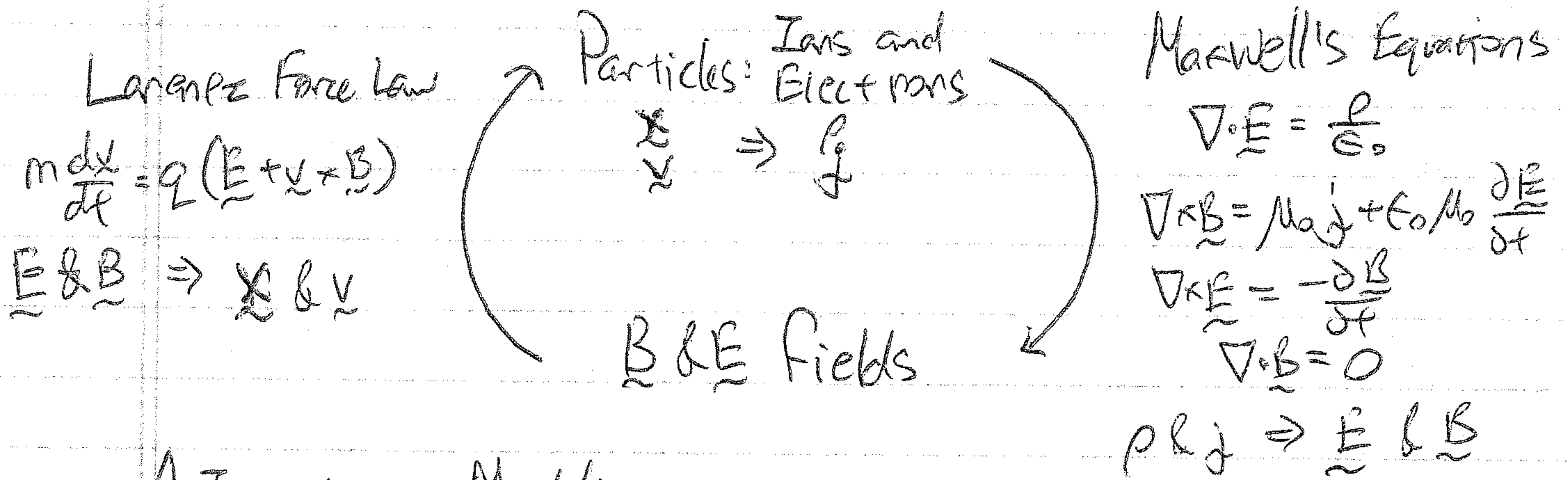


Lecture #12 Kinetic Description of a Plasma

Homework ①

I. Overview of Plasma Descriptions



A. Inconsistent Models:

1. Single Particle Motion: Lorentz force law for specified \mathbf{E} & \mathbf{B}

B. Consistent Models:

i. Kinetic Descriptions:

- a. Describe positions and velocities of all particles $\Rightarrow F(\mathbf{x}, \mathbf{v}, t)$

b. Klimontovich Equation:

- i. Exact description of every ion and electron in system
- ii. Together with Maxwell's Equations, completely deterministic
- iii. Too detailed for practical application

- c. Liouville Equation: Another exact description of system \Rightarrow too detailed.

d. Plasma Kinetic Equation (Boltzmann Equation)

- i. Statistical treatment of a plasma
- ii. Evolves six-dimensional distribution function $F(\mathbf{x}, \mathbf{v}, t)$ due to \mathbf{E} & \mathbf{B} fields and collisions.

- iii. Vlasov Equation: Limit of Plasma Kinetic Equation when collisions $\rightarrow 0$.

2. Fluid Descriptions:

- a. Velocity moments of the distribution function $F(\mathbf{x}, \mathbf{v}, t)$ [integrating over velocity space] lead to fluid variables [functions of (\mathbf{x}, t) only].

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I. 2. (Continued)

b. Evolution for each moment involves a higher moment

⇒ This leads to a closure problem

c. A physically motivated approximation is used to close equations.

