Lecture #16: The Frozen-in Flux Theorem

I. Review: Basic Concepts in MHD

A. MHD Equations

Continuity Eq. $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0$

Momentum Eq. $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla \left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$

Induction Eq. $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) + \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}$

Energy Eq. (Adiabatic) $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$

1. Resistive MHD when $\eta \neq 0$
2. Ideal MHD when $\eta = 0$

B. MHD Approximation

1. Strong Collisions: $\lambda_m \ll L$, $\tau \gg \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \nu_{ei}^{-1}$
2. Non-relativistic: $v_0 \ll c$
3. Magnetized: $r_{Li} \ll L$

C. Properties of MHD

1. Quasineutrality: $\sum_s n_s q_s \simeq 0$
2. Ohm's Law: $\underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} = \eta \underset{\sim}{J}$

a. Typically, the conductivity of plasmas is very high, so η is small.

b. Usually, $|\eta \underset{\sim}{J}| \ll |\underset{\sim}{U} \times \underset{\sim}{B}|$, so $\underset{\sim}{E} \simeq -\underset{\sim}{U} \times \underset{\sim}{B}$

Thus, the fluid velocity $\underset{\sim}{U}$ corresponds to the $\underset{\sim}{E} \times \underset{\sim}{B}$ velocity

II. Dimensionless Numbers in Fluid Dynamics

A. General

1. In fluid dynamics, a common way to characterize the behavior of fluids in different physical systems is to calculate dimensionless numbers characteristic of the flow.
2. These dimensionless numbers typically characterize the magnitude of the ratio of two terms in the dynamical equations.

B. Example: Reynolds Number in Hydrodynamics, Re

1. Navier-Stokes Equation: $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla p + \mu \nabla^2 \underset{\sim}{U}$

   where μ is the coefficient of shear viscosity.

2. We can divide by the density to yield:

$$\frac{\partial \underset{\sim}{U}}{\partial t} + \overbrace{\underset{\sim}{U} \cdot \nabla \underset{\sim}{U}}^{\text{convection}} = -\frac{1}{\rho}\nabla p + \overbrace{\nu \nabla^2 \underset{\sim}{U}}^{\text{diffusion}}$$

   where we define $\boxed{\text{Kinematic Viscosity } \nu \equiv \frac{\mu}{\rho}}$

3. The Reynolds Number is defined as ratio of convection to diffusion term.

   a. $Re \equiv \frac{|\underset{\sim}{U} \cdot \nabla \underset{\sim}{U}|}{|\nu \nabla^2 \underset{\sim}{U}|} \sim \frac{(V_0^2/L)}{\nu V_0/L^2} \sim \frac{L V_0}{\nu}$

   b. Thus $\boxed{Re \equiv \frac{L V_0}{\nu}}$

4. ~~The~~ Low Re flows ($Re <$ few hundred) $\Rightarrow$ laminar

   High Re flows ($Re \gtrsim 10^3$) $\Rightarrow$ turbulent.

C. Magnetic Reynolds Number: Re_M

1. Induction Equation: $\frac{\partial \underset{\sim}{B}}{\partial t} = \underbrace{\nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})}_{\text{convection}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}}_{\text{diffusion}}$

II. (Continued)

2. The <u>Magnetic Reynolds Number</u> is the ratio of convection to diffusion term in the induction equation,

$$Re_M \equiv \frac{|\nabla\times(\underset{\sim}{U}\times\underset{\sim}{B})|}{|\frac{\eta}{\mu_0}\nabla^2\underset{\sim}{B}|} \sim \frac{\left(\frac{V_0 B}{L}\right)}{\frac{\eta B}{\mu_0 L^2}} \sim \frac{\mu_0 L V_0}{\eta}$$

Thus $\boxed{Re_M \equiv \frac{\mu_0 L V_0}{\eta}}$

3. a. In the limit $Re_M \gg 1$, the convection term dominates and diffusion can be ignored ⟹ Ideal MHD!

b. In the limit $Re_M \ll 1$, the diffusion term dominates.

III. <u>The Frozen-in Flux Theorem:</u>

Most plasmas satisfy the condition $Re_M \gg 1$, giving

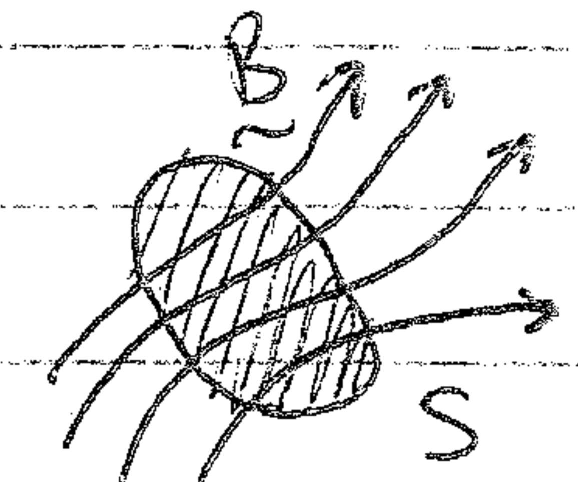
$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla\times(\underset{\sim}{U}\times\underset{\sim}{B}) = 0 \qquad \text{Ideal MHD Induction Eq.}$$

In this limit, a powerful theorem can be proven.

