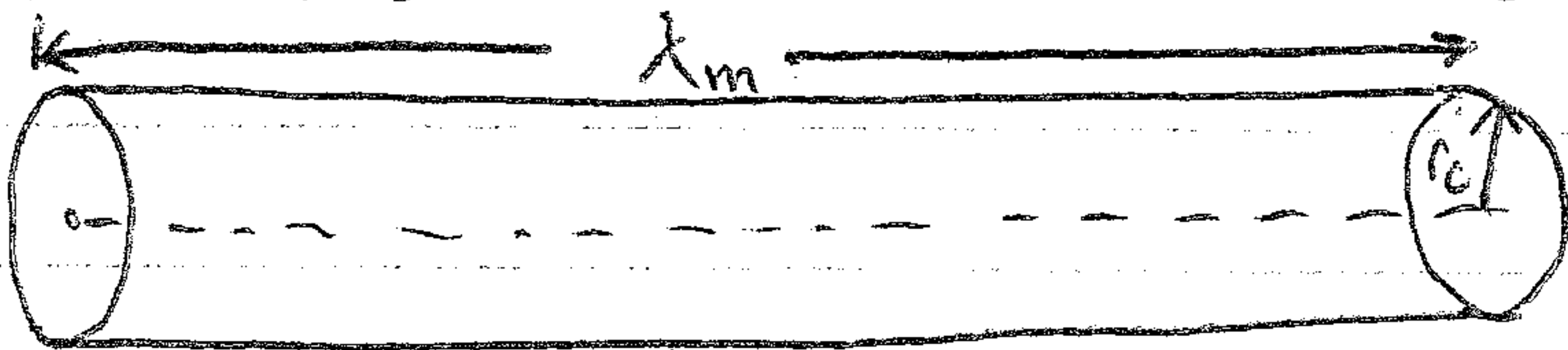
a.  $KE \sim PE$

$$\frac{3}{2} kT_e \sim \frac{e^2}{4\pi\epsilon_0 r_c}$$

b. Solve for  $r_c$ :  $r_c \sim \frac{2}{3} \left( \frac{e^2 N_0}{\epsilon_0 kT_e} \right)^{\frac{1}{3}} \frac{1}{4\pi N_0} = \frac{2}{3} \frac{\lambda_D}{3.34 \frac{4\pi}{3} \lambda_D^3 N_0} = \frac{2}{9} \frac{\lambda_D}{N_0}$

Distance of closest approach  $r_c \sim \frac{2}{9} \frac{\lambda_D}{N_0} \sim \frac{\lambda_D}{N_0}$

2. How far does electron travel before encountering ion within  $r_c$ ?



a. It must pass through a volume  $\pi r_c^2 \lambda_m$  such that  $\pi r_c^2 \lambda_m N_0 = 1$

b.  $\lambda_m = \frac{1}{\pi r_c^2 N_0} \sim \frac{N_0^2}{\pi N_0 \lambda_D^2} \sim \frac{N_0 \frac{4}{3} \lambda_D^3 N_0}{\pi N_0 \lambda_D^2} \sim \frac{4}{3} \lambda_D N_0 \gg \lambda_D$

