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Anomalous Diffusion Arising from Microinstabilities in a Plasma

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A plasma is considered in which a Maxwellian distribution of electrons with thermal velocity v_e and drift velocity v_D is drifting relative to a Maxwellian distribution of ions with thermal velocity v_i. For $v_D \lesssim v_e$ the usual ion acoustic waves are stable, however, electrostatic ion cyclotron waves with $\omega \cong \Omega_i$ are unstable for $v_D \gtrsim 5v_i$. In the case when $5v_i \lesssim v_D \lesssim v_e$, and $T_e/T_i < 2$ the electrostatic ion cyclotron waves grow to a nonlinear equilibrium spectrum. This spectrum of waves leads to a diffusion of electrons across the field lines with a diffusion coefficient $D = \alpha\rho_e^2\Omega_e$, where ρ_e is the electron Larmor radius and Ω_e is the electron Larmor frequency. α, the ratio of the resulting diffusion coefficient to the Bohm diffusion coefficient, is given by a constant $\times (v_D/v_e)^5(T_e/T_i)^2$.

I. INTRODUCTION

It is well known that a fluctuating electric field in a plasma leads to a diffusion of the particles across the magnetic field, and this is called anomalous diffusion to distinguish it from the usual collisional diffusion. This diffusion can be divided into two classes which depend on the character of the fluctuating electric fields. We denote the $\mathbf{k}$th Fourier space component of the electric field by $\mathbf{E}_\mathbf{k} = \hat{e}\,|E_\mathbf{k}|\exp[i\phi_\mathbf{k}(t)]$, where $\hat{e}$ is a unit vector in the direction of polarization. In one case the electric field is characterized by a phase $\phi_\mathbf{k}(t)$ which is a statistical function of time and as discussed by Spitzer[1] the diffusion coefficient depends on the correlation time of the electric field fluctuations. In the second case the electric field is made up of a superposition of coherent waves and $\phi_\mathbf{k}(t) = -\omega_\mathbf{k}t + \phi_{k_0}$, where ϕ_{k_0} is independent of time. In this case the diffusion arises from a resonance of particles and waves which move along the field lines with the same velocity.

It has been shown[2,3] that in the nonlinear limit certain types of microinstabilities lead to the establishment of an equilibrium electric fluctuation spectrum which is a superposition of coherent waves. It is thus possible to determine the anomalous diffusion arising from these microinstabilities by direct calculation and it is the purpose of this paper to do this for the case of a particularly strong instability—the two-stream ion-cyclotron instability.

In Sec. II we develop the linearized theory of the two-stream ion-cyclotron instability; the nonlinear theory of this instability is discussed in Sec. III. The diffusion due to the nonlinear equilibrium spectrum is calculated in Sec. IV and the results are discussed in Sec. V.

II. LINEARIZED THEORY

We consider a homogeneous infinite collisionless plasma in which the ions and electrons each have a Maxwellian distribution at a characteristic temperature with the center of the Maxwellians displaced by a drift velocity v_D and we restrict ourselves to the case of $T_e \cong T_i$ and $\beta = 8\pi nKT/B^2 \ll 1$. (Here T_e and T_i are the electron and ion temperatures respectively, n is the particle density, B is the magnetic field strength, and the drift velocity is along the magnetic field.) The usual theory of the two-stream instability, which considers only waves propagating parallel to the magnetic field, would predict stability until the electron drift velocity becomes comparable to the electron thermal velocity. Only in situations where $T_e \gg T_i$ does the critical velocity approach the ion thermal velocity. In the following we point out that for a collisionless plasma instability occurs at a much lower velocity for electrostatic waves near the ion cyclotron frequency and propagating at large angles to the field.

We will work in the frame where the ions are at rest and consider modes of the form $\exp(i\mathbf{k}\cdot\mathbf{r} - \omega t)$. If $v_D = 0$ the plasma is evidently stable. We would therefore expect that instability could only occur if $k_\parallel v_D > \omega$ where ω, is the wave frequency and $k_\parallel$ the wavenumber parallel to the field. This is the condition that the peak of electron distribution be moving slightly faster than the wave—the usual

[1] L. Spitzer, Jr., Phys. Fluids **3**, 659 (1960).
[2] W. E. Drummond and D. Pines, Salzburg Conference on Plasma Physics and Controlled Nuclear Fusion Research, September 4–9, 1961; paper No. 134, Nuclear Fusion (to be published).
[3] A. A. Vedenov, E. P. Vilikov, R. Z. Sagdeev, Salzburg Conference on Plasma Physics and Controlled Nuclear Fusion Research, September 4–9, 1961; paper No. 199, Nuclear Fusion (to be published).

condition for being able to put energy into the wave.

