Lecture #4 Boldyrev's Theory for Strong MHD Turbulence

I. Review of GS95 Strong MHD Turbulence Theory

A. Assumptions

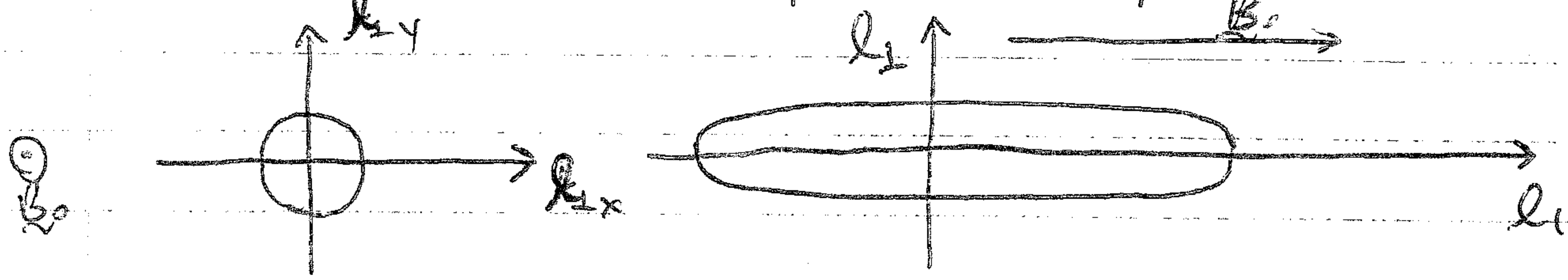
1. Kolmogorov Hypothesis: a. Local energy transfer
   b. Constant energy cascade rate
2. Anisotropic Cascade:
   Nonlinear turbulence frequency determined by perpendicular dynamics, $\omega_{nl} \sim k_\perp v_k$
3. Critical Balance between linear and nonlinear timescales,
   $\omega \sim \omega_{nl}$

B. Predictions:

1. $\omega_{nl} \sim k_\perp v_k$ [where $v_k \equiv v_\perp(k_\perp)$]
2. $v_k = \epsilon_0^{\frac{1}{3}} k_\perp^{-\frac{1}{3}}$
3. $E_{k_\perp} = \epsilon_0^{2/3} k_\perp^{-5/3}$, Goldreich-Sridhar spectrum $\propto k_\perp^{-5/3}$
4. $k_\parallel = k_0^{\frac{1}{3}} k_\perp^{2/3}$, Scale-dependent anisotropy, $k_\parallel \propto k_\perp^{\frac{2}{3}}$

C. NOTE:

1. Isotropic dynamics in perpendicular plane



   a. Thus, turbulence structures at small scales (with $k_\perp \gg k_\parallel$) are filamentary, or cigar-shaped, with elongation along the mean magnetic field.
2. The Boldyrev theory finds anisotropy in the perpendicular plane.

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II. Boldyrev's Theory

A. Motivation:

1. Although the GS95 theory for strong MHD turbulence accomplished a great stride forward in our fundamental understanding and the inevitability of anisotropy, detailed numerical simulations of MHD turbulence found spectra that appeared to scale like $k_\perp^{-3/2}$, **not** $k_\perp^{-5/3}$.
   (We'll review these simulation results in a later lecture).
2. This spectrum disagreed ~~both~~ with both:
   a. Goldreich-Sridhar 95, $E(k_\perp) \propto k_\perp^{-5/3}$, anisotropic
   b. Iroshnikov-Kraichnan $E(k) \propto k^{-3/2}$, isotropic.
3. Studies of decaying MHD turbulence (as opposed to driven turbulence) find a tendency towards dynamic alignment, where the fluctuations approach the state of either

$$\underset{\sim}{v}_\perp(\underset{\sim}{r}) = \underset{\sim}{b}_\perp(\underset{\sim}{r}) \quad \text{or} \quad \underset{\sim}{v}_\perp(\underset{\sim}{r}) = -\underset{\sim}{b}_\perp(\underset{\sim}{r})$$

   a. $\underset{\sim}{z}^+ = 0,\ \underset{\sim}{z}^- = 0$ or $\underset{\sim}{z}^+ = 0,\ \underset{\sim}{z}^- \neq 0$

   b. Nonlinear interaction is zero in either of these states!

