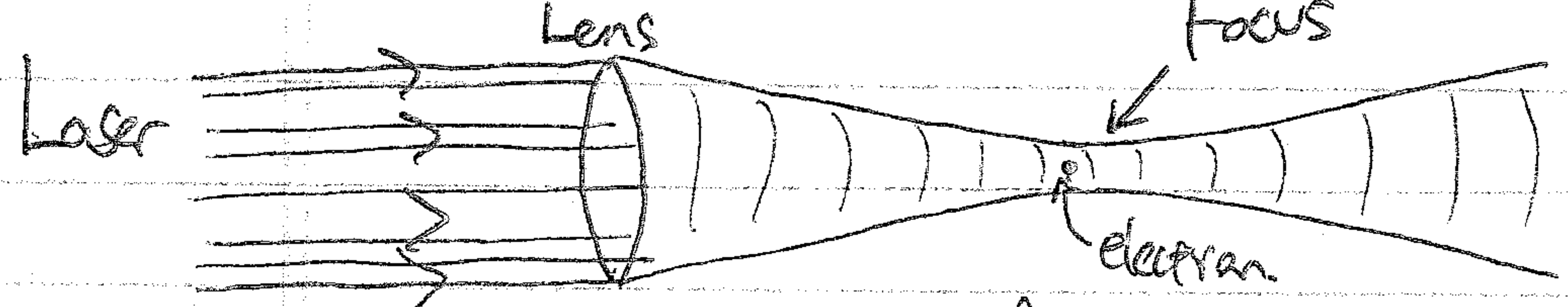
Lecture #10 The Ponderomotive Force

I. Particle Motion in High Frequency Electromagnetic Wave

A. Laser Plasma Interaction:



1. What is the motion of an electron in a high-frequency electromagnetic wave with variation in wave amplitude over space, ie., near the focal point of a laser?

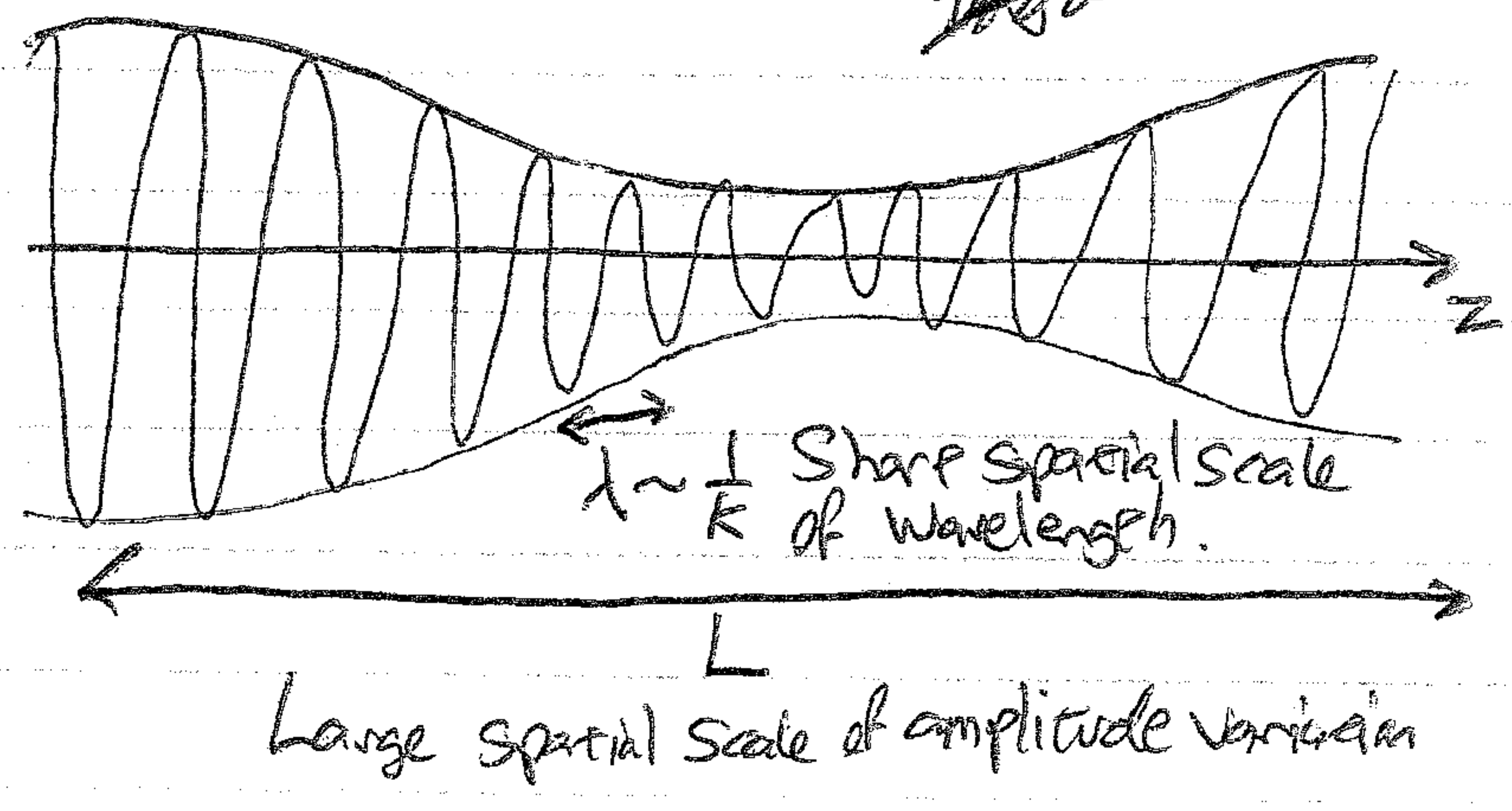
2. In this case, the plasma is unmagnetized. Only $\underline{E}$ and $\underline{B}$ from wave are present.
3. We'll use multiple timescale analysis to determine the lowest order nonlinear effect.