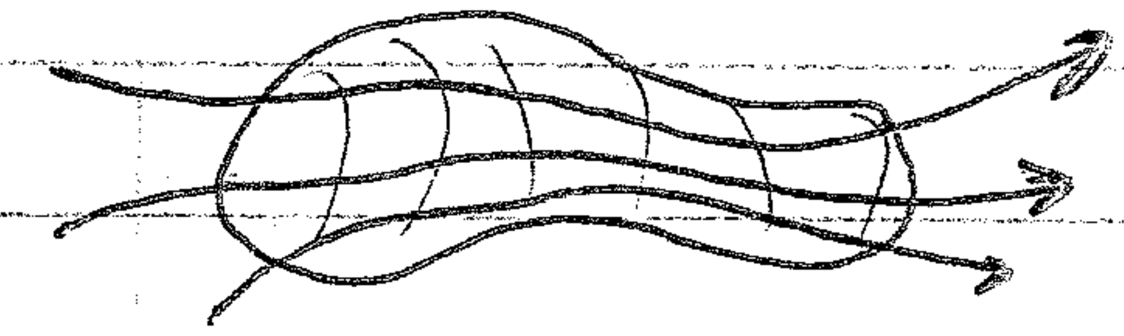
A. <u>The Frozen-in Flux Theorem:</u> The Magnetic flux through a surface moving with the plasma (at fluid velocity $\underset{\sim}{U}(\underset{\sim}{x},t)$) remains constant.

<u>Proof:</u>

1. First, we define Magnetic Flux $\boxed{\Phi_B = \int_S \underset{\sim}{B}\cdot d\underset{\sim}{A}}$



2.a. <u>Fact:</u> The flux through any closed surface is zero.

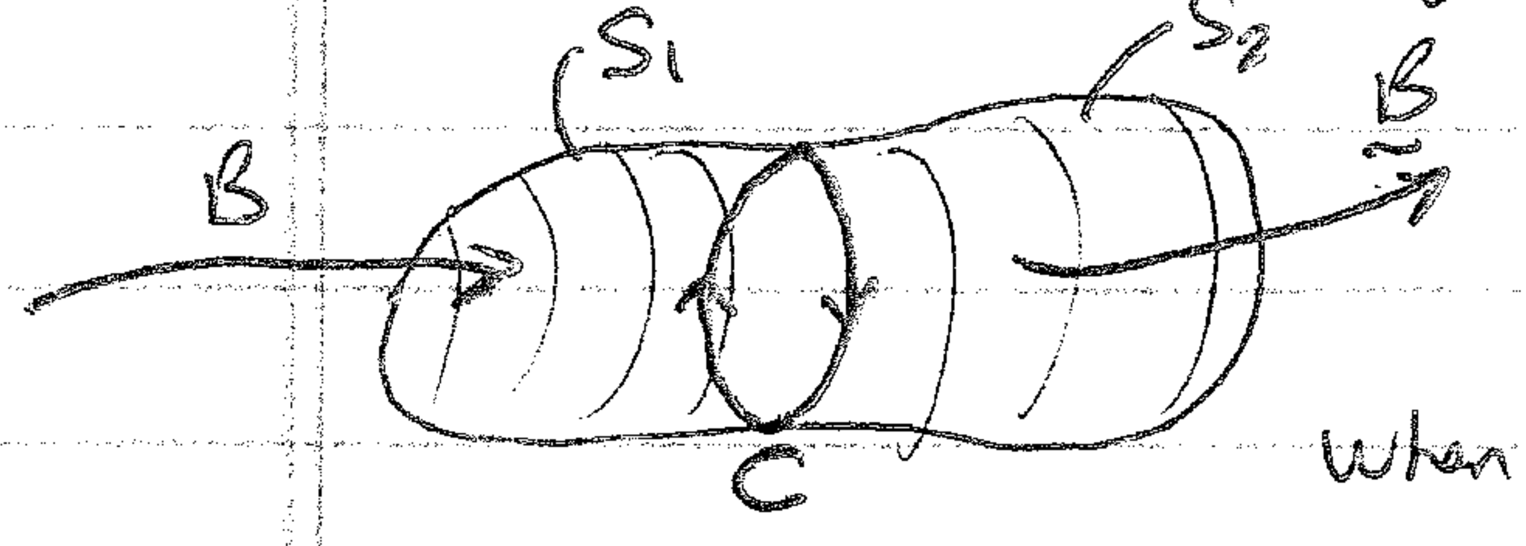


$$\oint_S \underset{\sim}{B}\cdot d\underset{\sim}{A} = \int_V \nabla\cdot\underset{\sim}{B}\, d^3\underset{\sim}{x} = 0$$

Divergence Theorem, NRL p.5 (28)

III. A.2. (Continued)

b. Fact: The flux through any surface spanning a curve C is the same.

