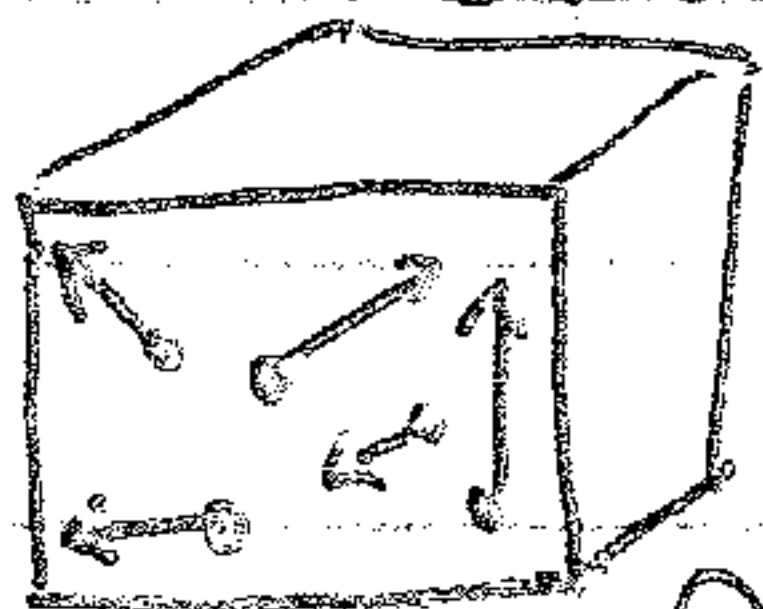


I. Review of Kinetic Description of Plasmas

A. Picture of a Kinetic Plasma.

1. At any point in space, a kinetic plasma contains particles moving with a distribution of velocities



2. The distribution function  $f_s(\underline{x}, \underline{v}, t)$  describes a statistical measure of the plasma in a volume of ~~size~~  $x_0^3$  such that
  - $x_0 \gg n^{-1/3}$  larger than particle spacing  $\Rightarrow$  many particles
  - $x_0 \ll \lambda_D$  smaller than Debye length

3.  $f_s(\underline{x}, \underline{v}, t)$  is a number density in 6-D phase space  $(\underline{x}, \underline{v})$

B. Evolution of distribution function

1. Along particle orbit,  $\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \frac{d\underline{x}}{dt} \cdot \frac{\partial f_s}{\partial \underline{x}} + \frac{d\underline{v}}{dt} \cdot \frac{\partial f_s}{\partial \underline{v}}$

2. The statistical ensemble average yields

Plasma Kinetic Equation

$$\frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = \left( \frac{df_s}{dt} \right)_{\text{coll}}$$

Statistically smooth description of particles  $f_s$  and fields  $\underline{E}$  &  $\underline{B}$  over volume  $x_0^3$

Effect of discrete particles on smaller scales  $\Rightarrow$  collisions.

3. Neglect the effect of collisions yields Vlasov Equation

$$\frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = 0$$

## Lecture #3 (Continued)

Pages ②

~~1. (Continued)~~

2. (Continued)

### 1.1. Moments of the Distribution Function

1. Summing over velocity space gives useful macroscopic quantities.

a. Density  $n_s(\underline{x}, t) = \int d^3\underline{v} f_s(\underline{x}, \underline{v}, t)$

b. Fluid Velocity  $\underline{U}_s(\underline{x}, t) = \frac{1}{n_s(\underline{x}, t)} \int d^3\underline{v} \underline{v} f_s(\underline{x}, \underline{v}, t)$

c. Kinetic Energy Density:  $\mathcal{E}_s(\underline{x}, t) = \int d^3\underline{v} \frac{1}{2} m |\underline{v}|^2 f_s(\underline{x}, \underline{v}, t)$

d. Pressure Tensor:  $\underline{P}_s(\underline{x}, t) = \int d^3\underline{v} m_s (\underline{v} - \underline{U}_s)(\underline{v} - \underline{U}_s) f_s(\underline{x}, \underline{v}, t)$

2. These are the velocity moments of the distribution function.

3. Taking moments of the kinetic distribution function gives "fluid" quantities, ~~or~~ observable quantities.

\*E.g. The temperature of a kinetic species  $s$  can be defined.

$$\mathcal{E}_s \equiv \frac{3}{2} n_s k T_s \quad \Rightarrow \quad T_s \equiv \frac{2 \mathcal{E}_s}{3 n_s k}$$

NOTE: From thermodynamics,  $p_s = n_s k T_s$ , so  $\mathcal{E}_s = \frac{3}{2} p_s$ , as expected.