B. Multiple Timescale Analysis:

1. Consider an electromagnetic wave of high frequency ω whose amplitude may vary on a long timescale and large spatial scale.

$$\underline{E}(\underline{x},t) = \underline{E}_0(\underline{x},\tau)\cos(\omega t - \underline{k}\cdot\underline{x})$$

a. Two spatial scales:

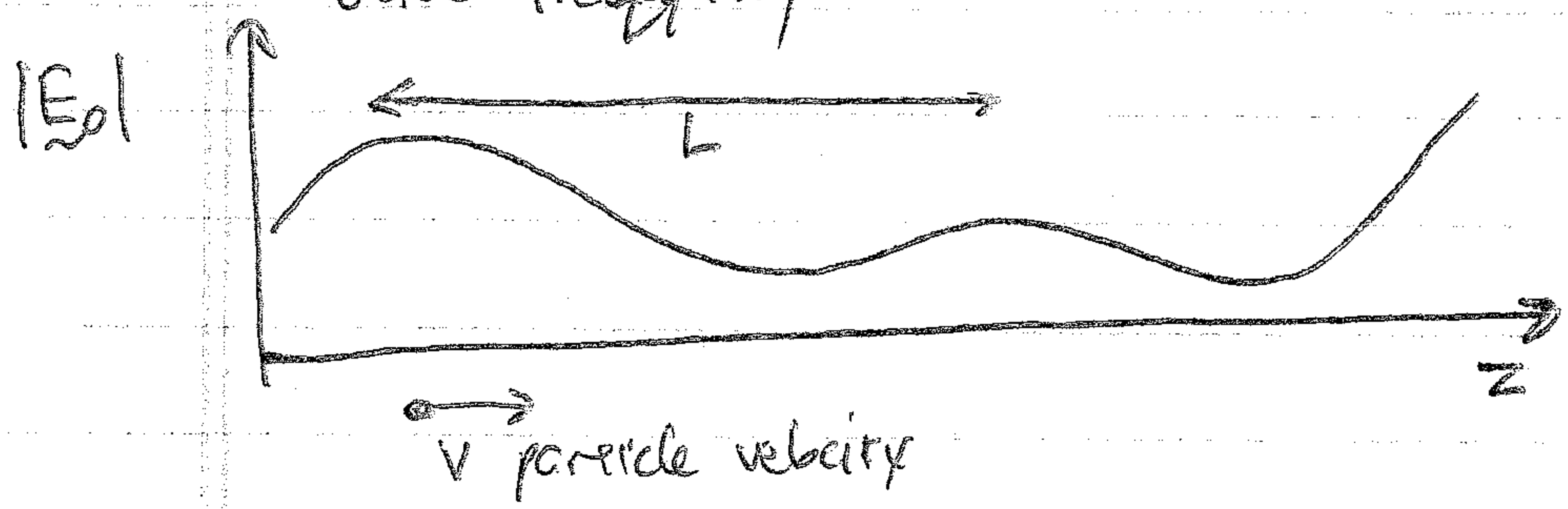


II. B. (Continued)

2. We'll show in HW#5 that $\underset{\sim}{E}(x,\tau,t)$ yields

$$\underset{\sim}{B}(\underset{\sim}{x},\tau,t) = -\frac{1}{\omega}\left\{\nabla\times\underset{\sim}{E}_0(\underset{\sim}{x},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{x}) - \underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{x},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{x})\right\}$$

3. We want a slow variation of EM wave magnitude due to motion of the particle through space compared to the wave frequency ω!