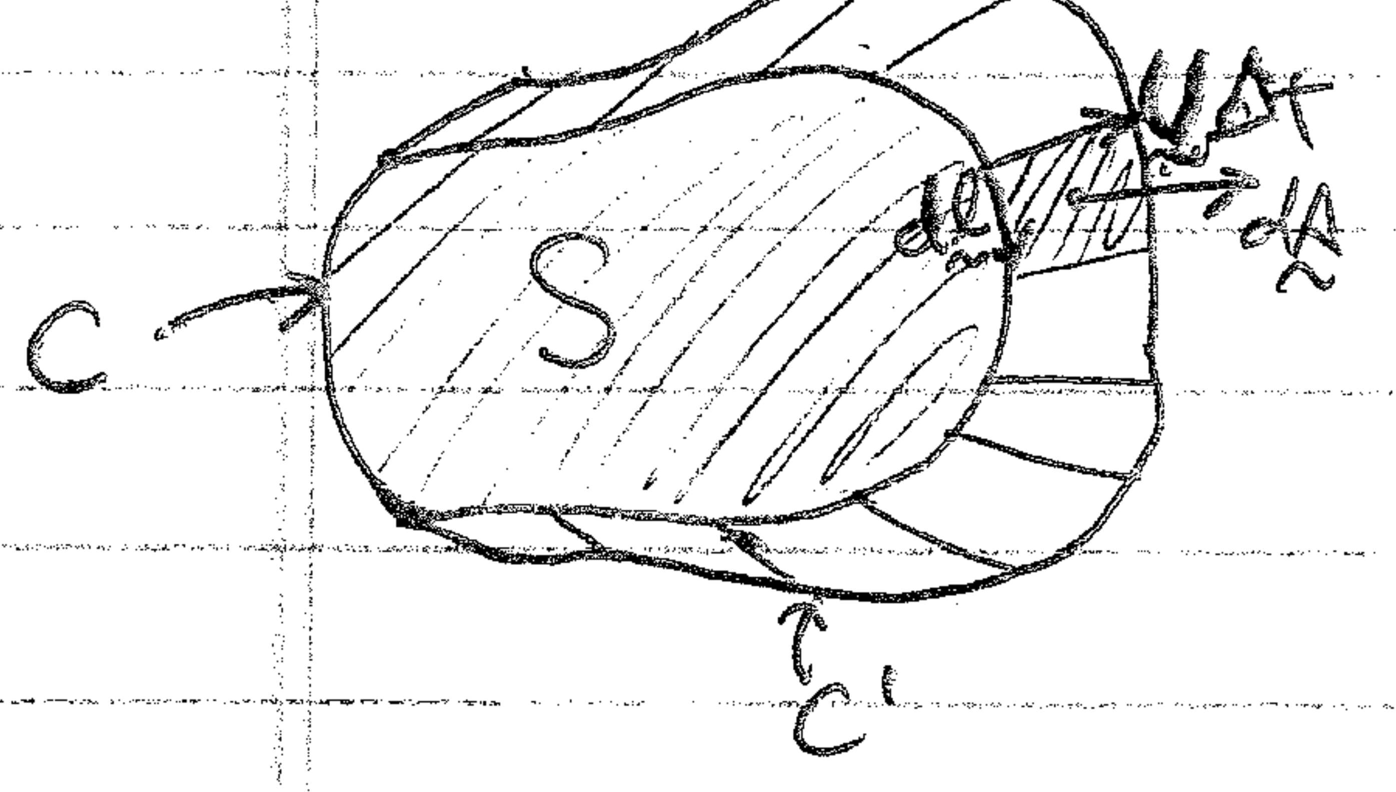


$$\int_{S_1} \underset{\sim}{B} \cdot d\underset{\sim}{A} = \int_{S_2} \underset{\sim}{B} \cdot d\underset{\sim}{A}$$

when $d\underset{\sim}{A}$ is taken in the same "sense".

3. a. Consider a surface S spanning curve C at time t.

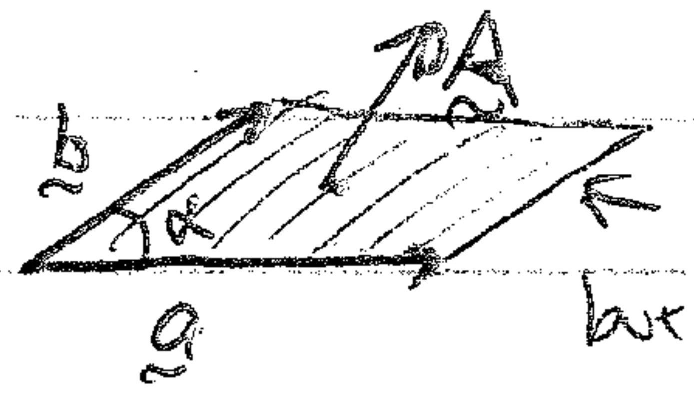
b. This moves to surface S' spanning curve C' at time $t' = t + \Delta t$



c. NOTE $S' = S' +$ ribbon connecting C to C'.

d. The area $d\underset{\sim}{A} = \underset{\sim}{U}\Delta t \times d\underset{\sim}{l}$

NOTE:



