

29:194 Homework #10

Reading: Gurnett & Bhattacharjee, Sec 4.1–4.3 (p.75–94), Sec 5.5 (p.155–162)

Supplemental Reading (optional): Chen, Sec 4.3–4.8 (p.82–100)

Due at the beginning of class, Thursday, December 9, 2010.

1. Resistive MHD Dispersion relation:

Calculate the dispersion relation for Resistive MHD equations below:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho &= -\rho \nabla \cdot \mathbf{U} \\ \rho \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) &= -\nabla \left(p - \frac{B^2}{2\mu_0} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0} \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{U} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} \\ \frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p &= -\gamma p \nabla \cdot \mathbf{U} \end{aligned}$$

Assume the equilibrium plasma conditions are constant in time and homogeneous with a mean equilibrium magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$. Take the wavevector to be of the form $\mathbf{k} = k_{\parallel} \hat{\mathbf{z}}$.

- (a) Find the linearized equations for Resistive MHD. State clearly any assumptions you have made and please box the final form of each equation.
- (b) Find the Fourier transform of the equations by assuming a plane wave solution of the form $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. Be sure to box the final form of each equation.
- (c) Eliminate all of variables except for \mathbf{U}_1 , writing the problem as a matrix equation of the form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = 0$$

in terms of ω , η , μ_0 , k_{\parallel} , $c_s = \sqrt{\gamma p_0 / \rho_0}$, and $v_A = B_0 / \sqrt{\mu_0 \rho_0}$.

- (d) Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$.
- (e) Solve for the frequency ω of the Alfvén wave mode.

2. Eigenfunctions for Resistive MHD with $\mathbf{k} = k_{\parallel} \hat{\mathbf{z}}$

- (a) Use your results from problem #2 to find the complete eigenfunction for a mode that has $\mathbf{U} = U_0 \hat{\mathbf{z}}$ (thus with $U_x = 0$ and $U_y = 0$). Your answer should provide the solutions for ρ_1 , p_1 , and \mathbf{B}_1 in terms of U_0 , B_0 , γ , p_0 , ρ_0 , η , μ_0 , k_{\parallel} , $c_s = \sqrt{\gamma p_0 / \rho_0}$, and $v_A = B_0 / \sqrt{\mu_0 \rho_0}$.
- (b) If $c_s < v_A$, does this mode correspond to the fast mode or the slow mode?

3. Shallow Water Equations

The Shallow Water Equations are a two-dimensional system of equations describing the dynamics of motions with wavelengths greater than the depth of the water H in a rotating frame of reference, valid for wavenumbers $kH \ll 1$. They are useful in describing ocean dynamics. The equations are

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{U}) = 0$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + f\hat{\mathbf{z}} \times \mathbf{U} = -g\nabla\eta$$

$$h = H + \eta$$

where the fluid velocity $\mathbf{U}(x, y) = U_x(x, y)\hat{\mathbf{x}} + U_y(x, y)\hat{\mathbf{y}}$ and the gradient $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}$ are two-dimensional. The term $f\hat{\mathbf{z}} \times \mathbf{U}$ represents the Coriolis force due to the rotation of the earth, with the frequency f representing the effective frequency of this rotation at a given latitude; we assume $f = f_0 = \text{constant}$. The height of the mean sea level above the ocean bottom is given by $H(x, y)$ and does not depend on time. The fluctuation of the sea surface due to waves from mean sea level is $\eta(x, y, t)$, so that $h(x, y, t)$ represents to total depth of water at any given position and time. We may assume $H \gg |\eta|$ and $\nabla H = 0$.

- Find the linearized Shallow Water Equations. State clearly any assumptions you have made and please box the final form of each equation.
- Find the Fourier transform of the equations by assuming a plane wave solution of the form $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. Be sure to box the final form of each equation.
- Put the system of equations into matrix form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \eta \\ U_x \\ U_y \end{pmatrix} = 0.$$

- Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$, using $c = \sqrt{gH}$ as the surface gravity wave velocity.
- For the non-trivial root, determine simplified dispersion relation in the short- and long-wavelength limits. Be sure to define each limit. (The short-wavelength solution corresponds to surface gravity waves and the long-wavelength solution corresponds to inertial waves.)
- Which of the above limits (short or long) gives non-dispersive wave behavior?

4. Reflection from a Plasma

Light waves are incident on a slab of cold, unmagnetized plasma at an angle θ as shown below. The fully-ionized plasma of protons and electrons has a uniform density. The index of refraction for a plasma is given by $n = ck/\omega$. Using Snell's law, find the critical angle θ_c for total internal reflection (within the free space region) as a function of the light wave frequency ω .