a. To particle, amplitude varies in time due to motion $\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0$

$\nabla\sim\frac{1}{L}$ large spatial scale $\quad |\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0|\sim\frac{v}{L}E_0$

b. Frequency of EM wave gives $|\omega\underset{\sim}{E}_0|\sim\omega E_0$

c. We want $\underbrace{|\underset{\sim}{v}\cdot\nabla\underset{\sim}{E}_0|}_{\text{Slow timescale}} \ll \underbrace{|\omega\underset{\sim}{E}_0|}_{\text{Fast timescale}} \Rightarrow \frac{v}{L}\ll\omega$ or $\boxed{\frac{v}{L\omega}\ll 1}$

d. This will be our ordering parameter

$$\boxed{\epsilon\sim\frac{v}{L\omega}\ll 1}$$

This separates fast oscillation timescale due to EM wave from slow drift timescale due to amplitude variation

4. $\frac{d\underset{\sim}{v}}{dt}=\frac{q}{m}(\underset{\sim}{E}+\underset{\sim}{v}\times\underset{\sim}{B})$

Compare these terms: $\quad \frac{|\underset{\sim}{v}\times\underset{\sim}{B}|}{|\underset{\sim}{E}|}\sim\frac{|v||B|\cancel{\sin\theta}}{|E|}\sim\frac{v\left(\frac{E_0}{L\omega}\right)}{E_0}\sim\frac{v}{L\omega}\sim\epsilon\ll 1$

$\frac{\partial \underset{\sim}{B}}{\partial t}=-\nabla\times\underset{\sim}{E} \Rightarrow \omega B\sim\frac{E_0}{L}$ or $B\sim\frac{E_0}{L\omega}$

I.B. (Continued)

5. Two Timescales: a. t Fast oscillation timescale

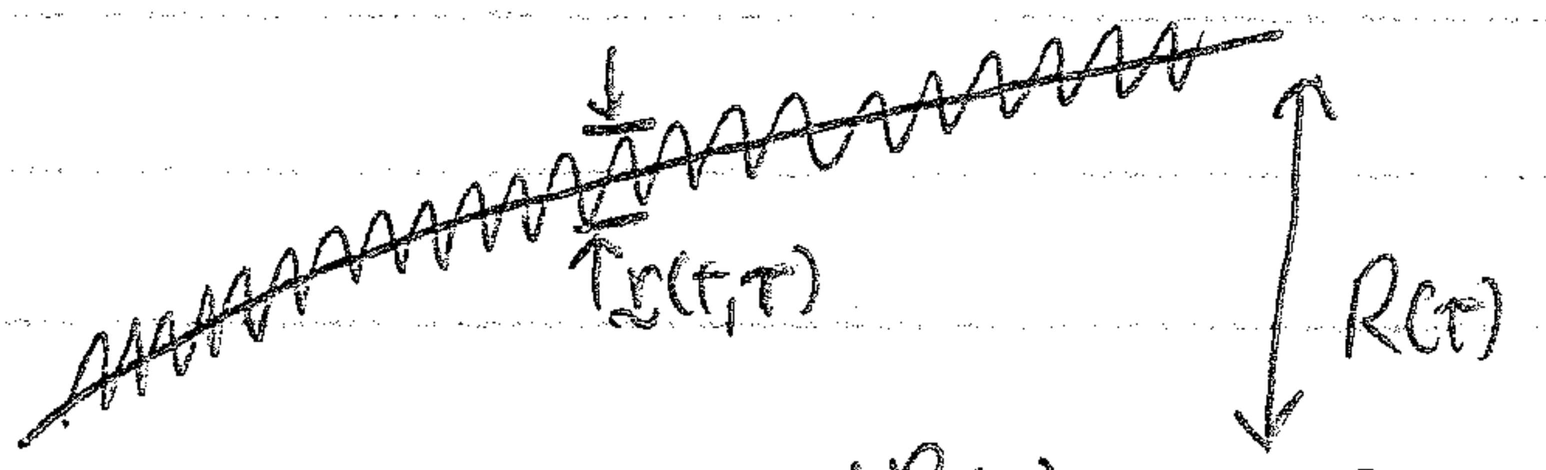
b. $\tau = \epsilon t$ Slow timescale of amplitude variation

c. Thus $\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}}$

d. Over each fast oscillation period, $\underset{\sim}{E}$ varies little in amplitude. But small changes each oscillation can sum to produce a long timescale change.