d. Two Fluid Equations

i. Ions and electrons are each evolved separately

e. MHD Equations

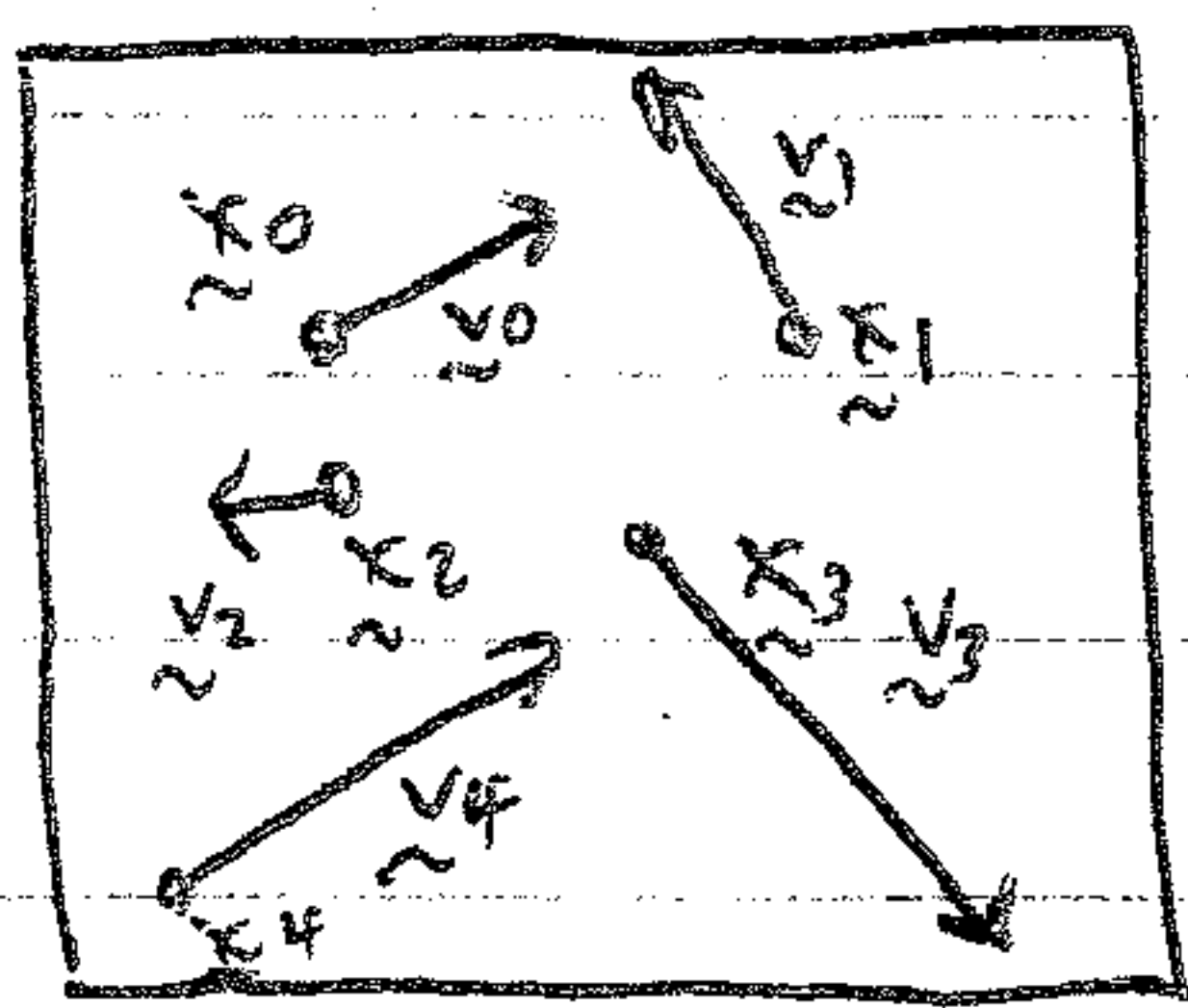
i. Ions and electrons move together, leading to a single fluid theory.

ii. Simple, consistent description of plasma behavior, but only valid when MHD approximation is satisfied.

II. Klimontovich Equation:

A. Phase Space:

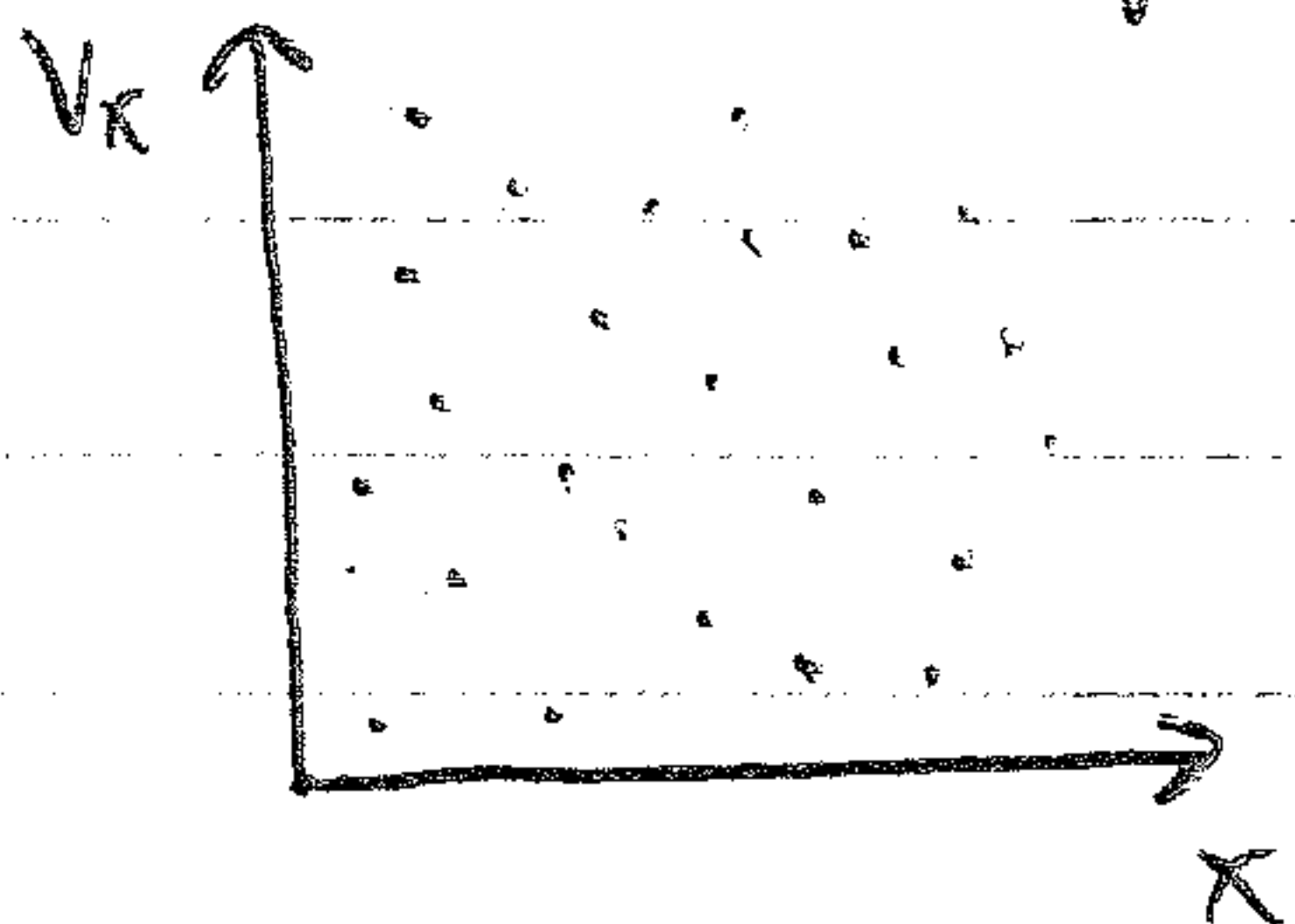
1. A plasma is completely described if we know the position \underline{x} and velocity \underline{v} of each particle at time t .