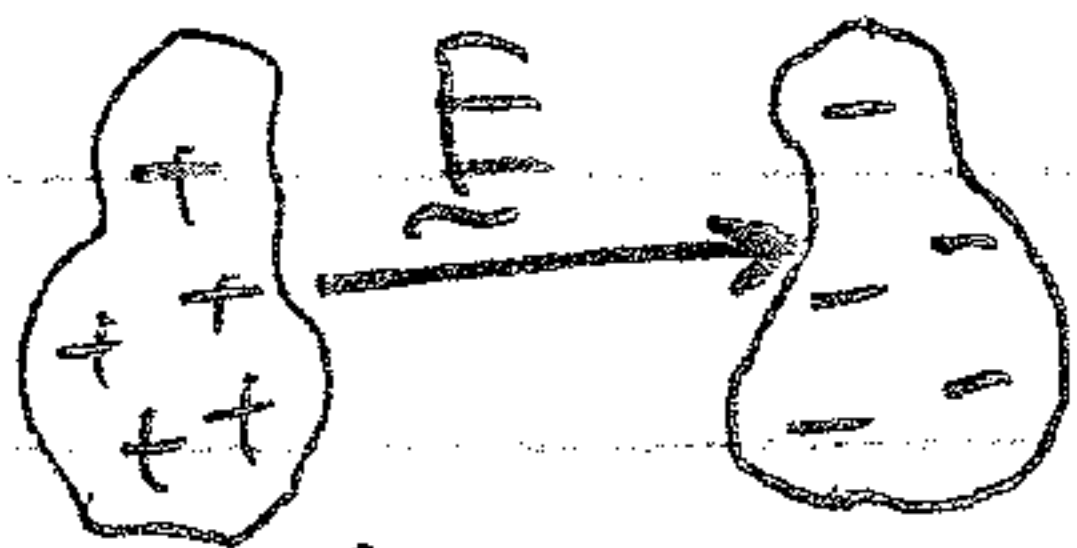
- c. Particles travel a long distance before colliding.

3. Systems are:  $\lambda_m \gg L$  "collisionless"  
 $\lambda_m \ll L$  collisional

Lecture #2 (Continued)  
I. (Continued)

E. Plasma Frequency:

1. Consider a quasi-neutral plasma ( $n_i = n_e = n_0$ ) with infinitely heavy ions ( $m_i/m_e \gg 1$ ).
2. a. Take the electrons in a small region and displace them



b. A strong electrostatic field will arise to restore neutrality.

c. This restoring force will lead to plasma oscillations at a characteristic frequency, the plasma frequency  $\omega_{pe} = \left( \frac{n_0 e^2}{\epsilon_0 m_e} \right)^{1/2}$

3. Any applied electric field with a frequency  $\omega < \omega_{pe}$  will be "shielded out" by the rapid electron response.

F. Observation Time:  $\tau$

1. a. If we observe the plasma for a time  $\tau$ , we are most sensitive to dynamics on that timescale, or at frequency  $f \sim \frac{1}{\tau}$  ( $\text{or } \omega \sim \frac{2\pi}{\tau}$ ).

b. Dynamics occurring on a slower timescale will not be apparent.

G. Collision Frequency:  $\nu$

1. If the mean free path  $\lambda_m$  and typical velocity are known, we can determine the frequency of collisions.

2. Typical velocity: Thermal Velocity

a. Take  $\frac{1}{2} m_s v^2 = kT_s$  for a species  $s$ .