6. Write particle position as slowly varying oscillation center $\underset{\sim}{R}(\tau)$ plus small, rapidly oscillating position $\underset{\sim}{r}(t,\tau)$

$$\boxed{\underset{\sim}{x} = \underset{\sim}{R}(\tau) + \epsilon \underset{\sim}{r}(t,\tau)}$$



7. Velocity:

$$\underset{\sim}{v} = \frac{d\underset{\sim}{x}}{dt} = \frac{d\underset{\sim}{R}(\tau)}{dt} + \epsilon \frac{d\underset{\sim}{r}(t,\tau)}{dt}$$

Define $\underset{\sim}{U} \equiv \frac{d\underset{\sim}{R}(\tau)}{dt} = \epsilon \frac{\partial \underset{\sim}{R}(\tau)}{\partial \tau}$

$\underset{\sim}{u} \equiv \frac{d\underset{\sim}{r}(t,\tau)}{dt}$,

$$\underset{\sim}{v} = \epsilon \underset{\sim}{U}(\tau) + \epsilon \underset{\sim}{u}(t,\tau)$$

8. Acceleration:

$$\frac{dv}{dt} = \epsilon \frac{d\underset{\sim}{U}(\tau)}{dt} + \epsilon \frac{d\underset{\sim}{u}(t,\tau)}{dt} = \cancel{\epsilon \frac{\partial \underset{\sim}{U}(\tau)}{\partial t}}^{\,0} + \epsilon^2 \frac{\partial \underset{\sim}{U}(\tau)}{\partial \tau} + \epsilon \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial \tau}$$

9. Thus, we find:

$$\epsilon \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{U}(\tau)}{\partial \tau} + \epsilon^2 \frac{\partial \underset{\sim}{u}(t,\tau)}{\partial \tau} = \epsilon \frac{q}{m}\left[\underset{\sim}{E}(\underset{\sim}{R}+\epsilon\underset{\sim}{r},\tau,t) + \left(\epsilon \underset{\sim}{U}(\tau) + \epsilon \underset{\sim}{u}(t,\tau)\right) \times \underset{\sim}{B}(\underset{\sim}{R}+\epsilon\underset{\sim}{r},\tau,t)\right]$$

a. NOTE that highest order nonzero term of LHS is $O(\epsilon)$. Thus, highest term on RHS must be $O(\epsilon)$ to balance. Hence, we multiply RHS by ϵ to give balance

I. B. (Continued)

10. Taylor Expand Fields about oscillation center $\underset{\sim}{R}$:

a. $\underset{\sim}{E}(\underset{\sim}{x},\tau,t) = \underset{\sim}{E}(\underset{\sim}{R},\tau,t) + (\underset{\sim}{x}-\underset{\sim}{R})\cdot\nabla\underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \frac{[(\underset{\sim}{x}-\underset{\sim}{R})\cdot\nabla]^2}{2!}\underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \ldots$

b. NOTE: $\underset{\sim}{x}-\underset{\sim}{R} = \underset{\sim}{r}$, so

$$\underset{\sim}{E}(\underset{\sim}{x},\tau,t) = \underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \epsilon\,\underset{\sim}{r}\cdot\nabla\underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \ldots$$

and likewise with $B(\underset{\sim}{x},\tau,t)$

11. Expand all variables and substitute:

a. $\underset{\sim}{U}(\tau) = \underset{\sim}{U}_1(\tau) + \epsilon\,\underset{\sim}{U}_2(\tau) + \ldots$

$\underset{\sim}{u}(\tau,t) = \underset{\sim}{u}_1(\tau,t) + \epsilon\,\underset{\sim}{u}_2(\tau,t) + \ldots$

$\underset{\sim}{r}(\tau,t) = \underset{\sim}{r}_1(\tau,t) + \epsilon\,\underset{\sim}{r}_2(\tau,t) + \ldots$

12. Thus, we get

$$\epsilon\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial t} + \epsilon^2\frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t} + \epsilon^2\frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau} + \epsilon^3\frac{\partial \underset{\sim}{U}_2(\tau)}{\partial \tau} + \epsilon^2\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial \tau} + \epsilon^3\frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial \tau}$$

$$= \frac{q}{m}\Big[\epsilon\,\underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \epsilon^2\underset{\sim}{r}_1\cdot\nabla\underset{\sim}{E}(\underset{\sim}{R},\tau,t) + \epsilon^2\underset{\sim}{U}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^3\underset{\sim}{U}_2\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)$$

$$+ \epsilon^3\underset{\sim}{U}_1\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^4\underset{\sim}{U}_2\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^2\underset{\sim}{u}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)$$

$$+ \epsilon^3\underset{\sim}{u}_2\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^3\underset{\sim}{u}_1\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \epsilon^4\underset{\sim}{u}_2\times(\underset{\sim}{r}_1\cdot\nabla)\underset{\sim}{B}(\underset{\sim}{R},\tau,t) + \ldots\Big]$$