a. We can introduce the six-dimensional phase space $(\underline{x}, \underline{v})$

such that each of the N particles occupies a position $(\underline{x}_i, \underline{v}_i)$ for $i=1, \dots, N$.

b. In this 6-D phase space, each particle occupies a point.



c. In a real plasma, the number of particles is huge, $\sim 10^{10} - 10^{30}$ particles.

2. We can describe this N -particle plasma with the

a. Klimontovich Distribution

$$F = \sum_{i=1}^N \delta[\underline{x} - \underline{x}_i(t)] \delta[\underline{v} - \underline{v}_i(t)]$$

b. $\underline{x}_i(t)$ = position of i^{th} particle

$\underline{v}_i(t)$ = velocity of i^{th} particle

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 II. A2. (Continued)

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c. $\frac{d\tilde{r}_i}{dt} = \tilde{v}_i$ $\frac{d\tilde{v}_i}{dt} = \tilde{a}_i = \frac{q_s}{m_s} (\underline{E} + \tilde{v}_i \times \underline{B})$

d. \mathcal{F} describes the density of particles in phase space. If we integrate over all velocities and position, we get number of particles

$$N = \int d^3x \int d^3v \mathcal{F}(\underline{x}, \underline{v}, t)$$

- e. \mathcal{F} depends on detailed initial positions and velocities of each particle
1. \mathcal{F} is far too detailed for practical use
 2. This level of detail is not needed for most important macroscopic results.

f. How does \mathcal{F} change in time? $\frac{d\mathcal{F}}{dt} = ?$

1. $\mathcal{F}(\underline{x}, \underline{v}, t)$ depends on 3 position, 3 velocity, and 1 time variables

2. $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y} + \frac{dz}{dt} \frac{\partial}{\partial z} + \frac{dv_x}{dt} \frac{\partial}{\partial v_x} + \frac{dv_y}{dt} \frac{\partial}{\partial v_y} + \frac{dv_z}{dt} \frac{\partial}{\partial v_z}$

3. If we evaluate $\frac{d\mathcal{F}}{dt}$ along the particle orbits.

i. $\frac{dx}{dt}|_{\text{orbit}} = v_x$, $\frac{dy}{dt}|_{\text{orbit}} = v_y$, etc. $\Rightarrow \frac{d\underline{x}}{dt} = \underline{v}$

ii. Similarly $\frac{dv_x}{dt}|_{\text{orbit}} = a_x$, etc. $\Rightarrow \frac{d\underline{v}}{dt} = \underline{a}$

4. Thus

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla + \underline{a} \cdot \frac{\partial}{\partial \underline{v}}$$