In this case if we put $\omega \cong \Omega_i$, the ion-cyclotron frequency, and $v_i < v_D \ll v_e$, the case we wish to discuss, we obtain the condition $k\rho_i \cong 1$, where v_e and v_i are the electron and ion thermal velocities respectively and $\rho_i = v_i/\Omega_i$ is the ion Larmor radius. The significance of this is that we can show that for such k and low β only pure electrostatic waves are possible. Thus if we write down the dispersion matrix obtained from $\nabla \times \nabla \times \mathbf{E} + \ddot{\mathbf{E}} = -4\pi\dot{\mathbf{j}}$ in a coordinate system in which one of the axes is parallel to k, determining $\mathbf{j}$ from the Boltzmann equation, we find the dispersion matrix has the following structure:

$$\begin{vmatrix} c^2k^2-\omega^2+(\alpha_{11}) & (\alpha_{12}) & (\alpha_{13}) \\ (\alpha_{21}) & c^2k^2-\omega^2+(\alpha_{22}) & (\alpha_{23}) \\ (\alpha_{31}) & (\alpha_{32}) & -\omega^2+(\alpha_{33}) \end{vmatrix} = 0, \quad (1)$$

where the quantities (α_{ij}) arise from the plasma currents and are all of order $\omega^2/(k\lambda_D)^2$, where $\lambda_{De,i} = v_{e,i}/\omega_{pe,i}$. We note that for $c^2k^2 \gg \omega^2$, $\omega^2/(k\lambda_D)^2$, the only possible root is $\omega^2 = (\alpha_{33})$, the pure electrostatic mode in which $\mathbf{E} \parallel \mathbf{k}$. If we put $\omega \cong \Omega_i$, $k \cong \Omega_i/v_i$ then $(k\lambda_D)^2 \cong B^2/4\pi nm_ic^2 \ll 1$ and $c^2k^2(k^2\lambda_D^2)/\omega^2 \cong B^2/4\pi nm_iv_i^2 = 1/\beta \gg 1$. Thus we need only consider pure electrostatic modes.

For this case the dispersion relation, which is worked out in the Appendix, is

$$\sum_{j=i,e} \sum_n \frac{\Gamma_n(k_\perp^2\rho_j^2)}{(k\lambda_{D_j})^2} \cdot\left\{W\left(\frac{-\omega + k_\parallel u_j + n\Omega_j}{k_\parallel v_j}\right) - \frac{n\Omega_j}{-\omega + k_\parallel u_j + n\Omega_j} \cdot\left[1 + W\left(\frac{-\omega + k_\parallel u_j + n\Omega_j}{k_\parallel v_j}\right)\right]\right\} = 1. \quad (2)$$

Here T_j is temperature, u_j the drift velocity of the j species, $\Gamma_n(x) = e^{-x} I_n(x)$, I_n is the usual Bessel function of imaginary argument, and

$$W(x) = -1 + \frac{x}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\exp(\frac{1}{2}y^2)}{x + y}\, dy. \quad (3)$$

The contour of the integral may be taken along the real axis for x in the lower half-plane (growing waves). Limiting forms are given by

$$\begin{aligned} W &\approx -1 + i(\tfrac{1}{2}\pi)^{\frac{1}{2}}x, & |x| &\ll 1, \\ W &\approx 1/x^2, & |x| &\gg 1. \end{aligned} \quad (4)$$

For all real x, Im $(W) = i(\frac{1}{2}\pi)^{\frac{1}{2}}x \exp(-\frac{1}{2}x^2)$. The limiting form given for large x is not valid for highly damped waves which are of no concern here.

Since we are concerned with wavelengths comparable to the ion-gyroradius $k_\perp\rho_e \ll 1$; $\Gamma_{0,e} = 1$; $\Gamma_{n\neq0,e} = 0$. Moreover we see that since $v_D/v_e \ll 1$ the argument of the electron W function is very small. Since we are concerned with a wave nearly at resonance with the ion-gyrofrequency we may neglect all the ion terms except $n = 1$. This gives us

$$0 = \left[-1 + i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(-\omega + k_\parallel v_D)}{k_\parallel v_e}\right] + \frac{T_e}{T_i}\Gamma_1\left\{W\left(\frac{-\omega + \Omega_i}{k_\parallel v_i}\right) + \frac{\Omega_i}{\omega - \Omega_i}\left[1 + W\left(\frac{-\omega + \Omega_i}{k_\parallel v_i}\right)\right]\right\}, \quad (5)$$

where we have neglected $(k\lambda_{De})^2 \ll 1$.

We may obtain an approximate solution by noting that a condition for solution is a large argument for the ion W function as otherwise the large imaginary part (ion cyclotron damping) will give a damped solution.

If

$$|(\omega - \Omega_i)/k_\parallel v_i| \gg 1, \quad (6)$$

we have simply

$$\begin{aligned} \frac{\Omega_i}{\omega - \Omega_i}\frac{T_e}{T_i}\Gamma_1 &= \left[1 - i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{-\Omega_i + k_\parallel v_D}{k_\parallel v_e}\right)\right], \\ \omega - \Omega_i &= \Omega_i\frac{T_e}{T_i}\Gamma_1\left[1 + i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{-\Omega_i}{k_\parallel v_e} + \frac{v_D}{v_e}\right)\right]. \end{aligned} \quad (7)$$

Here we have chosen $k_\parallel$ positive as the direction of propagation for instability and used $\omega \cong \Omega_i$.

We note that Γ_1 has a very flat maximum at $k_\perp^2 R_L^2 \cong 1.5$ attaining there a value 0.22.