$$(\underset{\sim}{z}^- \cdot \nabla)\underset{\sim}{z}^+ \quad \text{or} \quad (\underset{\sim}{z}^+ \cdot \nabla)\underset{\sim}{z}^-.$$

4. The Boldyrev theory proposes that this tendency to approach dynamic alignment occurs also in driven MHD turbulence, but ~~the~~ the turbulence may only achieve an imperfect (and scale-dependent) alignment while maintaining a constant energy flux to small scales.

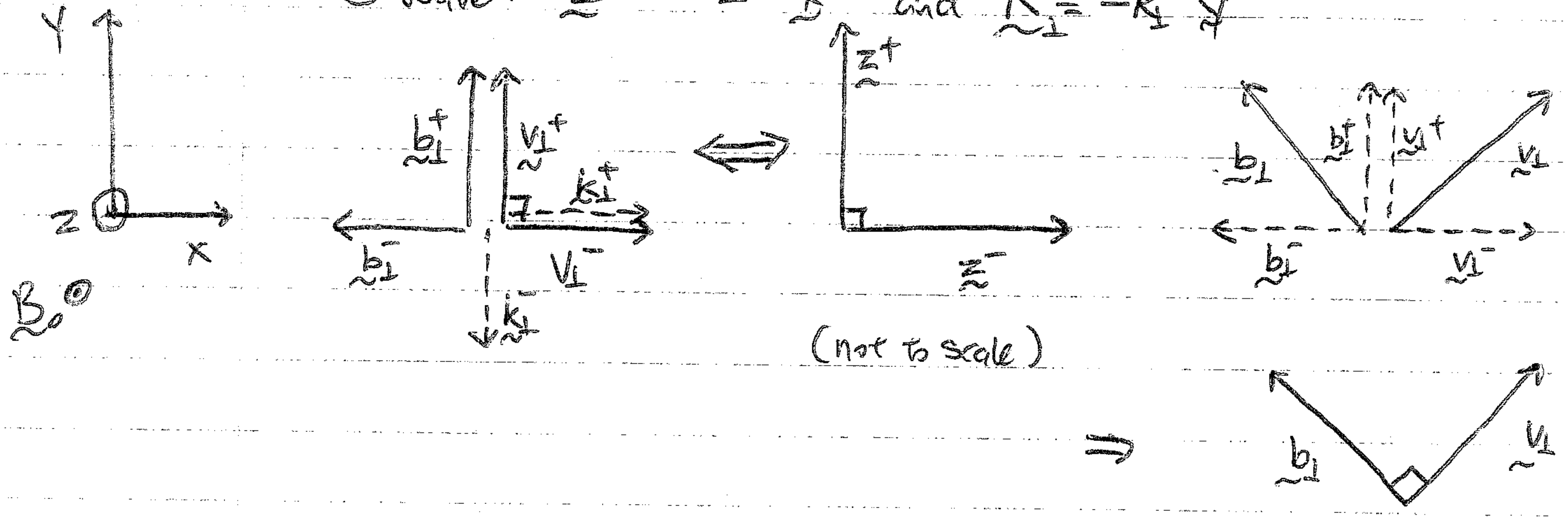
II (Continued)