13. Lowest Order: $\mathcal{O}(\epsilon)$

$$\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial t} = \frac{q}{m}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{R})$$

$$\frac{\partial \underset{\sim}{r}_1}{\partial t} = \underset{\sim}{u}_1(t,\tau) = \frac{q}{m\omega}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{R}) \longleftarrow \text{Oscillation velocity}$$

$$\underset{\sim}{r}_1(t,\tau) = \frac{-q}{m\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{R}) \longleftarrow \text{oscillation position}$$

Lecture #10 (Continued)

I. B. (Continued)

14. ~~Next~~ Order: $O(\epsilon^2)$

a. $$\underbrace{\frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t}}_{①} + \underbrace{\frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}}_{②} + \underbrace{\frac{\partial \underset{\sim}{u}_1(t,\tau)}{\partial \tau}}_{③} = \frac{q}{m}\Big[\underbrace{\underset{\sim}{r}_1\cdot\nabla \underset{\sim}{E}(\underset{\sim}{R},\tau,t)}_{④} + \underbrace{\underset{\sim}{U}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)}_{⑤} + \underbrace{\underset{\sim}{u}_1\times\underset{\sim}{B}(\underset{\sim}{R},\tau,t)}_{⑥}\Big]$$

b. We now average over oscillation period to annihilate terms $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} dt$.

i) Assume $\underset{\sim}{u}_2(t,\tau)$ is periodic over $T=\frac{2\pi}{\omega}$.
This should be checked a posteriori.

c. Term ①: $\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial \underset{\sim}{u}_2(t,\tau)}{\partial t}\,dt = 0$ by assumed periodicity

d. Term ②: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}\,dt = \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}\cancelto{1}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} dt} = \frac{\partial \underset{\sim}{U}_1(\tau)}{\partial \tau}$$

e. Term ③: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial}{\partial \tau}\Big[\frac{q}{m\omega}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big]dt$$

$$= \frac{q}{m\omega}\frac{\partial \underset{\sim}{E}_0(\underset{\sim}{R},\tau)}{\partial \tau}\,\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\,dt = 0$$

f. Term ④: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{q}{m}\Big[\frac{-q}{m\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big]\cdot\nabla \underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\,dt$$

$$= \frac{-q^2}{m^2\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cdot\nabla \underset{\sim}{E}_0(\underset{\sim}{R},\tau)\,\frac{\omega}{2\pi}\underbrace{\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\,dt}_{=\frac{\pi}{\omega}} = \frac{-q^2}{2m^2\omega^2}\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cdot\nabla \underset{\sim}{E}_0(\underset{\sim}{R},\tau)$$

g. Term ⑤: $$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{q}{m}\underset{\sim}{U}_1(\tau)\times\Big[\frac{-1}{\omega}\Big\{\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R}) - \underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\Big\}\Big]dt$$

$$= \frac{-q}{m\omega}\Big[\underset{\sim}{U}_1(\tau)\times\nabla\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cancelto{0}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\,dt} - \underset{\sim}{U}_1(\tau)\times\underset{\sim}{k}\times\underset{\sim}{E}_0(\underset{\sim}{R},\tau)\cancelto{0}{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\cos(\omega t-\underset{\sim}{k}\cdot\underset{\sim}{R})\,dt}\Big]$$

$= 0$

I. B.4 (Continued)

h. Term ⑥:

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{q}{m}\left[\frac{q}{m\omega}\mathbf{E}_0(\mathbf{R},\tau)\sin(\omega t-\mathbf{k}\cdot\mathbf{R})\right]\times\left\{\frac{-1}{\omega}\Big[\nabla\times\mathbf{E}_0(\mathbf{R},\tau)\sin(\omega t-\mathbf{k}\cdot\mathbf{R}) - \mathbf{k}\times\mathbf{E}_0(\mathbf{R},\tau)\cos(\omega t-\mathbf{k}\cdot\mathbf{R})\Big]\right\}dt$$

$$=\frac{-q^2}{m^2\omega^2}\left\{\mathbf{E}_0(\mathbf{R},\tau)\times\nabla\times\mathbf{E}_0(\mathbf{R},\tau)\underbrace{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t-\mathbf{k}\cdot\mathbf{R})\,dt}_{\to\frac{\pi}{\omega}} - \mathbf{E}_0(\mathbf{R},\tau)\times\mathbf{k}\times\mathbf{E}_0(\mathbf{R},\tau)\underbrace{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t-\mathbf{k}\cdot\mathbf{R})\cos(\omega t-\mathbf{k}\cdot\mathbf{R})\,dt}_{\to 0}\right\}$$

$$=-\frac{q^2}{2m^2\omega^2}\mathbf{E}_0(\mathbf{R},\tau)\times\nabla\times\mathbf{E}_0(\mathbf{R},\tau)$$