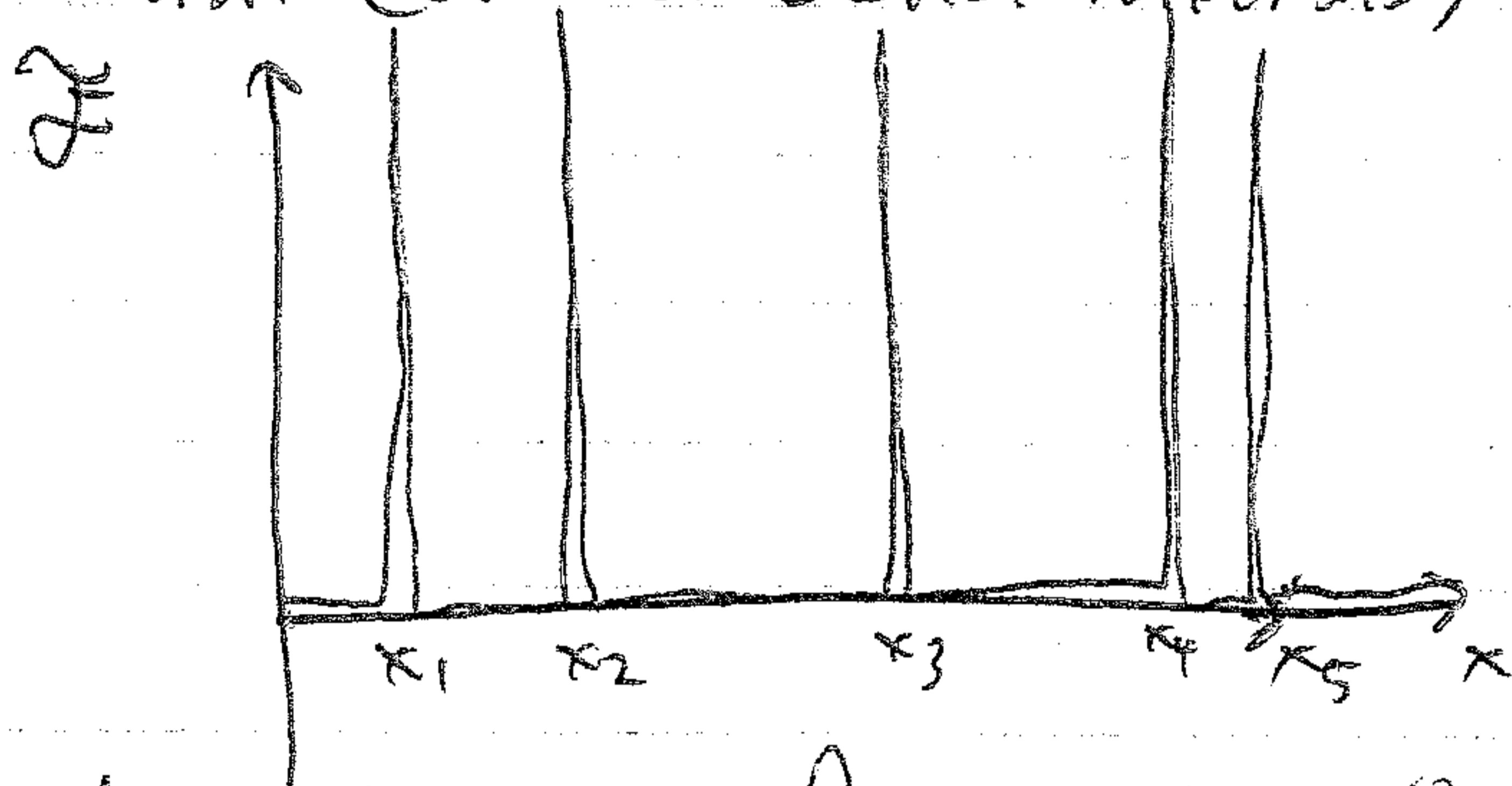
g. For a single species s , we get

$$\frac{d\mathcal{F}_s}{dt} = \left[\frac{\partial \mathcal{F}_s}{\partial t} + \underline{v} \cdot \nabla \mathcal{F}_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial \mathcal{F}_s}{\partial \underline{v}} \right] = 0$$

Klimontovich Equation

II. (Continued)

B. 1. The Klimontovich Distribution is a very spiky function (sum of delta functions) due to particle discreteness.



2. Along the orbit of particles, the distribution does not change $\frac{df}{dt} = 0$. Either you are on a particle, and density is "infinite", or not on a particle, density is zero.