← area $A = ab\sin\alpha$

but $|\underset{\sim}{a}\times\underset{\sim}{b}| = ab\sin\alpha$,

so $\underset{\sim}{A} = \underset{\sim}{a}\times\underset{\sim}{b}$

4. Now, calculate Φ_B at times t & $t' = t + \Delta t$

a. $\Phi_B(t) = \int_S \underset{\sim}{B}(t) \cdot d\underset{\sim}{A}$

b. $\Phi_B(t+\Delta t) = \int_{S'} \underset{\sim}{B}(t+\Delta t)\cdot d\underset{\sim}{A} = \int_S \underset{\sim}{B}(t+\Delta t)\cdot d\underset{\sim}{A} + \int_{\text{ribbon}} \underset{\sim}{B}(t+\Delta t)\cdot d\underset{\sim}{A}$

i) Expand $\underset{\sim}{B}(t+\Delta t) = \underset{\sim}{B}(t) + \Delta t \frac{\partial B}{\partial t} + \ldots$ NRL p.4 (1)

ii) For the ribbon $\int_{\text{ribbon}} \underset{\sim}{B}\cdot d\underset{\sim}{A} = \oint_C \underset{\sim}{B}\cdot(\underset{\sim}{U}\Delta t \times d\underset{\sim}{l}) = \Delta t \oint_C (\underset{\sim}{B}\times\underset{\sim}{U})\cdot d\underset{\sim}{l}$

$$= \Delta t \int_S [\nabla\times(\underset{\sim}{B}\times\underset{\sim}{U})]\cdot d\underset{\sim}{A}$$

c. Dropping higher order terms,

$$\Phi_B(t+\Delta t) = \int_S \underset{\sim}{B}(t)\cdot d\underset{\sim}{A} + \Delta t \int_S \frac{\partial B}{\partial t}\cdot d\underset{\sim}{A} - \Delta t \int_S [\nabla\times(\underset{\sim}{U}\times\underset{\sim}{B})]\cdot d\underset{\sim}{A}$$

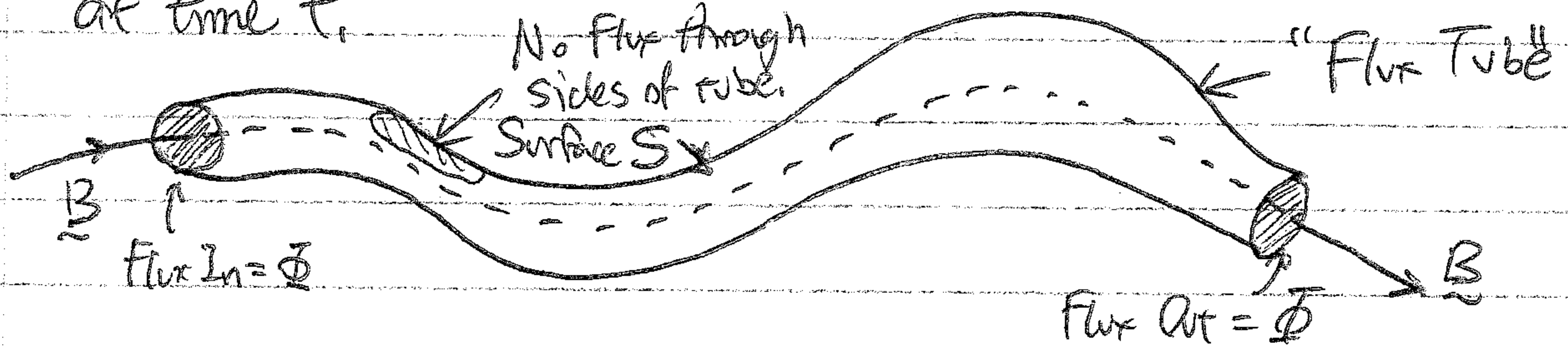
III. A. (Continued)

5. $\frac{d\Phi_B}{dt} = \lim_{\Delta t \to 0} \frac{\Phi_B(t+\Delta t) - \Phi_B(t)}{\Delta t} = \int_S \left[\frac{\partial B}{\partial t} - \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})\right] \cdot d\underset{\sim}{A} = 0$

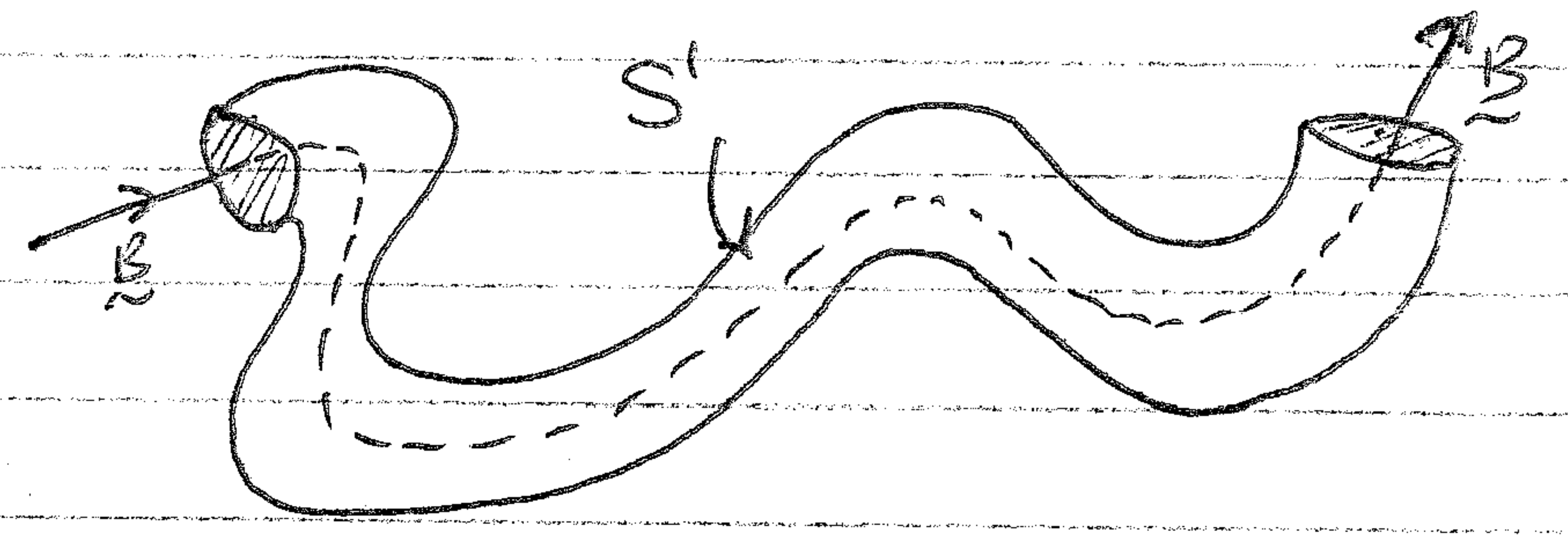
Q.E.D.