i. Putting solution together, we find

$$\frac{\partial\mathbf{u}_1(\tau)}{\partial\tau}=-\frac{q^2}{2m^2\omega^2}\left[\mathbf{E}_0(\mathbf{R},\tau)\cdot\nabla\mathbf{E}_0(\mathbf{R},\tau)+\mathbf{E}_0(\mathbf{R},\tau)\times\nabla\times\mathbf{E}_0(\mathbf{R},\tau)\right]$$

j. NOTE: NRL p. 4 (12) gives

$$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$$

Taking $\mathbf{A}=\mathbf{B}=\mathbf{E}$, we get $\nabla\left(\frac{|\mathbf{E}|^2}{2}\right)=\mathbf{E}\times(\nabla\times\mathbf{E})+(\mathbf{E}\cdot\nabla)\mathbf{E}$

k. Thus, we find:

$$\boxed{\frac{\partial\mathbf{u}_1(\tau)}{\partial\tau}=\frac{-q^2}{2m^2\omega^2}\nabla\left(\frac{|\mathbf{E}_0(\mathbf{R},\tau)|^2}{2}\right)}$$

Ponderomotive Force

Lecture #10 (Continued)

I. (Continued)

C. Properties of the Ponderomotive Force

1. $$\boxed{\underset{\sim}{F}_{pond} = m \frac{d\underset{\sim}{U}}{dt} = \frac{-q^2}{4 m \omega^2} \nabla |\underset{\sim}{E}_0|^2}$$ Pushes away from regions of intense field.

a. We can write this as a potential force

$$\underset{\sim}{F}_{pond} = - \nabla \Phi_{pond}$$

where $$\boxed{\Phi_{pond} = \frac{q^2}{4 m \omega^2} |\underset{\sim}{E}_0|^2}$$

b. Note that the average fast oscillation ~~velocity~~ energy is

$$\frac{1}{2} m \overline{|\underset{\sim}{u}|^2} = \frac{m}{2} \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \frac{q^2}{m^2 \omega^2} |\underset{\sim}{E}_0|^2 \overbrace{\sin^2(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{R})\, dt}^{=\frac{\pi}{\omega}} = \frac{m}{2} \frac{\omega}{2\pi} \frac{q^2}{m^2\omega^2} |\underset{\sim}{E}_0|^2 \frac{\pi}{\omega}$$

$$= \frac{q^2}{4 m \omega^2} |\underset{\sim}{E}_0|^2$$

c. Thus

$$\Phi_{pond} = \frac{1}{2} m \overline{|\underset{\sim}{u}|^2}$$

Ponderomotive potential is the average fast oscillation kinetic energy.

d. For Center of Oscillation $\boxed{\mathcal{E}_{osc} = \frac{1}{2} m U^2 + \Phi_{pond} = \frac{1}{2} m U^2 + \frac{1}{2} m \overline{|\underset{\sim}{u}|^2}}$

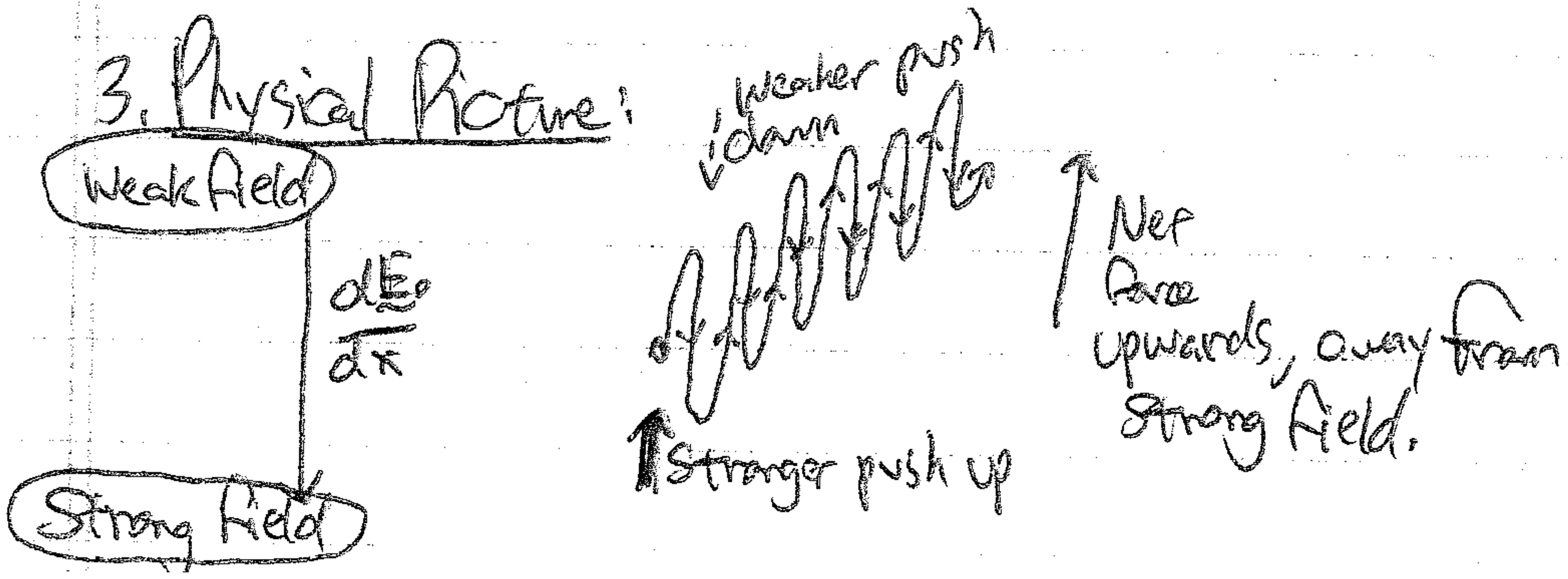
2. a. Force is independent of ~~sign of~~ charge