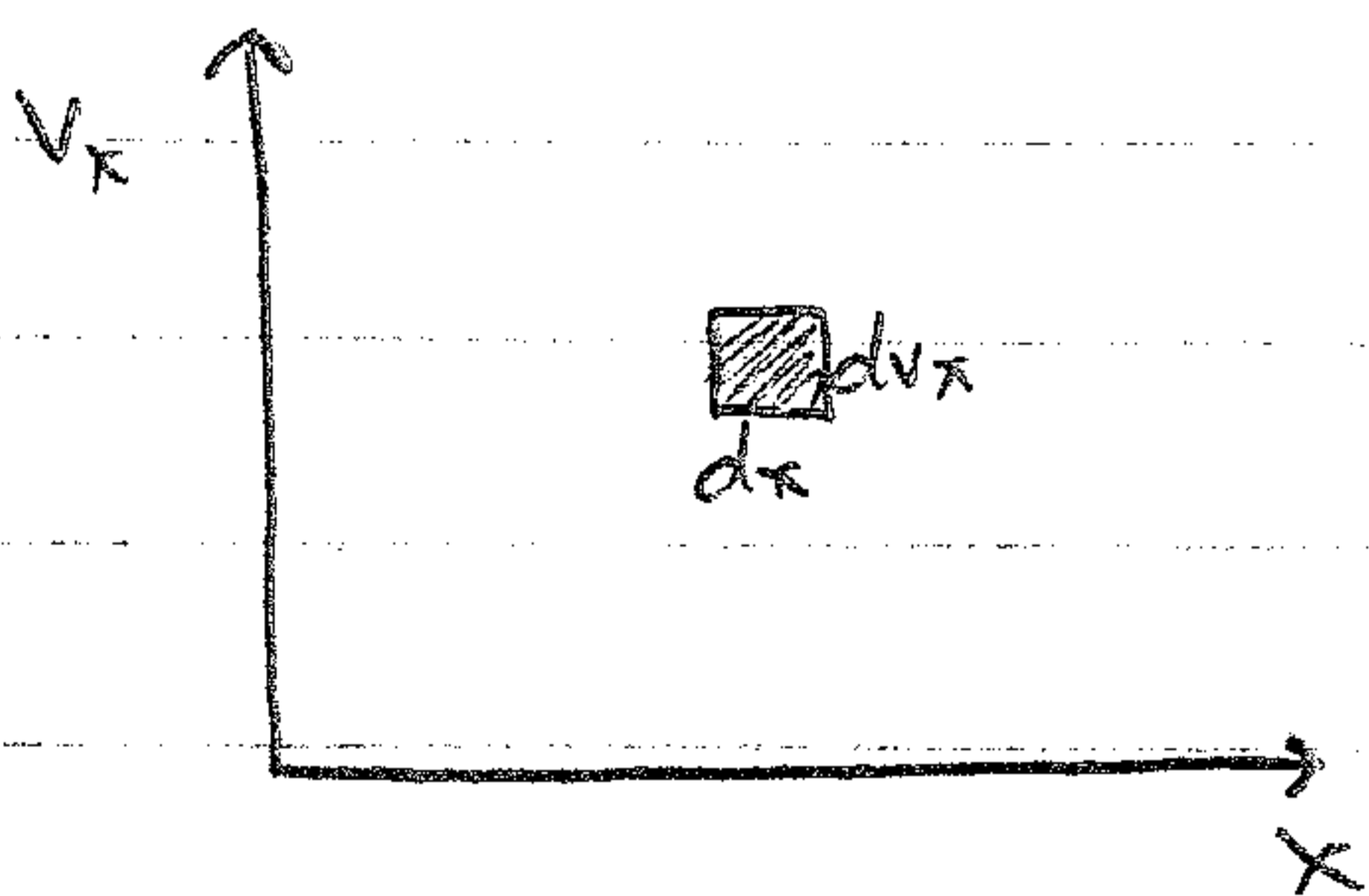
3. Evolving the Klimontovich Distribution is equivalent to an N-body problem.

a. For practical situations with $N \sim 10^{20}$, this is not possible

⇒ 4. We prefer a statistical description that averages over many particles, smoothing out the spiky-ness due to discrete particles, yielding a smooth solution.

III. Averaging to Yield Plasma Kinetic Equations

A. Coarse Grained Average:



1. Create a smoothed distribution by averaging over a small volume $d^3x \times d^3v$
2. This volume contains many particles, but is small enough that average doesn't change much over the volume.

3. For example, average with spherical exponential weighting

"Weight Function" $W(x', v') = e^{-\left(\frac{|x'|}{x_0}\right)^2 - \left(\frac{|v'|}{v_0}\right)^2}$ (characteristic sizes x_0, v_0)
 $x_0 \ll \lambda_D$ but $x_0 \gg n^{-1/3}$

4.

$$a. \underbrace{f(\underline{x}, \underline{v}, t)}_{\text{Distribution Function}} = \int d^3x' \int d^3v' W(\underline{x}', \underline{v}') \mathcal{F}(\underline{x} - \underline{x}', \underline{v} - \underline{v}', t)$$

Distribution Function

b. The distribution function $f(\underline{x}, \underline{v}, t)$ is a statistical density in 6-D phase space over small volume $d^3x d^3v$
 \Rightarrow Smooths out "discreteness" of Klimontovich.

c. Enough particles in $d^3x d^3v$ to yield a good statistical average $f(\underline{x}, \underline{v}, t)$.

5. a. An alternative approach uses the Liouville Equation to describe a system of particles

b. An ensemble average of such systems leads to a similar statistical description of kinetic plasmas.

B. Separating Smooth and Fluctuating parts:

1. a. The Klimontovich Equation describes the evolution of all plasma particles exactly.

b. We wish to find an evolution equation for the distribution function $f(\underline{x}, \underline{v}, t)$

2. Moments of Klimontovich Distribution: Charge and Current Densities.

a. Charge Density

$$\rho(\underline{x}, t) = \sum_s \int d^3v q_s \mathcal{F}_s(\underline{x}, \underline{v}, t)$$

"Zeroth" velocity moment

b. Current Density

$$\underline{j}(\underline{x}, t) = \sum_s \int d^3v q_s \underline{v} \mathcal{F}_s(\underline{x}, \underline{v}, t)$$

"First" velocity moment

Lecture #12 (Continued)

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III. B. (Continued)

3. Maxwell's Equations: ρ and \mathbf{j} are sources for $\underline{\underline{E}}$ & $\underline{\underline{B}}$

$$\nabla \cdot \underline{\underline{E}} = \frac{\rho}{\epsilon_0} \quad \nabla \times \underline{\underline{B}} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \underline{\underline{E}}}{\partial t}$$