B. Geometry of Nonlinear Interactions:

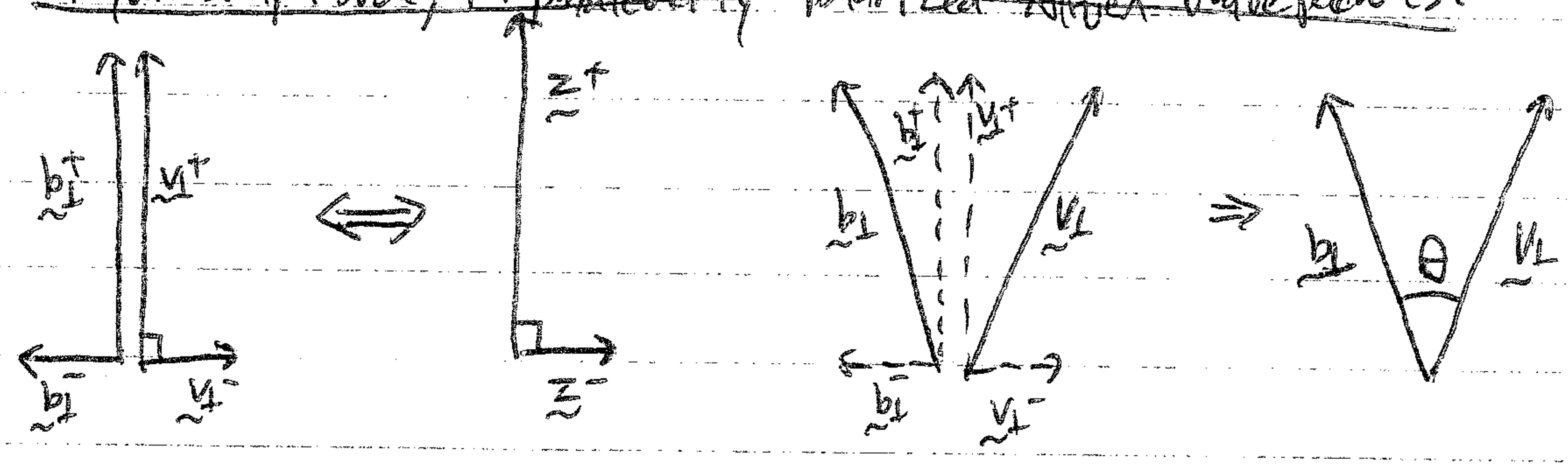
1. Equal Amplitude, Perpendicularly Polarized Alfven Wavepacket Collisions

a. ⊕ wave: $\mathbf{z}^+ = z^+ \hat{y}$ and $\mathbf{k}_\perp^+ = k_\perp^+ \hat{x}$

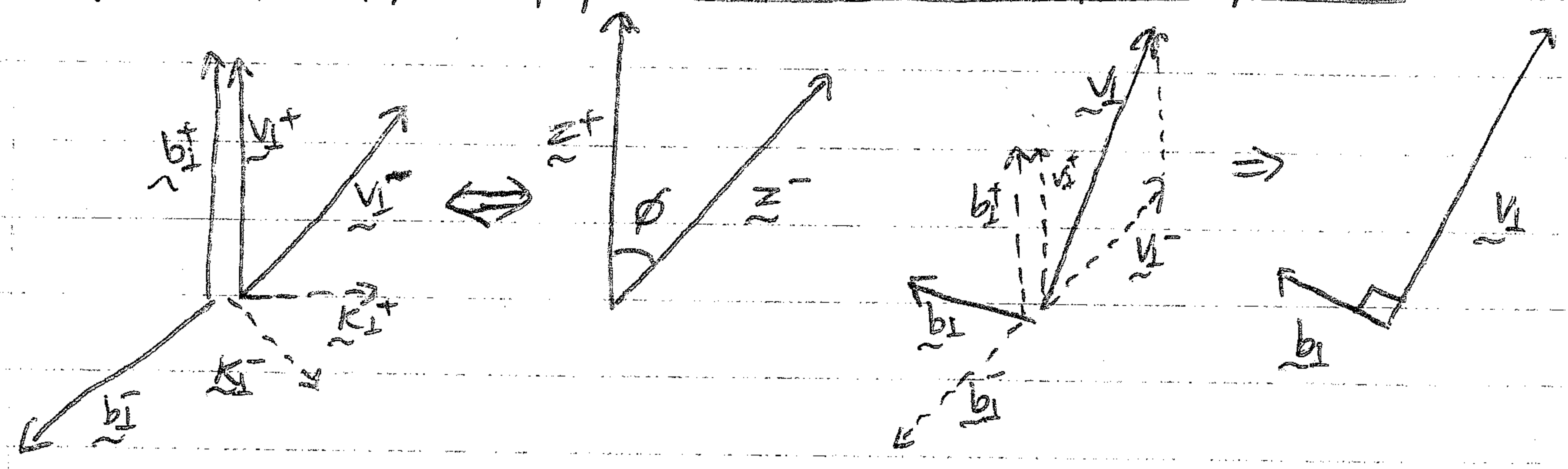
⊖ wave: $\mathbf{z}^- = z^- \hat{x}$ and $\mathbf{k}_\perp^- = -k_\perp^- \hat{y}$