$\Rightarrow$ Repels both ions and electrons from high field regions.

b. Because $m_e \ll m_i$, electrons are pushed aside much more easily.

$\Rightarrow$ Resulting polarization electric field acts to pull ions out.

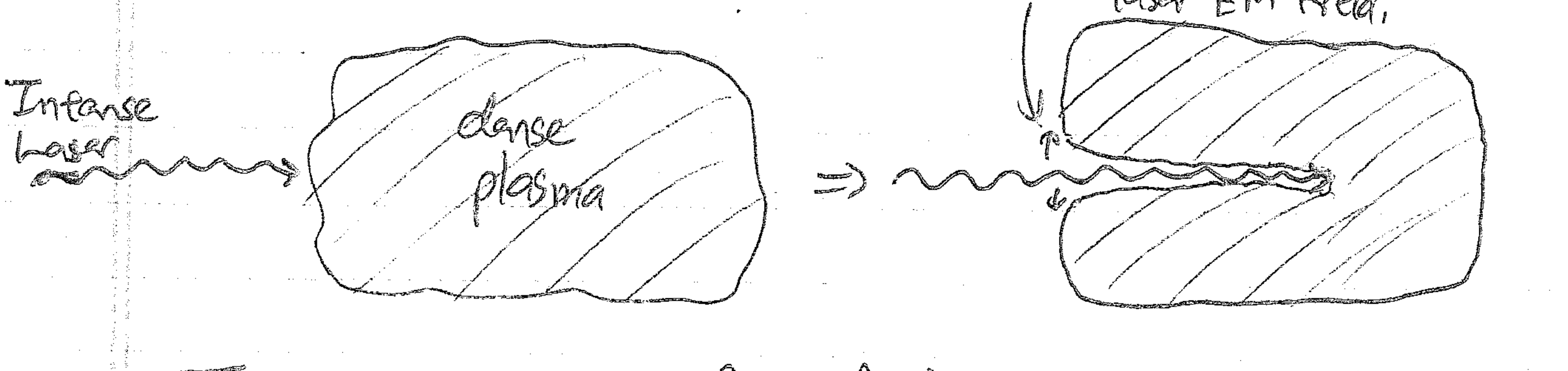
3. Physical Picture:



Lecture #10 (Continued)

IX (Continued)