## II. From a Kinetic to a Fluid Description

### A Fluid Description:

1. Often, we don't care about detailed distribution of velocities.
2. Thus, all we want are velocity moments,  $n_s$ ,  $\underline{U}_s$ ,  $\mathcal{E}_s$ , etc.
3. The velocity moments just define the dependent variables of a fluid description, which depend only on  $\underline{x}$  and  $t$ .

II. (Continued)

B. Comparison of Kinetic vs. Fluid Descriptions

Kinetic

$$1. \frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = 0$$

Charge Density:  $\rho = \sum_s q_s \int d^3v f_s(\underline{x}, \underline{v}, t)$

Current Density:  $\underline{j} = \sum_s q_s \int d^3v \underline{v} f_s(\underline{x}, \underline{v}, t)$

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0} \quad \nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad \nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

2. Everything depends on 6-D distribution function  $f_s(\underline{x}, \underline{v}, t)$

Fluid:

1. Moments:  $n_s(\underline{x}, t)$

$U_s(\underline{x}, t)$

$\tilde{E}_s(\underline{x}, t)$

etc.

2. Evolution equations (derived <sup>to be</sup> next)


3. A number of fluid variables (moments), all of which only depend on  $\underline{x}$  and  $t$ .

C. When is a fluid description appropriate?

ANSWER: When the details of the velocity distribution do not significantly affect the evolution.

1. Ex: Fluid description is not valid when particles are certain velocity become resonant with fluctuating EM fields.

$\Rightarrow$  Landau Damping 

$\Rightarrow$  Cyclotron Damping 

2. Ex: Fluid description is appropriate for strongly collisional plasmas.

Collisions drive towards Local Thermodynamic Equilibrium

$\Rightarrow$  Velocity distribution becomes Maxwellian.

## II. (Continued)

D. Maxwellian (and Bi-Maxwellian) Plasmas:

1. Fluid equations are a ~~an~~<sup>good</sup> description of a plasma when the velocity distributions remain Maxwellian

2. Maxwellian Distribution

$$f_{s,m}(\mathbf{x}, \mathbf{v}, t) = \frac{n_s(\mathbf{x}, t)}{\pi^{3/2} v_{Ts}^3} e^{-\frac{m_s |\mathbf{v} - \mathbf{U}(\mathbf{x}, t)|^2}{2kT_s(\mathbf{x}, t)}}$$

a. Fluid is completely described by three fluid variables:

Density  $n_s(\mathbf{x}, t)$

Fluid velocity  $\mathbf{U}_s(\mathbf{x}, t)$

Temperature  $T_s(\mathbf{x}, t)$

b. Here, as usual, the thermal velocity  $v_{Ts} = \sqrt{\frac{2kT_s}{m_s}}$

## 3. Zeroth Moment: Density

a.  $\int d^3v f_{s,m}(\mathbf{x}, \mathbf{v}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{n_s}{\pi^{3/2} v_{Ts}^3} e^{-\frac{|\mathbf{v} - \mathbf{U}|^2}{v_{Ts}^2}}$

b. NOTE:  $|\mathbf{v} - \mathbf{U}|^2 = (v_x - U_x)^2 + (v_y - U_y)^2 + (v_z - U_z)^2$

c. We can split this into three integrals which are identical

$$\frac{1}{\pi^{3/2} v_{Ts}^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(v_x - U_x)^2}{v_{Ts}^2}} = \frac{1}{\pi^{3/2} v_{Ts}^3} \int_{-\infty}^{\infty} \underbrace{v_{Ts} dy}_{= v_{Ts} \sqrt{\pi}} e^{-y^2} = \frac{\sqrt{\pi} v_{Ts}}{\pi^{3/2} v_{Ts}^3} = 1$$

$y = \frac{v_x - U_x}{v_{Ts}} \quad dy = \frac{dv_x}{v_{Ts}}$

d. Thus  $\int d^3v f_{s,m}(\mathbf{x}, \mathbf{v}, t) = n_s(\mathbf{x}, t)$

4. First Moment: Fluid Velocity

$$\int d^3v \underline{v} f_{sm}(\underline{x}, \underline{v}, t) = n_s(\underline{x}, t) \underline{U}_s(\underline{x}, t)$$