2. Unequal Amplitude, Perpendicularly Polarized Alfven Wavepackets:



3. Equal Amplitude, Non-perpendicular Polarized Alfven wave packets.



II. (Continued)

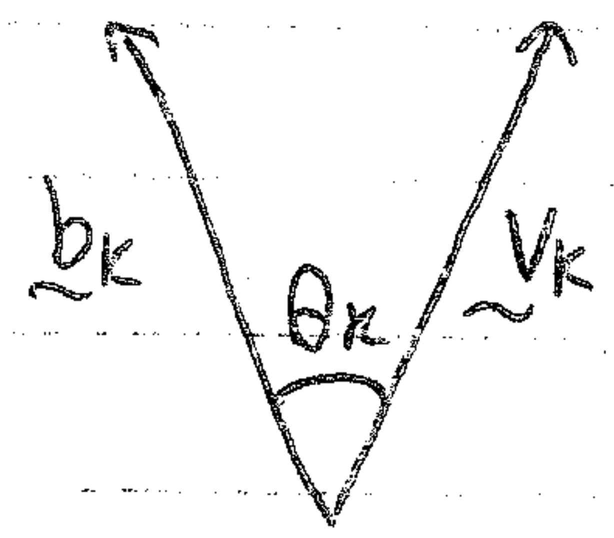
C. Boldyrev's Approach:

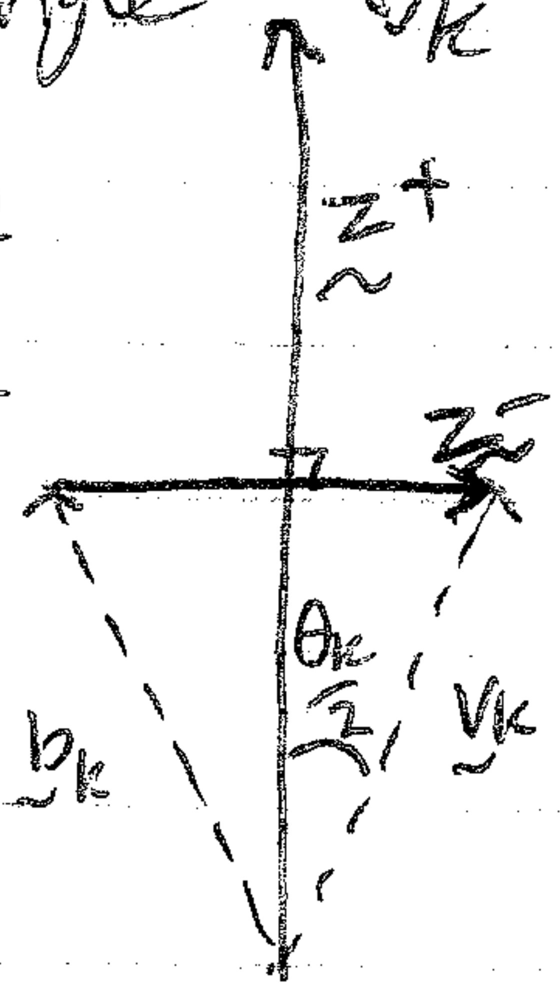
1a. Assume that, for fluctuations at same perpendicular scale $k_\perp$, magnetic (b_k) & velocity (v_k) field fluctuations are aligned by some angle θ_k