4. Split into Smoothed and Fluctuating parts.

a. Take
$$\mathcal{F}_s = \underbrace{f_s}_{\text{Smoothed part}} + \underbrace{(\mathcal{F}_s - f_s)}_{\text{Fluctuating due to particle discreteness}}$$

b. Thus
$$\rho(\mathbf{x}, t) = \bar{\rho}(\mathbf{x}, t) + \tilde{\rho}(\mathbf{x}, t)$$

where
$$\bar{\rho}(\mathbf{x}, t) = \int d^3v q_s f_s(\mathbf{x}, \mathbf{v}, t)$$

$$\tilde{\rho}(\mathbf{x}, t) = \int d^3v q_s \left[\mathcal{F}_s(\mathbf{x}, \mathbf{v}, t) - f_s(\mathbf{x}, \mathbf{v}, t) \right]$$

c. Similarly
$$\mathbf{j}(\mathbf{x}, t) = \bar{\mathbf{j}}(\mathbf{x}, t) + \tilde{\mathbf{j}}(\mathbf{x}, t)$$

5. Maxwell's Equations are linear, so we can perform the same separation for the fields $\underline{\underline{E}}$ & $\underline{\underline{B}}$

a. For example
$$\nabla \cdot \underline{\underline{E}} = \frac{\rho}{\epsilon_0} \Rightarrow \begin{cases} \nabla \cdot \bar{\underline{\underline{E}}} = \frac{\bar{\rho}}{\epsilon_0} \\ \nabla \cdot \tilde{\underline{\underline{E}}} = \frac{\tilde{\rho}}{\epsilon_0} \end{cases} \quad \text{and the same for } \underline{\underline{B}}.$$

C. Averaging the Klimontovich Equation:

1. Now, we perform the same averaging procedure on $\frac{d\mathcal{F}}{dt} = 0$ to yield.

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\bar{\underline{\underline{E}}} + \mathbf{v} \times \bar{\underline{\underline{B}}}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \frac{q_s}{m_s} \left\langle (\tilde{\underline{\underline{E}}} + \mathbf{v} \times \tilde{\underline{\underline{B}}}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} \right\rangle = \left(\frac{\partial f}{\partial t} \right)_{\text{collisions}}$$

Plasma Kinetic Equation (or Boltzmann Equation)

III C. (Continued)

2. a. The LHS contains only terms that vary smoothly in $(\underline{x}, \underline{v})$ space, i.e. $f_s, \underline{E}, \underline{B}$.
 \Rightarrow Collective effects of plasma

b. The RHS contains very spiky quantities due to particle discreteness, \Rightarrow Collisional effects of plasma

3. Recall the ratio of $\frac{\text{Collective effects}}{\text{Collisional effects}} \sim \frac{\omega_{pe}}{v_c} \sim N_0$

Thus, collisional effects are typically weak.

4. We can neglect the effect of collisions, a good approximation for most plasma, so give

$$\frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_s}{\partial \underline{v}} = 0$$

Vlasov Equation

$$\nabla \cdot \underline{E} = \frac{1}{\epsilon_0} \sum_s \int d^3 \underline{v} \underbrace{q_s f_s(\underline{x}, \underline{v}, t)}_{=\rho}$$

Maxwell's

$$\nabla \times \underline{B} = \mu_0 \sum_s \int d^3 \underline{v} \underbrace{q_s \underline{v} f_s(\underline{x}, \underline{v}, t)}_{=\underline{j}} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

Equation

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \cdot \underline{B} = 0$$

These Vlasov-Maxwell Equations describe the collisionless kinetic evolution of a plasma.

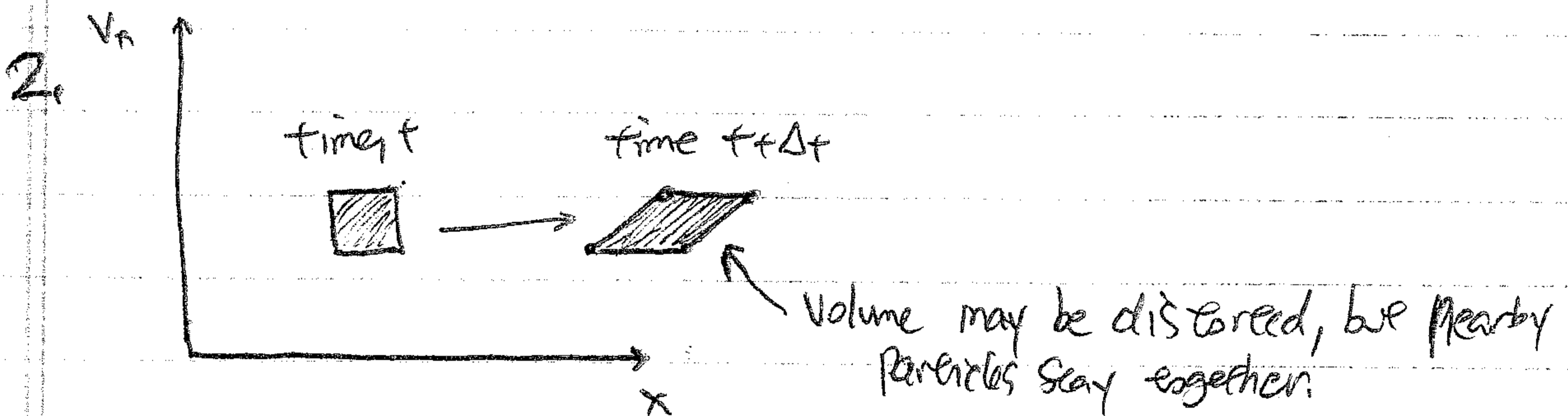
"Integro-differential" system of equations.