We conclude therefore that the maximum growth rate is given by

$$\gamma \cong 0.3(T_e/T_i)\,\Omega_i(v_D/v_e) \quad (8)$$

occurring for $k_\perp^2 R_L^2 \cong 1.5$ and $k_\parallel \gg \Omega_i/v_D$.

Moreover from Eq. (6) we must have

$$\frac{\omega - \Omega_i}{k_\parallel v_i} \approx 0.2\,\frac{T_e}{T_i}\frac{\Omega_i}{k_\parallel v_i} > 1$$

and also $\Omega/k_\parallel v_e < v_D/v_e$, so that a rough criterion for instability is given as

$$v_D/v_i > 5.$$

To refine the stability criterion we return to Eq. (5) and look for a critical value of v_D which will lead to real frequency ω. The imaginary part of Eq. (5) then becomes

$$\frac{\omega}{k_\parallel v_e} - \frac{v_D}{v_e} + \frac{T_e}{T_i}\,\Gamma_1\,\frac{\omega}{k_\parallel v_i}\exp\{-\tfrac{1}{2}[(\omega-\Omega_i)/k_\parallel v_i]^2\} = 0, \tag{9a}$$

and the real part

$$(T_e/T_i)\Gamma_1[\Omega_i/(\omega-\Omega_i)] = 1. \tag{9b}$$

Substituting (9b) into (9a) we have

$$\frac{v_D}{v_e} = \left(1+\frac{T_e}{T_i}\,\Gamma_1\right)\left[\frac{\omega-\Omega_i}{k_\parallel v_i}\,\frac{v_i}{v_e}\,\frac{T_i}{\Gamma_1 T_e} + \frac{\omega-\Omega_i}{k_\parallel v_i}\right.$$
$$\left.\cdot\exp\{-\tfrac{1}{2}[(\omega-\Omega_i)/k_\parallel v_i]^2\}\right].$$

The minimum comes for

$$\frac{1}{2}\left(\frac{\omega-\Omega}{k_\parallel v_i}\right)^2 \approx -\ln\frac{v_i}{v_e}\,\frac{T_i}{\Gamma_1 T_e} \approx \frac{1}{2}\ln\frac{m_i}{m_e}$$

or

$$\frac{\Omega}{k_\parallel v_i} \approx \left(\ln\frac{m_i}{m_e}\right)^{\frac{1}{2}}\frac{T_i}{T_e\Gamma_1},$$

and the critical drift is then

$$\frac{v_D}{v_i} \approx \left(\frac{T_i}{\Gamma_1 T_e}+1\right)\left(\ln\frac{m_i}{m_e}\right)^{\frac{1}{2}} \approx 14\,\frac{T_i}{T_e} + 3. \tag{10}$$

This formula is not reliable for T_e much greater than T_i, as then higher n values must be considered in Eq. (9b). Nonetheless one can see by inspection that as T_e/T_i decreases, the root moves closer to Ω_i and the critical drift increases. Conversely as T_e/T_i increases, the critical drift decreases.

It would appear then that this instability near the ion-cyclotron frequency reduces the value of v_D/v_e necessary for instability by about an order of magnitude in the case of equal temperatures as compared to previous theories which consider only $k_\perp = 0$.

III. NONLINEARIZED THEORY

From the derivation it is clear that the dispersion relation $(\omega-\Omega_i)/\Omega_i = \Gamma(T_e/T_i)$ is insensitive to small changes in the distribution function while the growth rate depends critically on the distribution function. In particular, for a more general electron distribution, the growth rate γ is given by

$$\gamma = \alpha\,\frac{\partial g}{\partial v_\parallel}\bigg]_{v_\parallel=\omega/k_\parallel} \tag{11}$$

where

$$g = \int f N_\perp\,dN_\perp\,d\varphi,$$

$$\alpha = 2\Omega_i\Gamma(T_e/T_i)(\pi v_e^2/n).$$

In order to apply the nonlinear theory[2] it is required that the growth rate in the linearized theory be proportional to $\partial g/\partial v_\parallel$ and $\gamma_{\mathbf{k}}/\omega_{\mathbf{k}} \ll 1$. In addition, the dispersion relation must be such that $\omega_{\mathbf{k}} + \omega_{\mathbf{q}} \neq \omega_{\mathbf{k}+\mathbf{q}}$. The ion-cyclotron instability considered in Sec. II satisfies all these necessary conditions.

In the nonlinear theory $f_0(\mathbf{v})$ is replaced by a slowly varying function $f(\mathbf{v}, t)$ for which $f(\mathbf{v}, 0) = f_0(\mathbf{v})$, and $f(\mathbf{v}, t)$ varies in time according to

$$\frac{\partial f(\mathbf{v}, t)}{\partial t} = \left(\frac{e}{m}\right)\sum_{\mathbf{k}}\mathbf{E}_{-\mathbf{k}}\cdot\nabla_{\mathbf{v}} f_{\mathbf{k}}. \tag{12}$$