B. Theorem II: The Magnetic field lines are frozen to the plasma/fluid flow.

1. Consider a tube of plasma surrounding a magnetic field line at time t.



2. Let tube move with fluid at velocity $\underset{\sim}{U}(\underset{\sim}{x}, t)$.
3. By Frozen-In Flux Theorem, at time t', the flux through S' must still be zero. This is true for all parts of the tube surface S'.



4. The flux through the ends must still be Φ.

$\Rightarrow$ 5. The field line still goes down the flux tube.

6. If we shrink the tube to infinitesimal size, flux tube encompasses a single field line.

Thus, $\boxed{\text{The magnetic field lines are frozen to the fluid flow}}$

III. (Continued)

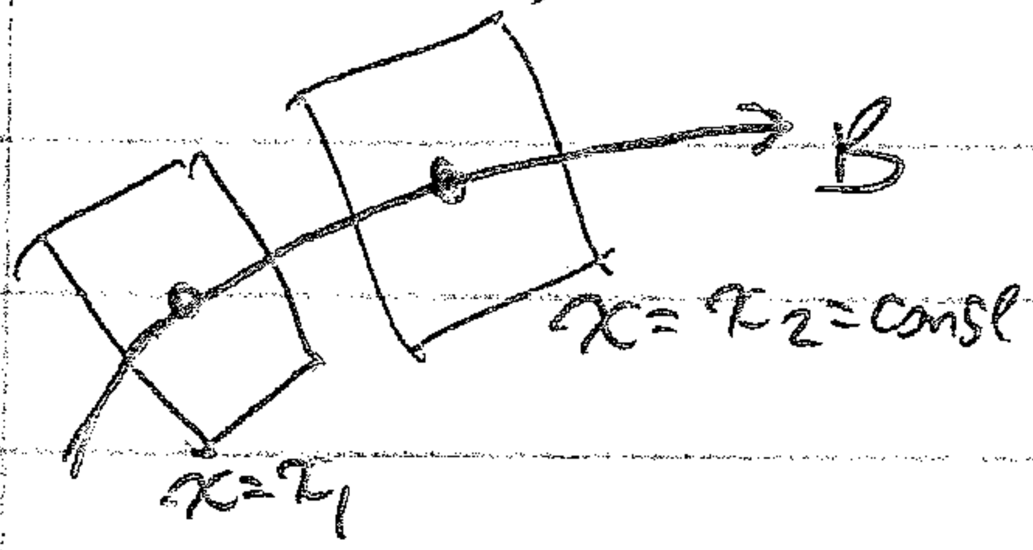
C. Clebsch Coordinates

Often representing the magnetic field in terms of scalar "potentials" is useful.

1. Example: Vacuum Scalar Potential for $\underset{\sim}{B}$.

a. In vacuum, $j=0$, so $\nabla \times \underset{\sim}{B} = 0 \Rightarrow \boxed{\underset{\sim}{B} = -\nabla \chi}$

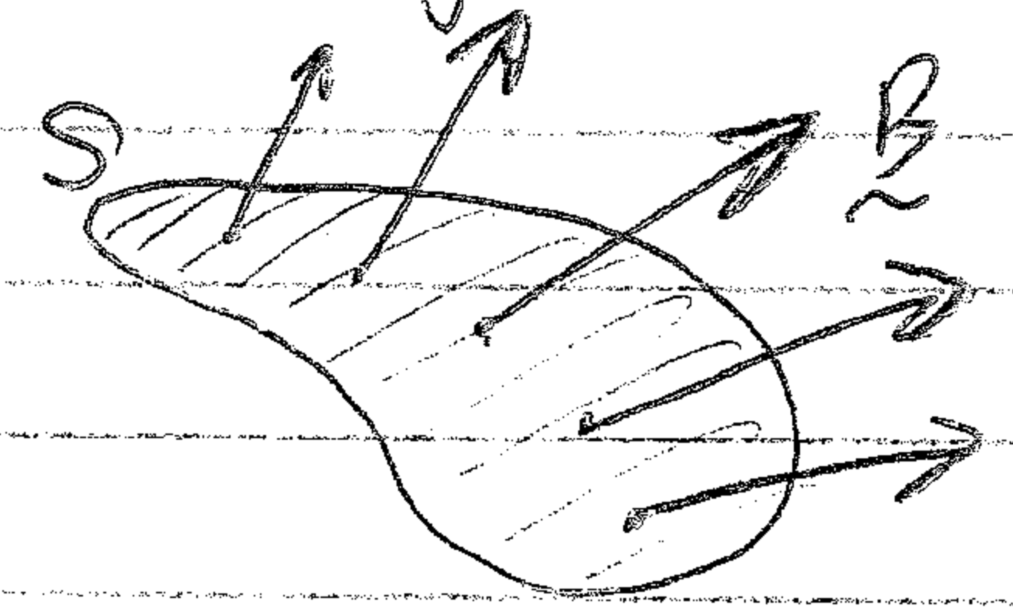
b. The constraint $\nabla \cdot \underset{\sim}{B} = 0$ then gives $\boxed{\nabla^2 \chi = 0}$ Laplace's Equation