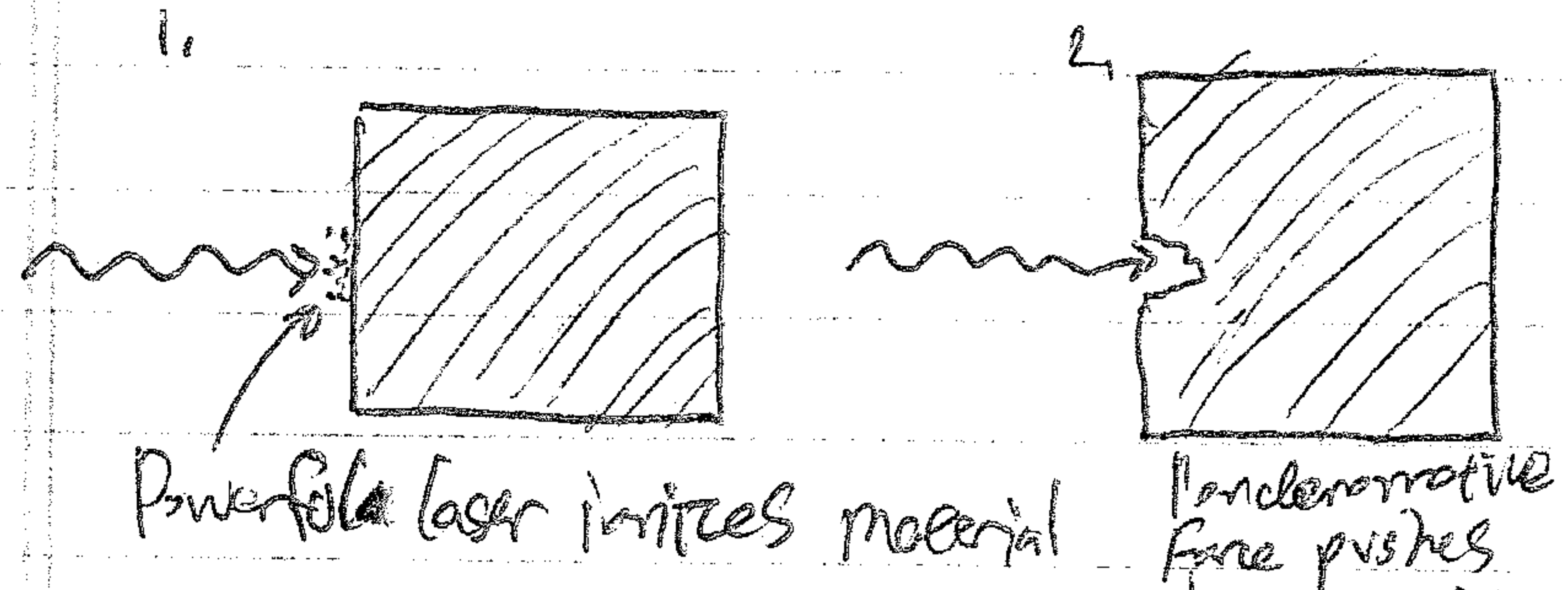
4. Laser Plasma Interaction:



a. This can lead to self-focus of laser light in a plasma.

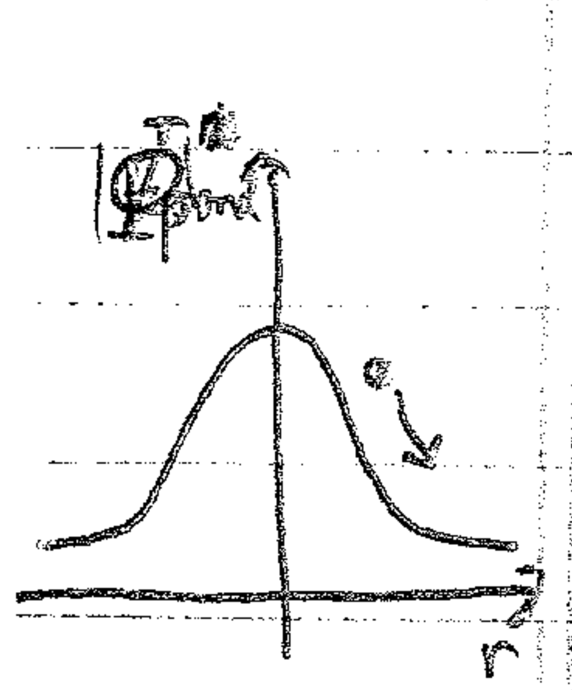
1. Powerful laser pushes aside electrons (and ions) due to ponderomotive force.
2. The resulting depression in plasma density acts as a convex lens, focusing the laser light into the evacuated channel.

b. Lasers can bore holes in materials by this mechanism.



3. Laser can propagate if ~~...~~ light frequency $\omega > \omega_{pe}$ in plasma.

5. Example: A particle of charge q & mass m is initially at rest at the center of a Gaussian laser beam with $|\mathbf{E}_0(\mathbf{x})| = E_0 e^{-\frac{r^2}{r_0^2}}$. Find Center of oscillation velocity as a function of position.



$$\mathcal{E}_0 = \frac{1}{2} m u^2 + \Phi_{pond} \quad \text{where} \quad \Phi_{pond} = \frac{q^2}{4m\omega^2}|\mathbf{E}_0|^2 = \frac{q^2}{4m\omega^2} E_0^2 e^{-\frac{2r^2}{r_0^2}}$$

At $r=0$, $\mathcal{E}_0 = \frac{1}{2} m \cancel{u^2}^{\,0} + \frac{q^2}{4m\omega^2} E_0^2$.

Thus
$$u = \sqrt{\frac{2\mathcal{E}_0}{m} - \frac{2\Phi_{pond}}{m}} = \sqrt{\frac{2\mathcal{E}_0}{m} - \frac{2\mathcal{E}_0}{m} e^{-\frac{2r^2}{r_0^2}}} = \sqrt{\frac{2\mathcal{E}_0}{m}}\left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}$$

$$\boxed{u(r) = \frac{q E_0}{m\omega\sqrt{2}}\left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}}$$