5. Second Moment: Energy

$$\int d^3v \frac{1}{2} m_s v^2 f_{sm}(\underline{x}, \underline{v}, t) = \underbrace{\frac{3}{2} n_s k T_s}_{\text{Thermal Energy}} + \underbrace{\frac{1}{2} m_s n_s |\underline{U}_s|^2}_{\text{Kinetic Energy of Fluid Flow}}$$

6. The presence of a Mean Magnetic Field  $B_0$  often leads to an anisotropic distribution function

Bi-Maxwellian Distribution: 
$$f_{s, \text{Bin}}(\underline{x}, \underline{v}, t) = \frac{n_s(\underline{x}, t)}{\pi^{3/2} v_{Ts}^2 v_{T||s}} e^{-\frac{mv_{\perp}^2}{2kT_{\perp s}} - \frac{mv_{||}^2}{2kT_{||s}}}$$

where

$$v_{Ts} = \sqrt{\frac{2kT_{\perp s}}{m_s}} \quad \text{and} \quad v_{T||s} = \sqrt{\frac{2kT_{||s}}{m_s}}$$

a. This can lead to the

Chew-Goldberger Law, or Double Adiabatic Equation of State

III. Moment Equations

Moments of the Plasma Kinetic Equation determine evolution of velocity moments.

A. Zeroth Moment: No. eqn  $\int_{\underline{v}} = \int d^3v$

$$\int_{\underline{v}} \frac{\partial f_s}{\partial t} + \int_{\underline{v}} \underline{v} \cdot \nabla f_s + \int_{\underline{v}} \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = \int_{\underline{v}} \left( \frac{\partial f_s}{\partial t} \right)_{\text{coll}}$$

i. ① =  $\frac{\partial}{\partial t} \int_{\underline{v}} f_s = \frac{\partial}{\partial t} n_s$

Lecture 13 (Continued)

Homework 6

III. A. (Continued)

$$2. \textcircled{2} = \int_V \underline{v} \cdot \nabla f_s = \nabla \cdot \int_V \underline{v} f_s = \nabla \cdot (n_s \underline{U}_s)$$

a. NOTE:  $x$  and  $v$  are independent variables, so  $\nabla \cdot (\underline{v} f_s) = \underline{v} \cdot \nabla f_s + f_s \nabla \cdot \underline{v}$

$$3. \textcircled{3} = \frac{q_s}{m_s} \int_V (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = \frac{q_s}{m_s} \int_V \nabla \cdot [(\underline{E} + \underline{v} \times \underline{B}) f_s]$$

a. NOTE:  $\underline{E}(x,t)$  &  $\underline{B}(x,t)$  are independent of  $\underline{v}$ :  $\frac{\partial}{\partial \underline{v}} \cdot (\underline{E} f_s) = \underline{E} \cdot \frac{\partial f_s}{\partial \underline{v}} + f_s \nabla \cdot \underline{E}$

b.  $\frac{\partial}{\partial \underline{v}} \cdot (\underline{v} \times \underline{B}) f_s = \underline{v} \times \underline{B} \cdot \frac{\partial f_s}{\partial \underline{v}} + f_s \frac{\partial}{\partial \underline{v}} \cdot (\underline{v} \times \underline{B})$

Since each component is like  $\frac{\partial}{\partial v_x} (v_y B_z - v_z B_y) = \frac{\partial}{\partial v_x} (v_y B_z - v_z B_y) = 0$

c. Using NRL p.5 (28) Gauss's Thm:  $\int_V d^3x \nabla \cdot \underline{A} = \int_S d\underline{S} \cdot \underline{A}$

$$\Rightarrow \textcircled{3} = \frac{q_s}{m_s} \int d\underline{S}_V \cdot [(\underline{E} + \underline{v} \times \underline{B}) f_s] = 0$$

d. At  $v \rightarrow \pm\infty$ ,  $f_s \rightarrow 0$ , so surface integral vanishes at infinity.

$$4. \textcircled{4} = \left( \frac{\partial}{\partial t} \int_V f_s \right)_{\text{coll}} = \left( \frac{\partial n_s}{\partial t} \right)_{\text{coll}} = 0$$