a. $\underset{\sim}{B}$ is perpendicular to surfaces of constant χ.

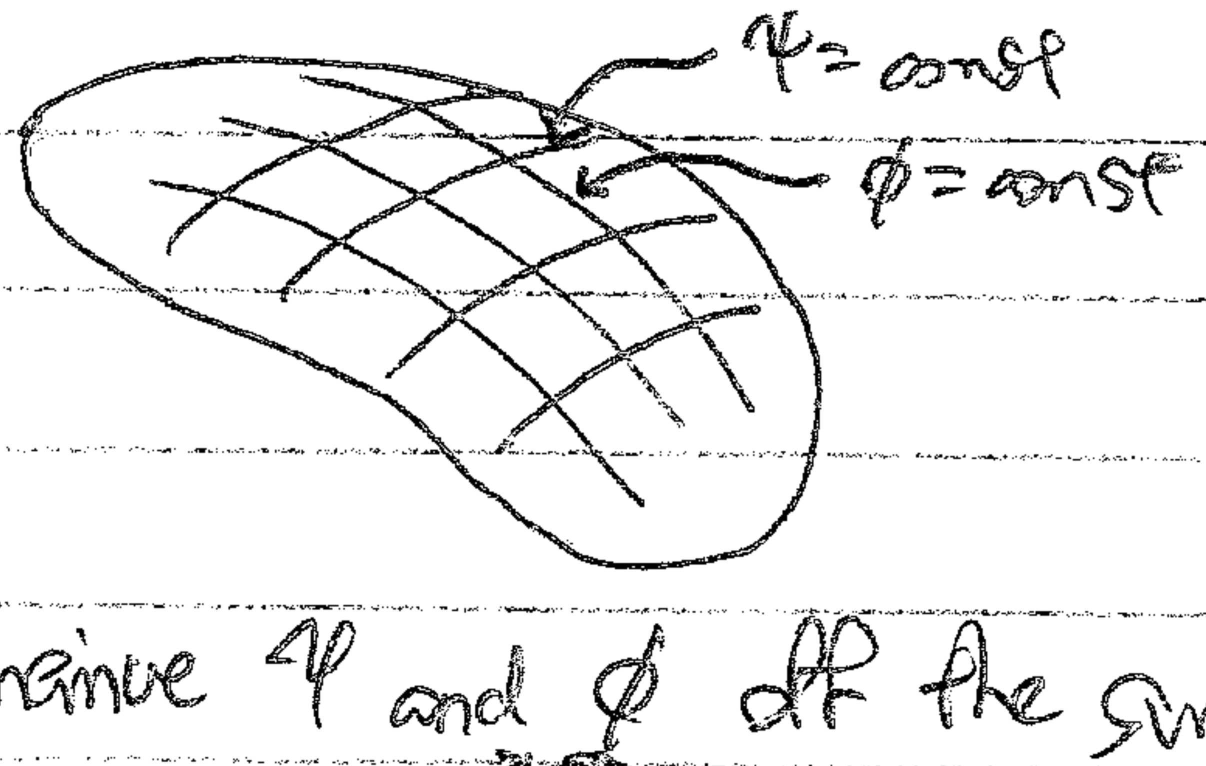
2. Clebsch Coordinates: (also called Flux Coordinates)

a. Consider a reference surface that is everywhere perpendicular to the magnetic field

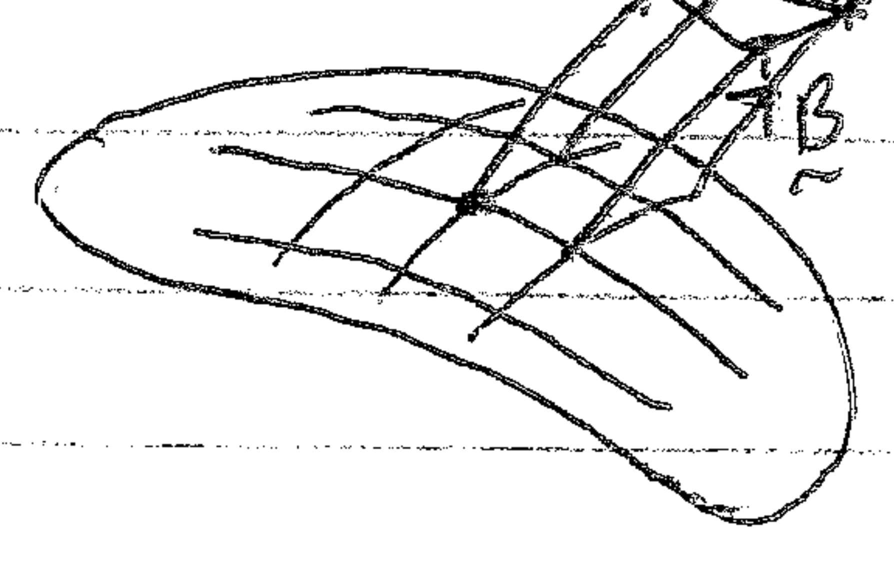


$\underset{\sim}{B}$ perpendicular to surface S

b. Construct a coordinate system on S. For example, label each point on S with ψ and ϕ.