b. Define Thermal Velocity

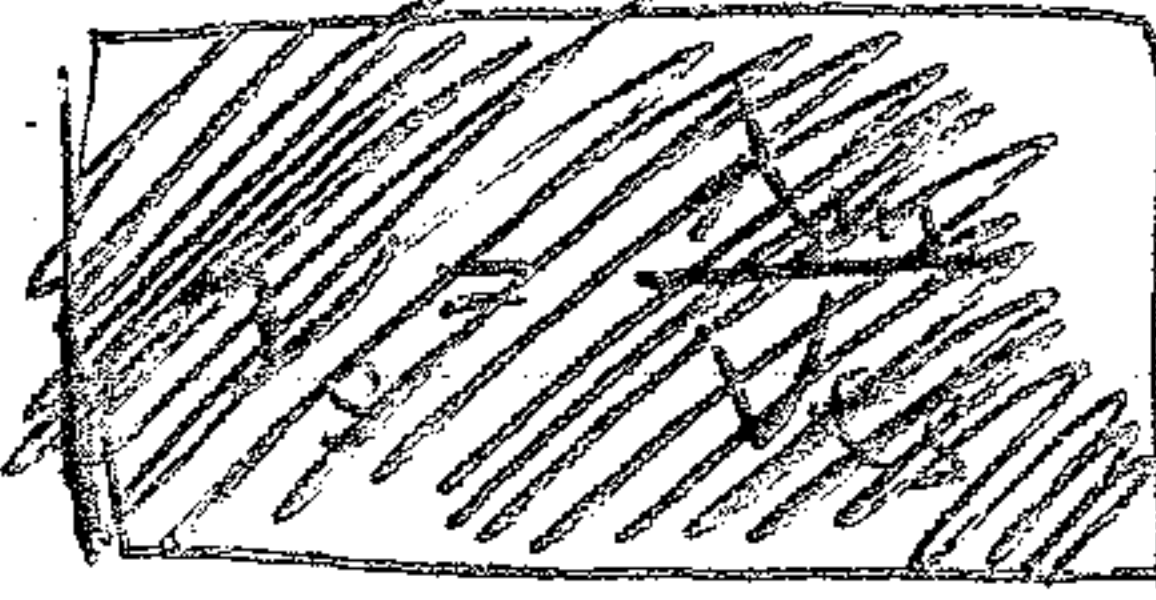
$$v_{T_s} \equiv \left( \frac{2kT_s}{m_s} \right)^{1/2}$$

Lecture 12 (Continued)

Howes (5)

1. G: (Continued)

3. Thus, collision frequency for species s

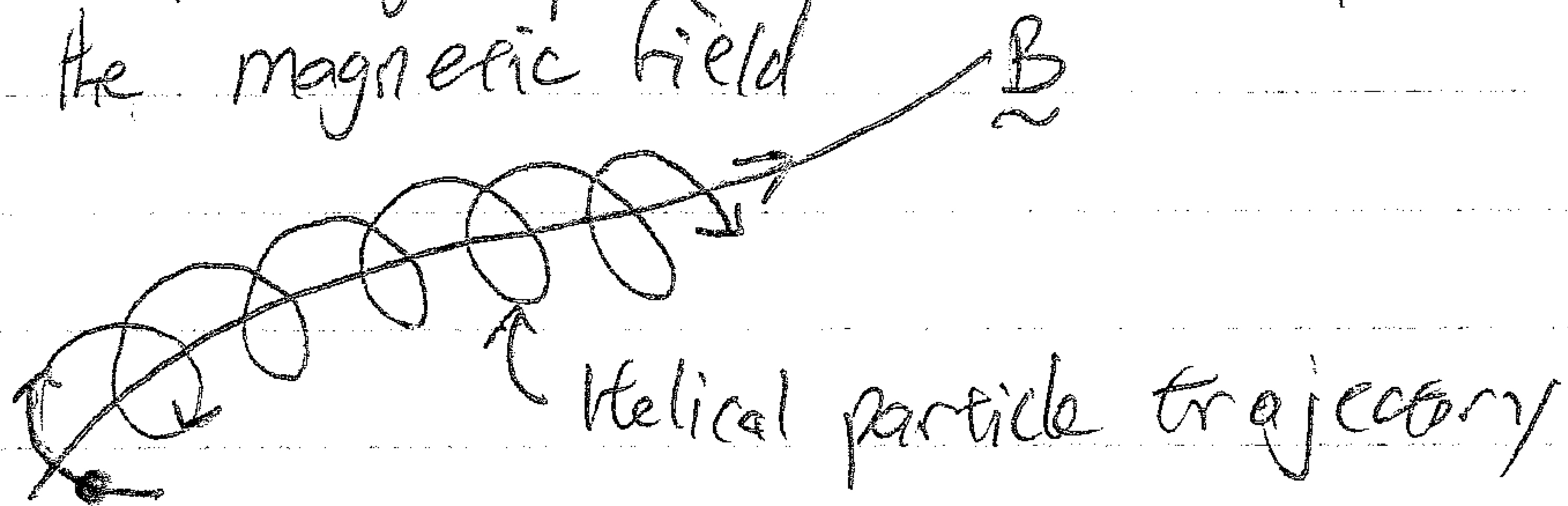


$$\nu_s = \frac{v_{ts}}{\lambda_m}$$

## H. Magnetized Plasmas:

1. Two additional important scales occur when a plasma is magnetized.

a. As we know, charged particles exhibit cyclotron motion about the magnetic field