Collisions don't create or destroy particles (number is conserved)

5. Thus, we are left with

$$\boxed{\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{U}_s) = 0}$$

Continuity Equation

a. Expresses conservation of particles

b. Integrate over a small volume  $V$ :  $\int_V d^3x \frac{\partial n_s}{\partial t} = - \int_V d^3x \nabla \cdot (n_s \underline{U}_s)$

i) Once again, use Gauss's Thm on RHS.

ii)  $N = \text{number of particles in } V \Rightarrow \frac{\partial N}{\partial t} = - \int_S d\underline{S} \cdot (n_s \underline{U}_s)$   
Particles moving through surface.

Lecture 13 (Continued)

Pages 7

III. B. First Moment:  $\int \underline{v} \underline{v}$  on Kinetic Equation

$$\int \underline{v} \underline{v} \frac{\partial f_s}{\partial t} + \int \underline{v} \underline{v} \cdot \nabla f_s + \int \underline{v} \underline{v} \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = \int \underline{v} \underline{v} \left( \frac{df_s}{dt} \right)_{coll} \quad (4)$$

1. (1) =  $\frac{\partial}{\partial t} \int \underline{v} f_s = \frac{\partial}{\partial t} (n_s \underline{U}_s)$

2. This term involves Tensor mathematics.

(2) =  $\int \underline{v} \underline{v} \cdot \nabla f_s = \nabla \cdot \left( \int \underline{v} \underline{v} f_s \right)$  since  $\underline{x}$  &  $\underline{v}$  are independent.

a. Use  $\underline{v} = \underline{U}_s + \underline{v} - \underline{U}_s$  to get  $\underline{v} \underline{v} = \underbrace{\underline{U}_s \underline{U}_s}_{(a)} + \underbrace{\underline{U}_s (\underline{v} - \underline{U}_s)}_{(b)} + \underbrace{(\underline{v} - \underline{U}_s) \underline{U}_s}_{(c)} + \underbrace{(\underline{v} - \underline{U}_s) (\underline{v} - \underline{U}_s)}_{(d)}$

b. (a) =  $\int \underline{U}_s \underline{U}_s f_s = n_s \underline{U}_s \underline{U}_s$

c. (b) =  $\int \underline{U}_s (\underline{v} - \underline{U}_s) f_s = \underline{U}_s \int \underline{v} f_s - \underline{U}_s \underline{U}_s n_s = \underline{U}_s \underline{U}_s n_s - \underline{U}_s \underline{U}_s n_s = 0$

d. (c) same as (b) = 0

e. (d) Using  $\underline{P}_s = \int \underline{v} \underline{v} m_s (\underline{v} - \underline{U}_s) (\underline{v} - \underline{U}_s) f_s$ , (d) =  $\frac{\underline{P}_s}{m}$

Final (2) =  $\nabla \cdot (n_s \underline{U}_s \underline{U}_s) + \frac{1}{m_s} \nabla \cdot \underline{P}_s$

3. (3) =  $\frac{q_s}{m_s} \int \underline{v} \underline{v} \frac{\partial}{\partial \underline{v}} \cdot [(\underline{E} + \underline{v} \times \underline{B}) f_s]$  using same trick as Zeroth Moment.

a. Nine integrals for each pairing of  $v_x, v_y, v_z$  &  $\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}$

i) Six have  $v_i \frac{\partial}{\partial v_j}$  where  $i \neq j$ , thus are similar to

$$\int dv_x \int dv_y \int dv_z v_i \frac{\partial}{\partial v_j} [(\underline{E} + \underline{v} \times \underline{B}) f_s]$$

(ii)  $v_z$  integral gives  $(\underline{E} + \underline{v} \times \underline{B}) f_s \Big|_{-\infty}^{\infty} = 0$  as  $v \rightarrow \infty$  since  $f_s \rightarrow 0$ .