IV. The Distribution Function $f_s(\mathbf{x}, \mathbf{v}, t)$

A. Intuitive Picture:

1. $d^3x d^3v f_s(\mathbf{x}, \mathbf{v}, t)$ is the number of particles of species s in a infinitesimal volume in 6-D phase space, $\Delta v_x \Delta v_y \Delta v_z \Delta x \Delta y \Delta z$.

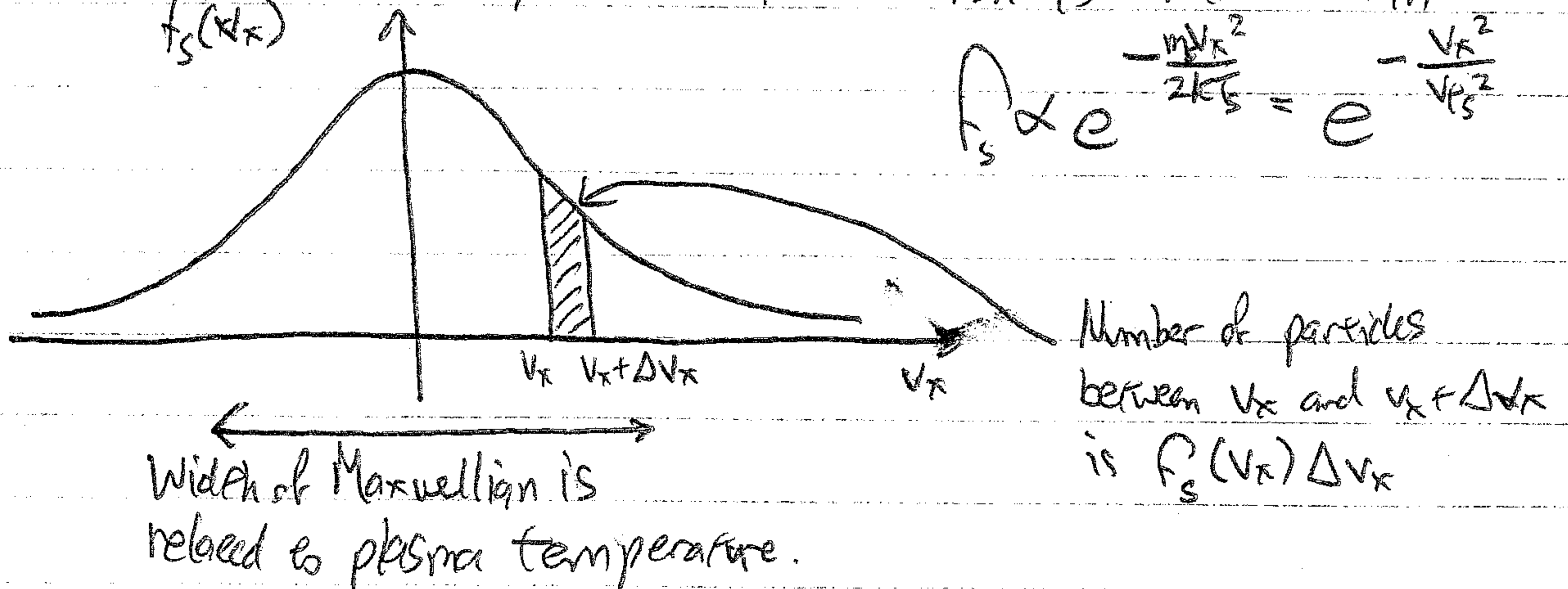
Thus $f_s(\mathbf{x}, \mathbf{v}, t) = \frac{\text{Number}}{\text{Volume}}$ is a number density in 6-D phase space.



- a. Particles at nearby points in phase space move together along their trajectories

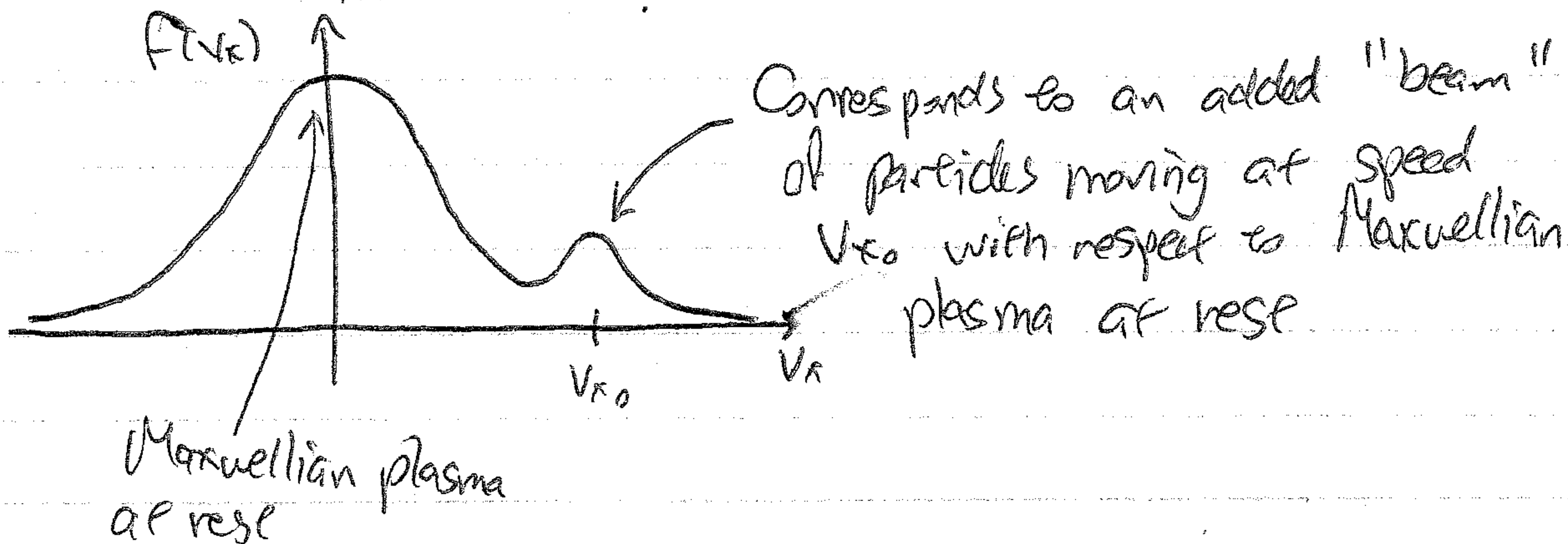
- b. $\frac{df_s}{dt} = 0$ (collisionless) suggests the distribution function is the density of an incompressible, 6-D fluid.