c. Continue ψ and ϕ off the surface S by making them constant along $\underset{\sim}{B}$ field lines



Thus $\boxed{\underset{\sim}{B} \cdot \nabla \psi = 0}$

and $\boxed{\underset{\sim}{B} \cdot \nabla \phi = 0}$

III.C.2 (Continued)

d. Since $\underset{\sim}{B}$ is perpendicular to $\nabla\psi$ and $\nabla\phi$, we can write

$$\underset{\sim}{B} = a(\psi,\phi,\ell)\,\nabla\psi\times\nabla\phi$$

coordinate along field line

e. Now apply $\nabla\cdot\underset{\sim}{B}=0$

1. $\nabla\cdot\left[a\,\nabla\psi\times\nabla\phi\right] \underset{\text{NRL p.4 (7)}}{=} a\,\cancel{\nabla\cdot(\nabla\psi\times\nabla\phi)}^{\,0} + (\nabla\psi\times\nabla\phi)\cdot\nabla a$

2. NOTE: $\nabla\cdot(\nabla\psi\times\nabla\phi) \underset{\text{NRL p.4 (9)}}{=} \nabla\phi\cdot\cancel{(\nabla\times\nabla\psi)}^{\,0} - \nabla\psi\cdot\cancel{(\nabla\times\nabla\phi)}^{\,0}$ — 0 by NRL p.4 (15)

3. $\nabla a = \frac{\partial a}{\partial\psi}\nabla\psi + \frac{\partial a}{\partial\phi}\nabla\phi + \frac{\partial a}{\partial\ell}\nabla\ell$

4. Thus, we get $= \frac{\partial a}{\partial\ell}\underbrace{(\nabla\psi\times\nabla\phi)\cdot\nabla\ell}_{\neq 0 \text{ in general, so } \frac{\partial a}{\partial\ell}=0.} = 0$

5. Thus, $a = a(\psi,\phi)$ [a function only of ψ and ϕ.]

f. To finish up derivation of Clebsch coordinates,

1. Let $\phi=\beta$.
2. Define $\alpha = \int^{\psi} a(\psi',\beta)\,d\psi'$

Thus $\nabla\alpha = \int^{\psi}\left(\frac{\partial a}{\partial\psi'}\nabla\psi' + \frac{\partial a}{\partial\beta}\nabla\beta\right)d\psi' = a\nabla\psi + \nabla\beta\int_0^{\psi}\frac{\partial a}{\partial\beta}d\psi'$

3. Thus $\underset{\sim}{B} = \nabla\alpha\times\nabla\beta = \left(a\nabla\psi + \cancel{\nabla\beta\int_0^{\psi}\frac{\partial a}{\partial\beta}d\psi'}^{\,0}\right)\times\nabla\beta = a\,\nabla\psi\times\nabla\phi$ ✓

g. General Clebsch Representation: $\boxed{\underset{\sim}{B} = \nabla\alpha\times\nabla\beta}$

1. Field lines are solutions of $\alpha(\underset{\sim}{x}) =$ constant, $\beta(\underset{\sim}{x}) =$ constant.
2. NOTE: $\nabla\times\alpha\nabla\beta \underset{\text{NRL p.4 (5)}}{=} \nabla\alpha\times\nabla\beta + \alpha\,\cancel{\nabla\times\nabla\beta} = \nabla\alpha\times\nabla\beta = \underset{\sim}{B}$

Lecture #16 (Continued)