Inserting $\mathbf{f}_{\mathbf{k}}(\mathbf{v})$ from Eq. (A2) of the Appendix we obtain

$$\frac{\partial f}{\partial t} = \left(\frac{e}{m}\right)^2\sum_{\mathbf{k}} E_{-\mathbf{k}}\cdot\nabla_{\mathbf{v}}$$
$$\cdot\left[E_\perp\,\frac{\sin(\phi-\theta)}{\Omega_e}\,\frac{\partial}{\partial v_\perp} + \frac{E_\parallel}{(s+ik_\parallel v_\parallel)}\,\frac{\partial}{\partial v_\parallel}\right] f(\mathbf{v}, t)$$
$$= \left(\frac{e}{m}\right)^2\sum_{\mathbf{k}}\left[E_\perp\cos(\phi-\theta)\,\frac{\partial}{\partial v_\perp} + E_\parallel\,\frac{\partial}{\partial v_\parallel}\right]$$
$$\cdot\left[E_\perp\,\frac{\sin(\phi-\theta)}{\Omega_e}\,\frac{\partial}{\partial v_\perp} + \frac{E_\parallel}{(s+ik_\parallel v_\parallel)}\,\frac{\partial}{\partial v_\parallel}\right] f(\mathbf{v}, t), \tag{13}$$

where θ is the azimuthal angle of $\mathbf{k}$ and

$$s = s(k) = -i\,\frac{k_\parallel}{|k_\parallel|}\,\omega(k) + \gamma(k).$$

Integrating over ϕ we obtain

$$\int\frac{\partial f}{\partial t}\,d\phi = \left(\frac{e}{m}\right)^2\sum_{\mathbf{k}}|E_\parallel|^2\,\frac{\partial}{\partial v_\parallel}\,\frac{1}{(s+ik_\parallel v_\parallel)}\,\frac{\partial}{\partial v_\parallel}\int f\,d\phi,$$

and therefore,

$$\frac{\partial g}{\partial t} = \left(\frac{e}{m}\right)^2\sum_{\mathbf{k}}|E_\parallel(\mathbf{k})|^2\,\frac{\partial}{\partial v_\parallel}\,\frac{1}{(s+ik_\parallel v_\parallel)}\,\frac{\partial g}{\partial v_\parallel}. \tag{14}$$

Replacing $\sum_{k_\parallel}$ by $L/2\pi\int dk_\parallel$ we have, since $|E_\parallel(\mathbf{k})|^2$ and the real part of $1/(s_{\mathbf{k}} + ik_\parallel v_\parallel)$ are even functions of $k_\parallel$ and the imaginary part of $1/(s_{\mathbf{k}} + ik_\parallel v_\parallel)$ is an odd function of $(k_\parallel)$

$$\left(\frac{e}{m}\right)^2\frac{L}{2\pi}\int dk_\parallel\,\frac{|E_\parallel(\mathbf{k}_\perp, k_\parallel)|^2}{(s_{\mathbf{k}}+ik_\parallel v_\parallel)}$$
$$= \left(\frac{e}{m}\right)^2\frac{L}{2\pi}\,\frac{\pi}{v_\parallel}\int_{-\infty}^{\infty}dk_\parallel\,|E_\parallel(\mathbf{k}_\perp, k_\parallel)|^2\,\delta\!\left(k_\parallel - \frac{\omega_{\mathbf{k}}}{v_\parallel}\right)$$
$$= \left(\frac{e}{m}\right)^2\frac{L}{v_\parallel}\left|E_\parallel\!\left(\mathbf{k}_\perp, \frac{\omega_{\mathbf{k}}}{v_\parallel}\right)\right|^2. \tag{15}$$

Denoting $E_\parallel(\mathbf{k}_\perp, \omega_{\mathbf{k}}/v_\parallel)$ by $E_\parallel(\mathbf{k}_\perp, v_\parallel)$ the equations of motion become

$$\frac{\partial\,|E(\mathbf{k}_\perp, v_\parallel)|^2}{\partial t} = \alpha\,|E|^2\,\frac{\partial g}{\partial v_\parallel}, \tag{16}$$

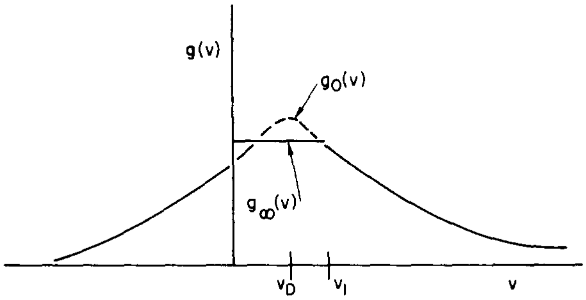

FIG. 1. Plot of initial velocity distribution along the magnetic field $g_0(v)$ and the final velocity distribution $g_\infty(v)$.

$$\frac{\partial g(v_\parallel, t)}{\partial t} = \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \frac{L}{v_\parallel} \sum_{\mathbf{k}\perp} |E(\mathbf{k}_\perp, v_\parallel)|^2 \left(\frac{k_\parallel}{k}\right)^2 \frac{\partial g}{\partial v_\parallel}. \quad (17)$$