## 2. Cyclotron Frequency: $\omega_{cs}$

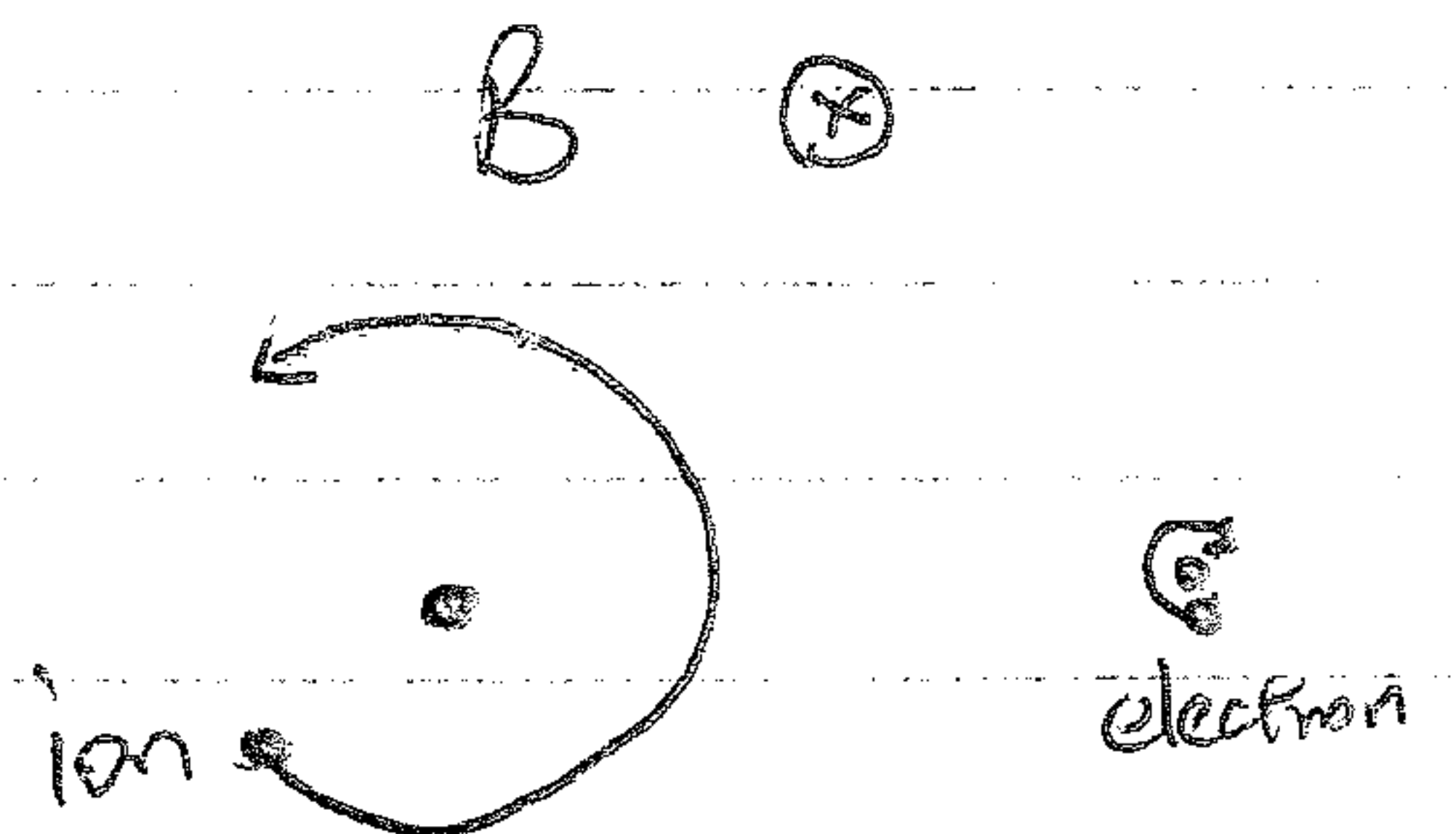
a. The characteristic frequency of gyration is given by