(iii) Three have  $v_i \frac{\partial}{\partial v_i} \Rightarrow \int dv_x \int dv_y \int dv_z v_i \frac{\partial}{\partial v_i} [(\underline{E} + \underline{v} \times \underline{B}) f_s]$

Lesson #13 (Continued)  
 III. B3, (Continued)

Answer 8

$$\int dv_z v_z \frac{d}{dv_z} \left[ (\underline{E} + \underline{v} \times \underline{B}) f_s \right] = v_z (\underline{E} + \underline{v} \times \underline{B}) f_s \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (\underline{E} + \underline{v} \times \underline{B}) f_s dv_z$$

$$u = v_z \quad dv_z = \frac{d}{dv_z} (\underline{E} + \underline{v} \times \underline{B}) f_s dv_z$$

$$du = dv_z \quad v = (\underline{E} + \underline{v} \times \underline{B}) f_s$$

b. We get ③ =  $-\frac{n_s q_s}{m_s} (\underline{E} + \underline{U}_s \times \underline{B})$

$$4. \textcircled{4} = \frac{1}{m_s} \int \underline{v} m_s \underline{v} \left( \frac{df_s}{dt} \right)_{\text{coll}} = \frac{1}{m_s} \left( \frac{\partial}{\partial t} \int \underline{v} m_s \underline{v} f_s \right)_{\text{coll}} = \frac{1}{m_s} \left( \frac{\partial n_s m_s \underline{U}_s}{\partial t} \right)_{\text{coll}}$$

- a. This represents a DRAG FORCE due to collisions between species
- b. Same species collisions conserve momentum, so produce no drag.

5. Thus,

$$\frac{\partial}{\partial t} (n_s \underline{U}_s) + \nabla \cdot (n_s \underline{U}_s \underline{U}_s) = -\frac{1}{m_s} \nabla \cdot \underline{P}_s + \frac{n_s q_s}{m_s} (\underline{E} + \underline{U}_s \times \underline{B}) + \text{Collisional } (\underline{F}_{ps})$$

a. NOTE: LHS can be written, using tensor identity  $\nabla \cdot (\underline{A} \underline{B}) = (\nabla \cdot \underline{A}) \underline{B} + (\underline{A} \cdot \nabla) \underline{B}$

$$n_s \frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \frac{\partial n_s}{\partial t} + \underline{U}_s \nabla \cdot (n_s \underline{U}_s) + n_s \underline{U}_s \cdot \nabla \underline{U}_s$$

$$= \underline{U}_s \left[ \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{U}_s) \right]$$

Continuity eq.

6. Multiplying by  $m_s$ , we get

Momentum Equation

$$n_s m_s \frac{\partial \underline{U}_s}{\partial t} + n_s m_s \underline{U}_s \cdot \nabla \underline{U}_s = -\nabla \cdot \underline{P}_s + n_s q_s (\underline{E} + \underline{U}_s \times \underline{B}) + \underline{F}_{ps}$$

c. Second Moment Taking  $\int \underline{v} \frac{1}{2} m_s v^2$  of kinetic equation gives,

$$\frac{\partial \mathcal{E}_s}{\partial t} + \nabla \cdot \underline{Q}_s - \underline{E} \cdot \underline{j}_s = \left( \frac{\partial \mathcal{E}_s}{\partial t} \right)_{\text{coll}}$$

Energy Equation



II. C. (Continued)

1. Here  $\underline{Q}_S$  is a heat flux

Heat Flux  $\underline{Q}_S = \int d^3v \frac{1}{2} m_s |v|^2 v f_s(x, v, t)$  (Third Velocity Moment)

2.  $E \cdot \underline{j}_s$  represents Joule heating of species S

3.  $\left(\frac{\partial E_s}{\partial t}\right)_{coll}$  is collisional heating due to collisions with other species.

a. Same-species conserve energy within species.

D. Closure Problem:

1. The evolution equation for the  $n^{th}$  moment involves the ~~(n)~~  $(n+1)^{th}$  moment.

a.  $\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{U}_s) = 0$   
 (Zeroth)                      (First)

b.  $n_s m_s \left[ \frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \cdot \nabla \underline{U}_s \right] = -\nabla \cdot \underline{P}_s + n_s q_s (\underline{E} + \underline{v} \times \underline{B}) + \underline{F}_{D_s}$   
 (First)                                      (Second)

2. For N moment equations, a closed system can only be specified if the  $(N+1)^{th}$  moment is related to the first N moments.

Ex: We may specify an equation of state relate pressure in terms of density.

Adiabatic Equation of State

~~$P_s = n_s k_B T_s$~~   $P_s n_s^{-\gamma} = \text{constant}$

$\Rightarrow \frac{d}{dt} \left( \frac{P}{n_s^\gamma} \right) = 0$