As discussed in Sec. II, γ has a broad maximum near $k_\perp = k_{\perp_0} \cong 1.5\Omega_i/v_i$ and this leads, after many e-folding times, to a very sharp resonance of $|E(\mathbf{k}_\perp, v_\parallel)|^2$ near $k_\perp = k_{\perp_0}$. Making use of this and

$$\frac{k_\parallel}{k} \cong \frac{k_\parallel}{k_\perp} \cong \frac{v_i}{1.5 v_\parallel}$$

we can sum Eqs. (16) and (17) over $\mathbf{k}_\perp$ to obtain

$$\partial \mathcal{E}(v_\parallel)/\partial t = \alpha_0 \mathcal{E}(v_\parallel)\, \partial g/\partial v_\parallel, \quad (18)$$

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial v} \beta \mathcal{E}(v_\parallel) \frac{\partial g}{\partial v}, \quad (19)$$

where

$$\mathcal{E}(v_\parallel) = \sum_{\mathbf{k}\perp} |E(\mathbf{k}_\perp, v_\parallel)|^2, \qquad \alpha_0 = \alpha(k_{\perp_0})$$

$$\beta = \frac{1}{2.2}\frac{L}{\pi}\left(\frac{e}{m}\right)^2\left(\frac{v_i}{v_\parallel}\right)^2. \quad (20)$$

The temporal behavior of the unstable waves can be described as follows.[2] At $t = 0$ the electric fluctuations are assumed to consist of random noise, and initially those waves with $0 \lesssim \omega_k/k_\parallel < v_D$ grow with the linearized growth rate,[4] and the spectrum is very peaked about the fastest growing waves. After $\mathcal{E}(\mathbf{k}_\parallel v_\parallel)$ has grown to a sufficient amplitude the distribution function begins to diffuse according to Eq. (19). This tends to flatten the distribution function at the velocity corresponding to the fastest growing wave, and consequently to steepen the distribution function on either side of this point. This in turn increases the growth rate of those waves on either side of the fastest growing wave while decreasing the growth rate of the fastest growing wave. Thus as the distribution function diffuses the spectrum of waves spreads.

As discussed in reference 2, the asymptotic result is

$$\lim_{t\to\infty} g(v_\parallel, t) = g_\infty \quad \text{for} \quad 0 < v_\parallel < v_1,$$

$$= g_0(v) \quad \text{for} \quad v_\parallel < 0, \quad v_\parallel > v_1, \quad (21)$$

where g_∞ is a constant and together with v_1 is determined by

$$\int_0^{v_1} [g_\infty - g_0(v)] = 0, \qquad g_\infty = g_0(v_1), \quad (22)$$

and

$$g_0(v_\parallel) = \int f_0(\mathbf{v}) v_\perp\, dv_\perp\, d\phi.$$

The accompanying equilibrium spectrum of waves is given by

$$\mathcal{E}_\infty(v) = \frac{\alpha}{\beta}\int_0^v [g_\infty - g_0(v')]\, dv. \quad (23)$$

The asymptotic distribution function and the equilibrium wave spectrum are illustrated in Figs. 1 and 2.

IV. DIFFUSION COEFFICIENT

The equilibrium spectrum of fluctuations gives rise to a diffusion of particles across the field lines and as will be shown this is a result of a resonance between particle and wave velocities of the type which leads to the velocity diffusion given by Eq. (19). This relation can be exploited to give a heuristic derivation of the diffusion coefficient.

It is well known that for sufficiently slowly varying fields the drift velocity across the field is

$$\frac{d\mathbf{r}}{dt} = c\frac{\mathbf{E}(t) \times \mathbf{B}}{B^2} = c\frac{E_\perp(t)}{B}$$

and

$$D_r t = \langle(\Delta r)^2\rangle = \left\langle\left(\frac{c}{B}\right)^2\left(\int_0^t E_\perp(t')\, dt'\right)^2\right\rangle, \quad (24)$$

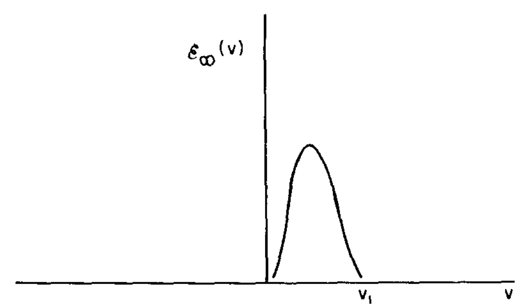

FIG. 2. Equilibrium spectrum of plasma waves as a function of their phase velocity, v.

[4] Actually there is a lower limit, v_0, such that waves with $\omega_k/k_\parallel < v_0$ do not grow. However v_0 is small compared to v_D if v_D is substantially greater than the critical value for instability and we thus neglect v_0 compared to v_D.

where D_r is the spatial diffusion coefficient perpendicular to the field lines, and $\langle\ \rangle$ indicates an ensemble average. Similarly the diffusion of particles in velocity space is determined by

$$dv_{\parallel}/dt = (e/m)E_{\parallel}(t)$$

and

$$D_v t = \langle(\Delta v)^2\rangle = \left(\frac{e}{m}\right)^2\left\langle\left(\int_0^t E_{\parallel}(t')\,dt'\right)^2\right\rangle \qquad (25)$$