III. (Continued)

C. 2 g. (Continued)

3. Thus Vector Potential $\boxed{\underset{\sim}{A} = \alpha \nabla \beta + \nabla \lambda}$

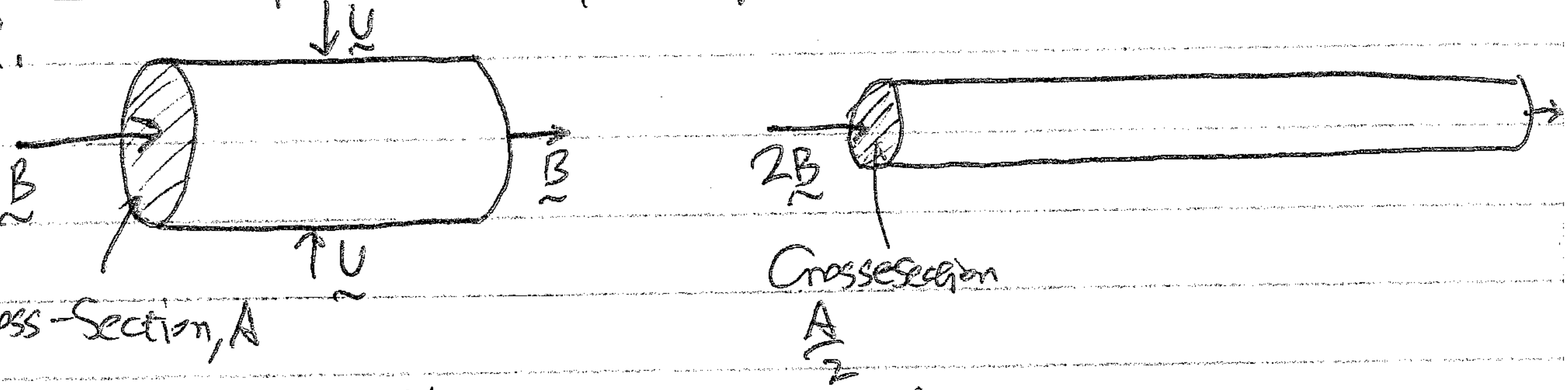
since $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$

λ = any Scalar.

IV. Applications of the Frozen-in Flux Theorem

A. Field Amplification

1.



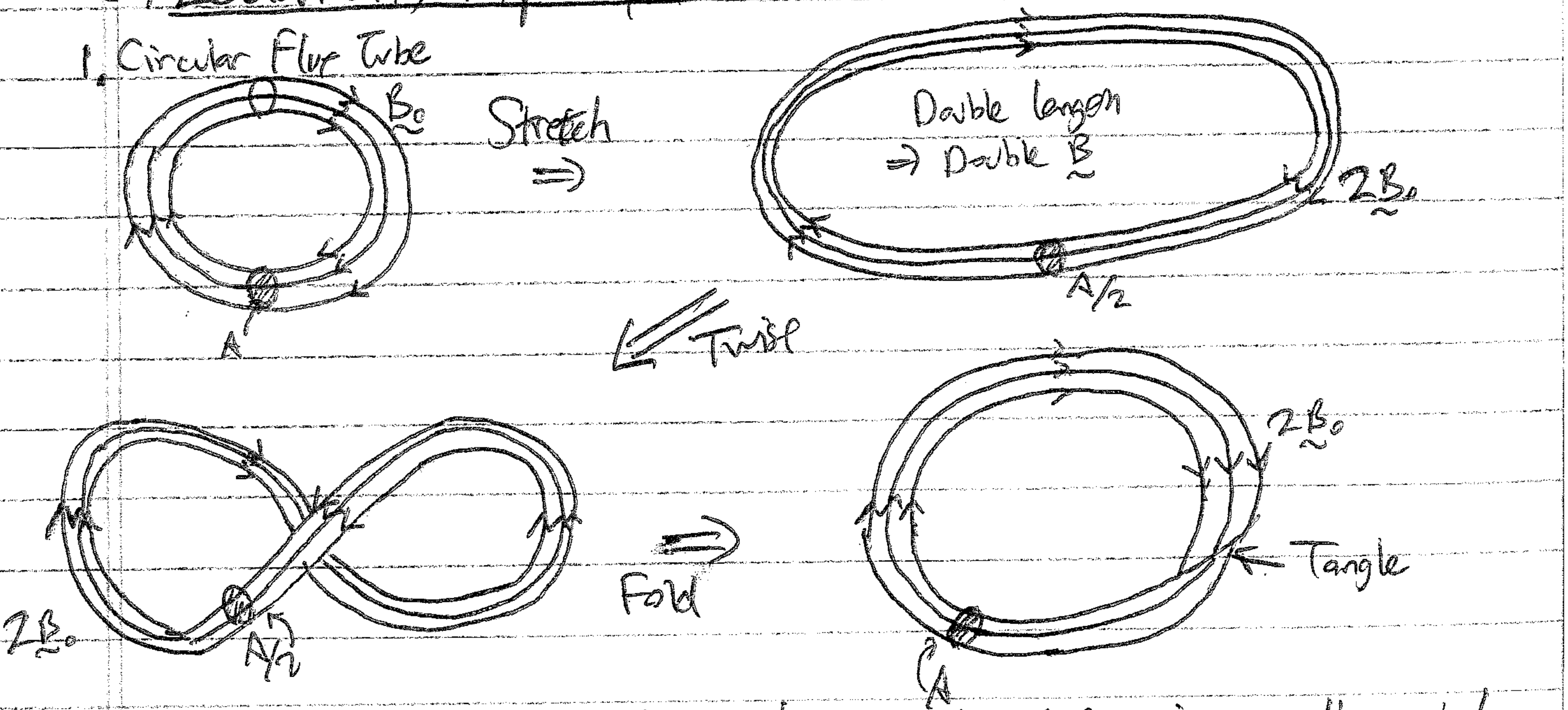
a. Incompressible Motion ⇒ Volume is the same.

b. Flux must be the same: $\Phi_1 = B_1 A \qquad \Phi_2 = B_2\left(\frac{A}{2}\right)$

$$\Rightarrow B_2 = \frac{B_1 A}{(A/2)} = 2B_1 \quad \text{Amplified Field.}$$

B. Zeldovich's Rope Dynamo

1. Circular Flux Tube



2. Nearly the same as start, but $\underset{\sim}{B}$ has doubled and there is a small tangle!