$$\omega_{cs} = \frac{q_s B}{m_s}$$

## 3. Thermal Larmor Radius: $r_{Ls}$

a. For particles with a characteristic thermal velocity  $v_{ts}$ , the radius of this gyration is

$$r_{Ls} = \frac{v_{ts}}{\omega_{cs}}$$



## II. Dimensionless Parameters of a Plasma

### A. Why dimensionless?

1. Plasma physicists often use dimensionless quantities to characterize the nature of a given plasma.
2. These dimensionless numbers connect directly to the dynamics of the plasma, regardless of the order of magnitude of the characteristic dimensional quantities.
3. Allows for comparison of plasmas in very different environments.
4. The plasma parameter  $N_D$  is one such dimensionless number.

### B. Plasma Beta

$$1. \beta \equiv \frac{\text{Thermal Pressure}}{\text{Magnetic Pressure}} = \frac{P_{th}}{P_m}$$

$$a. P_{th} = n_0 k (T_i + T_e) \quad (\text{for quasi neutral plasma } n_i = n_e)$$

$$b. P_m = \frac{B_0^2}{2\mu_0}$$

$$c. \beta \equiv \frac{2\mu_0 n_0 k (T_i + T_e)}{B_0^2}$$

2. Low beta plasmas,  $\beta \ll 1$  are magnetically dominated.

Ex. magnetic fusion plasmas, laboratory plasmas,  
solar corona

3. High beta plasmas,  $\beta \gg 1$ , have a magnetic field that can be highly deformed by the plasma motions

Ex. Black hole accretion disks

Lecture #2 (Continued)  
 II (Continued)

Hines (10)

C. Magnetization:

1. Whether a plasma can be considered magnetized is determined by  $\frac{v_{ti}}{L}$

a.  $\frac{v_{ti}}{L} \ll 1$  magnetized

b.  $\frac{v_{ti}}{L} \gg 1$  unmagnetized

D. Collisionality:

1. Compare collision frequency  $\nu$  to observation "frequency"  $\frac{1}{\tau}$

$\frac{\nu}{(1/\tau)} = \nu\tau$  a.  $\nu\tau \ll 1$  "collisionless"

b.  $\nu\tau \gg 1$  collisional

2. NOTE: Since  $v_s = \frac{v_{ts}}{\lambda_m}$  (and suppressing species subscript's)

a.  $\nu\tau = \frac{v_t}{\lambda_m} \tau = \frac{L}{\lambda_m}$  if  $v_t \equiv \frac{L}{\tau}$

b. This agrees with definitions of collisionality earlier in terms of  $L$  &  $\lambda_m$

III. Summary of length scales, time scales, and dimensionless quantities.

<u>Lengths:</u>		<u>Time/Frequency:</u>	<u>Dimensionless:</u>
Particle spacing	$n_0^{-1/3}$	Plasma Frequency $\omega_p$	Plasma parameter $N_0$
Debye Length	$\lambda_D$	Cyclotron Frequency $\omega_c$	Plasma Beta $\beta$
Larmor Radius	$r_L$	Collision Frequency $\nu$	Magnetization $v_{ti}/L$
Mean Free Path	$\lambda_m$	Observation "frequency" $\frac{1}{\tau}$	Collisionality $\nu\tau$
System Size	$L$		