Making use of the fact that $E_{\perp}/E_{\parallel} = k_{\perp}/k_{\parallel} \cong v/v_i$, we have

$$\begin{aligned} D_r &= \frac{1}{t}\left(\frac{c}{B}\right)^2\left\langle\left(\int \frac{E_{\perp}}{E_{\parallel}} E_{\parallel}\,dt'\right)^2\right\rangle \\ &= \frac{1}{t}\left(\frac{c}{B}\right)^2\left(\frac{k_{\perp}}{k_{\parallel}}\right)^2\left\langle\left(\int E_{\parallel}(t')\,dt'\right)^2\right\rangle \\ &= \rho_e^2\left(\frac{v}{v_e}\right)^2 \frac{1}{v_i^2} D_v. \end{aligned} \qquad (26)$$

Now $D_v \cong (\Delta v_{\parallel})^2/\tau$ when we expect $(\Delta v_{\parallel})$ to be of the order of v_D and $1/\tau$ to be of the order of γ. Thus $D_v \propto v_D^2\gamma$ and thus (taking $v \cong v_D$)

$$D_r \propto \rho_e^2\Omega_e(T_e/T_i)^2(v_D/v_e)^5. \qquad (27)$$

Since only those electrons with $0 < v < v_D$ can resonate with the plasma waves only these electrons diffuse. Since these represent only a fraction of the order of v_D/v_e of the total number of electrons the average diffusion coefficient goes more nearly as $(v_D/v_e)^6$. Thus we are able to obtain the dependence of D_r on the parameters in a simple way and we note that $D_r \propto \rho_e^2\Omega_i \equiv D_B$, the Bohm diffusion coefficient.

A more rigorous derivation of this result which also determines the coefficient of proportionality is given below. For electric fields varying slowly compared to the electron cyclotron frequency the electrons drift across the magnetic field with a velocity $v_{\perp} = cE_{\perp}/B$ and thus

$$\begin{aligned} r_{\perp} &= \frac{c}{B}\sum_{\mathbf{k}}\int_0^t E_{\perp}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}_0) \\ &\qquad \cdot\exp[i(k_{\parallel}v_{\parallel} - \omega_k)t]\,dt \\ &= \frac{c}{B}\sum_{\mathbf{k}} E_{\perp}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}_0) \\ &\qquad \cdot\frac{\exp[i(k_{\parallel}v_{\parallel} - \omega_k)t] - 1}{i(k_{\parallel}v_{\parallel} - \omega_k)}, \end{aligned} \qquad (28)$$

and

$$\langle(r_{\perp})^2\rangle = \left(\frac{c}{B}\right)^2 \sum_k |E_{\perp}(k)|^2\left\{\frac{\sin[\frac{1}{2}(k_{\parallel}v_{\parallel} - \omega_k)t]}{\frac{1}{2}(k_{\parallel}v_{\parallel} - \omega_k)}\right\}^2, \qquad (29)$$

where $\langle\ \rangle$ denotes an average over the initial phases and we have assumed that $\langle\sum_{\mathbf{k}'} E_{\mathbf{k}}\bar{E}_{\mathbf{k}'}\rangle = |E_{\mathbf{k}}|^2$. For large t the function

$$\{\sin[\tfrac{1}{2}(k_{\parallel}v_{\parallel} - \omega_k)t]/\tfrac{1}{2}(k_{\parallel}v_{\parallel} - \omega_k)\}^2$$

is sharply peaked about $k_{\parallel}v_{\parallel} - \omega_k = 0$ and

$$\sum_k \to L/2\pi \int dk_{\parallel}$$

can be evaluated by taking advantage of this to give

$$\langle r_{\perp}^2\rangle = 2(c/B)^2(L/v)\mathcal{E}(v)t, \qquad D_r = 2(c/B)^2(L/v)\mathcal{E}(v). \qquad (30)$$

We see here that only those terms for which $k_{\parallel}v_{\parallel} = \omega_k$ lead to time proportional diffusion and thus it is a resonance between particle and phase velocities which leads to diffusion.

Inserting $\mathcal{E}_{\infty}(v)$ from Eq. (23) we have

$$D_r = 8.9\pi\Gamma_0\left(\frac{T_e}{T_i}\right)^2\left(\frac{v_{\parallel}}{v_e}\right)^2\int\left(\frac{g_{\infty} - g_0}{n}\right)dv. \qquad (31)$$

For $v_D \ll v_e$

$$\int\frac{g_{\infty} - g_0}{n}\,dv \cong A\left(\frac{v_D}{v_e}\right)^3 \qquad (32)$$

and

$$D_r = A\left(\frac{v_D}{v_e}\right)^5\left(\frac{T_e}{T_i}\right)^2\rho_e^2\Omega_e, \qquad (33)$$

in agreement with our earlier results.

The diffusion of ions is more complicated and is not considered in the present paper. For a given physical situation, however, the ion diffusion and the need for charge neutrality must be included.

V. DISCUSSION

We have shown that this instability near the ion-cyclotron frequency reduces the critical current which can be drawn parallel to the field without producing instability by about an order of magnitude in the case of equal temperature as compared to previous theories which consider only $k_{\perp} = 0$. This appears to be the explanation of the results of D'Angelo and co-workers who have observed instabilities in a cesium plasma and the critical current they observe is in agreement with results of Sec. II. In addition, the measured frequency of the unstable oscillations was $\omega \cong 1.2\Omega_i$, also in agreement. Nonetheless, this explanation is not completely satisfactory since in the experiment $\Omega_i\tau_{\text{coll}} \cong 10$ where τ_{coll} is the ion–ion collision time and in view of the low predicted growth rates it is difficult to see why collisional damping would not dominate.