3. In Thermal Equilibrium, distribution function is Maxwellian

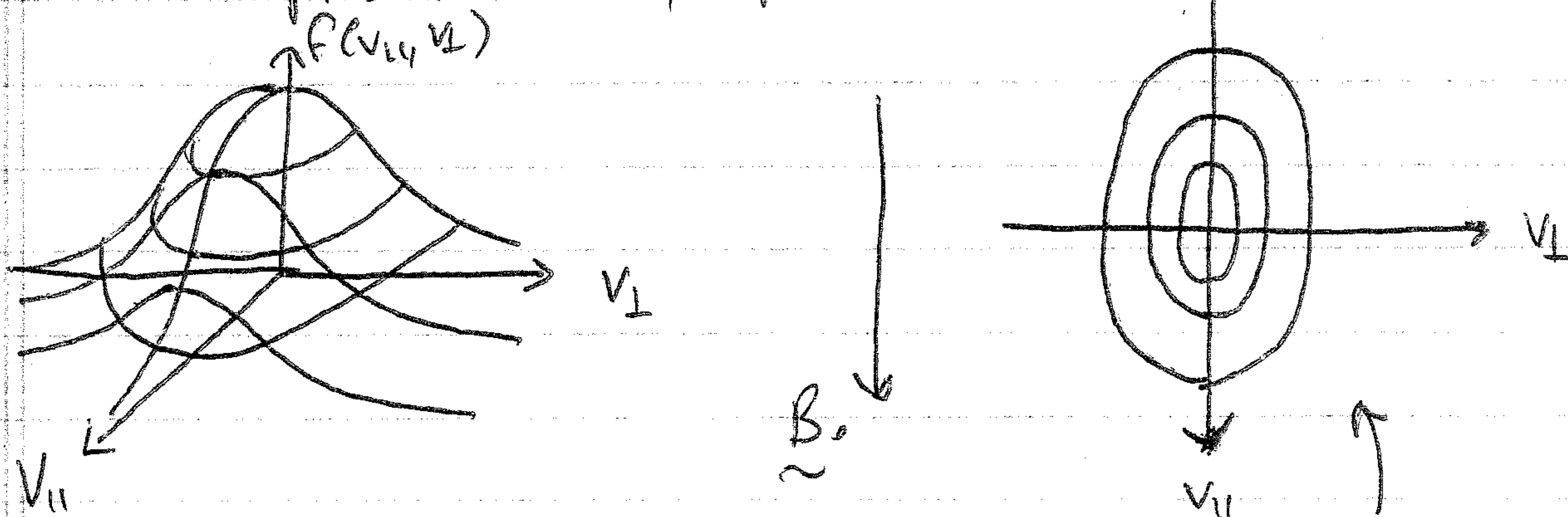


IV, A. (Continued)

4. Non-Maxwellian Distribution Function



5. Contour plots of velocity space



Here, temperature in the direction parallel to \underline{B}_0 is greater than perpendicular temp.

B. Moments of the distribution function:

1. Density: $n_s(\underline{x}, t) = \int d^3\underline{v} f_s(\underline{x}, \underline{v}, t) = \frac{\text{Number of particles}}{\text{unit volume}}$

2. Fluid Velocity: $\underline{U}_s(\underline{x}, t) = \frac{\int d^3\underline{v} \underline{v} f_s(\underline{x}, \underline{v}, t)}{n_s(\underline{x}, t)}$

3. Kinetic Energy Density: $\Sigma(\underline{x}, t) = \int d^3\underline{v} \frac{1}{2} m v^2 f_s(\underline{x}, \underline{v}, t)$

4. Pressure Tensor: $\underline{P}_s(\underline{x}, t) = \int d^3\underline{v} (\underline{v} - \underline{U}_s)(\underline{v} - \underline{U}_s) f_s(\underline{x}, \underline{v}, t)$

Next time, we'll use velocity moments to derive fluid equations.