The diffusion coefficient obtained is small compared to the "Bohm diffusion coefficient", $\rho_e \Omega_e^2$, although it has the same dependence on magnetic field, i.e., $D \propto 1/B$. Indeed for the stellarator $v_D/v_e \ll 1$ while the experimental result is $D_{ex} \cong \rho_e^2 \Omega_e$, and it thus appears that "pumpout" is not due to this type of instability, at least in its simplest form. There is, however, the possibility that the particle collisions couple the external electric field (which produces the drift) to the fluctuations, leading to a much higher fluctuation amplitude. To answer this question, however, one must solve the nonlinear Fokker–Planck equation.

In the absence of such an external source of energy it seems unlikely that any microinstabilities for which $\gamma/\omega \ll 1$ will produce large macroscopic diffusion for the simple reason that the energy fed into electric fluctuations is small, i.e., or order γ/ω. For the case at hand, it is *a fortiori* small since, as remarked in Appendix A, only the fraction $(kL_D)^2 \ll 1$ of the energy given up by the electrons goes into electric field, the bulk of the energy going into ion kinetic energy.

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APPENDIX A

We consider an infinite homogeneous plasma with a magnetic field B in the direction of the z axis. As discussed by many authors, e.g., Bernstein,[5] the perturbed electron distribution function is given in the linearized theory by

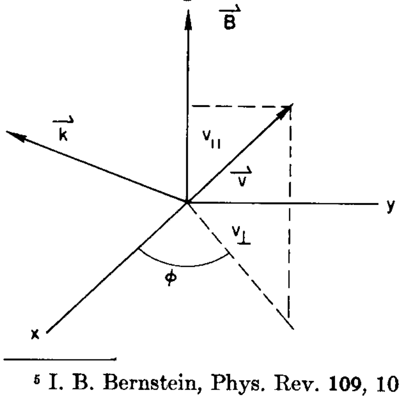

FIG. 3. Coordinate system showing the components of the velocity.

$$f_k = \frac{e}{m} \sum_n J_n(\lambda_e) \exp(in\phi - i\lambda_e \sin\phi) \cdot \left\{ \frac{E_\perp}{2} \left[\frac{e^{i\phi}}{s + ik_\parallel v_\parallel + i(n+1)\Omega} + \frac{e^{-i\phi}}{s + ik_\parallel v_\parallel + i(n-1)\Omega} \right] \frac{\partial}{\partial v_\perp} + \frac{E_\parallel}{s + ik_\parallel v_\parallel} \frac{\partial}{\partial v_\parallel} \right\} f_0(v_\perp, v_\parallel), \quad \text{(A1)}$$

where J_n is the usual Bessel function, $s = -i\omega$ and $\lambda_e = (k_\perp \rho_e)$. For $k\rho_i \cong 1$, $s \cong -i\Omega_i$ we have $\lambda_e \ll 1$, $s + ik_\parallel v_\parallel \ll i\Omega_e$ for all but the fastest electrons and we can neglect all terms but $n = 0$ to obtain

$$f_k = \frac{e}{m} \left[\frac{E_\perp}{\Omega_e} \sin\phi \frac{\partial}{\partial v_\perp} + \frac{E_\parallel}{(s + ik_\parallel v_\parallel)} \frac{\partial}{\partial v_\parallel} \right] f_0(v_\perp, v_\parallel). \quad \text{(A2)}$$

The electronic charge density is given by

$$\rho_{k_e} = -e \int d^3v f_k = -\frac{e^2}{m} E_\parallel \int_{-\infty}^{\infty} \frac{\partial g_0/\partial v_\parallel}{(s + ik_\parallel v_\parallel)} dv_\parallel \quad \text{(A3)}$$

where $g_0 = \int f_0(\mathbf{v}) v_\perp \, dv_\perp \, d\phi$.

Similarly the perturbed ion distribution function is given by

$$F_k = -\frac{e}{M} \sum_{n=-\infty}^{\infty} J_n(-\lambda_i) \exp(in\phi + i\lambda_i \sin\phi) \cdot \left\{ \frac{E_\perp}{2} \left[\frac{e^{i\phi}}{(s + ik_\parallel v_\parallel) - i(n+1)\Omega_i} + \frac{e^{-i\phi}}{(s + ik_\parallel v_\parallel) - i(n-1)\Omega_i} \right] \frac{\partial}{\partial v_\perp} + \frac{E_\parallel}{(s + ik_\parallel v_\parallel) - in\Omega_i} \frac{\partial}{\partial v_\parallel} \right\} F_0 \quad \text{(A4)}$$

Taking $F_0(v_\perp, v_\parallel)$ to be a Maxwellian yields

$$\rho_{k_i} = E_k \frac{\omega_{pi}^2}{4\pi} \sum_{n=-\infty}^{\infty} \Gamma_n[(k_\perp \rho_i)^2] \frac{i}{kv_i^2} \left\{ W\left(-\frac{is + n\Omega}{k_\parallel v_i} \right) + \frac{n\Omega}{is - n\Omega} \left[1 + W\left(-\frac{is + n\Omega}{k_\parallel v_i} \right) \right] \right\}, \quad \text{(A5)}$$

where Γ_n is defined in Sec. II.

If the particles have a drift velocity v_{D_i} we must replace s by $s + ik_\parallel v_{D_i}$. Thus using $\nabla \cdot \mathbf{E} \to ikE_k = 4\pi(\rho_{e_k} + \rho_{i_k})$, and $s = -i\omega$, we obtain the dispersion relation, Eq. (2).

It is worth noting that only a small fraction, $(kL_{D_e})^2 \ll 1$, of the energy which is given up by those electrons in resonance with the waves, i.e., those electrons with $v_\parallel = \omega/k_\parallel$, goes into electro-

[5] I. B. Bernstein, Phys. Rev. **109**, 10 (1958).

static energy, and the bulk of the energy goes into the kinetic energy of wave motion. This is in contrast to the case of electron plasma oscillation for which the energy from resonance particles is evenly divided between kinetic and potential energy.[2] To see this we note that the energy transfer is proportional to the "in-phase" part of the current $\mathbf{j}$ and that $j_k = -(s/ik)\rho_k$. The rate of change of energy is thus

$$\frac{\partial U}{\partial t} = -\frac{\partial}{\partial t} \sum_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} = \sum_{\mathbf{k}} \mathbf{E}_{-\mathbf{k}} \cdot \mathbf{j}_{\mathbf{k}}$$

$$= -\sum_{\mathbf{k}} |E_{\mathbf{k}}|^2 \frac{s}{4\pi} \sum_j \left(\frac{\omega_{pj}}{kv_j}\right)^2 \cdot \left\{-1 + i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{-\omega + ik_\parallel v_D}{v_\parallel v_e}\right) + \Gamma_i\left[\left(\frac{k_\perp v_i}{\Omega_i}\right)^2\right] \frac{\Omega_i}{is - \Omega_i}\right\}, \tag{A6}$$

where

$$U = \sum_j \frac{1}{2} m_j \int v^2 f_j(\mathbf{v})\, d^3v$$

is the kinetic energy of the particles and we have taken only $n = 0$ for electrons and $n = 1$ for ions. For each $\mathbf{k}$ the real part is given by

$$\left.\frac{dU}{dt}\right)_{\mathbf{k}} = \frac{|E_{\mathbf{k}}|^2}{4\pi(kL_{De})^2}\left\{+\gamma - \omega\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(-\frac{\omega + k_\parallel v_D}{k_\parallel v_e}\right) + \gamma\left(\frac{\Omega}{\omega - \Omega}\right)^2 \Gamma_1 \frac{T_e}{T_i}\right\}$$

$$= \frac{|E_{\mathbf{k}}|^2}{4\pi(kL_{De})^2}\left\{\gamma - \gamma \frac{\omega T_i}{\Omega \Gamma_1 T_e}[1 + (kL_{De})^2]^2 + \frac{\gamma}{\Gamma_1}\frac{T_i}{T_e}[1 + (kL_{De})^2]^2\right\}, \tag{A7}$$

where we have evaluated γ and ω from Eq. (2) without neglecting $(kL_{De})^2 \ll 1$.

$$\left(\frac{\omega - \Omega_i}{\Omega}\right) = \frac{T_e}{T_i}\frac{\Gamma_1}{1 + (kL_{De})^2},$$

$$\gamma = \frac{T_e}{T_i}\frac{\Gamma_1 \Omega}{[1 + (kL_{De})^2]^2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(-\frac{\omega + k_\parallel v_D}{k_\parallel v_e}\right). \tag{A8}$$

The first term in the curly brackets in Eq. (A7) comes from the bulk of the electrons and represents the energy fed into the electron kinetic energy of the waves. The second term comes from the resonant electrons and represents the energy drawn from these particles. The third term is the energy fed into the ion kinetic energy of the wave. Note that the terms from the resonant electrons and from the ions are both larger by a factor of $1/\Gamma_1$ than the electron kinetic energy term and their difference is $-\gamma[1 + (kL_{De})^2]$. Thus the total change in particle energy is just

$$\left.\frac{dU}{dt}\right)_{\mathbf{k}} = -\gamma\frac{|E_{\mathbf{k}}|^2}{4\pi} = -\frac{\partial}{\partial t}\frac{|E_{\mathbf{k}}|^2}{8\pi}. \tag{A9}$$

We may describe this as follows. The resonant electrons give up energy to the waves at a rate of order $\gamma/\Gamma_1(kL_{De})^2$. The waves in turn feed energy into the ions at a rate of $\gamma/\Gamma_1(kL_{De})^2$ and into the bulk of the electrons at a rate of order $\gamma/(kL_{De})^2$. The *net* rate of electrostatic energy change is, however, of order $\gamma \ll \gamma/(kL_{De})^2$. Thus the ion-cyclotron waves have only the fraction $(kL_{De})^2 \ll 1$ of their energy as potential energy.