$\underset{\sim}{z}^+ = \underset{\sim}{v}_\perp + \underset{\sim}{b}_\perp$

$\underset{\sim}{z}^- = \underset{\sim}{v}_\perp - \underset{\sim}{b}_\perp$

$\Rightarrow$





b. Thus $|\underset{\sim}{z}^+| = 2 v_K \cos\left(\frac{\theta_k}{2}\right)$

and $|\underset{\sim}{z}^-| = 2 v_k \sin\left(\frac{\theta_k}{2}\right)$

NOTE:

$\sin\theta$	θ
1	$\frac{\pi}{2} \simeq 1.57$
0.707	$\frac{\pi}{4} \simeq 0.785$
0.5	$\frac{\pi}{6} \simeq 0.524$
0.383	$\frac{\pi}{8} \simeq 0.393$

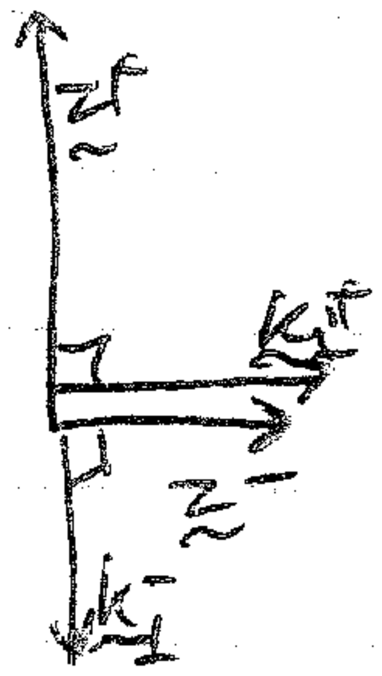
c. For $\theta_k \ll 1$, $\sin\theta_k \simeq \frac{\theta_k}{2}$

and $\cos\left(\frac{\theta_k}{2}\right) \simeq 1$

d. Thus, $z^+ \sim 2 v_k$

$z^- \sim v_K \theta_K$

2. The nonlinear term is $(\underset{\sim}{z}^\mp \cdot \nabla)\underset{\sim}{z}^\pm$:



a. Thus (dropping the 2), $(\underset{\sim}{z}^- \cdot \nabla)\underset{\sim}{z}^+ \sim v_k^2 k_\perp^+ \theta_K$

b. Equivalently, $\omega_{nl} \sim |\underset{\sim}{z}^- \cdot \nabla| \sim k_\perp^+ v_k \theta_K$

c. Since we have assume local interactions in scale-space, $k_\perp^+ \sim k_\perp^- \sim k_\perp$, so we have $\boxed{\omega_{nl} \sim k_\perp v_k \theta_K}$

NOTE: This differs from GS95 by the factor $\theta_k \ll 1$.

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II. C. (Continued)

3. Assume Constant Energy Cascade Rate:

a. $\epsilon \sim v_k^2 \omega_{nl} \sim v_k^3 k_\perp \theta_k = \epsilon_0$

b. $\boxed{v_k = \epsilon_0^{1/3} \theta_k^{-1/3} k_\perp^{-1/3}}$ Thus $v_k \propto \theta_k^{-1/3} k_\perp^{-1/3}$

4. Critical Balance: Linear $\sim$ Nonlinear frequencies

$\omega \sim \omega_{nl}$

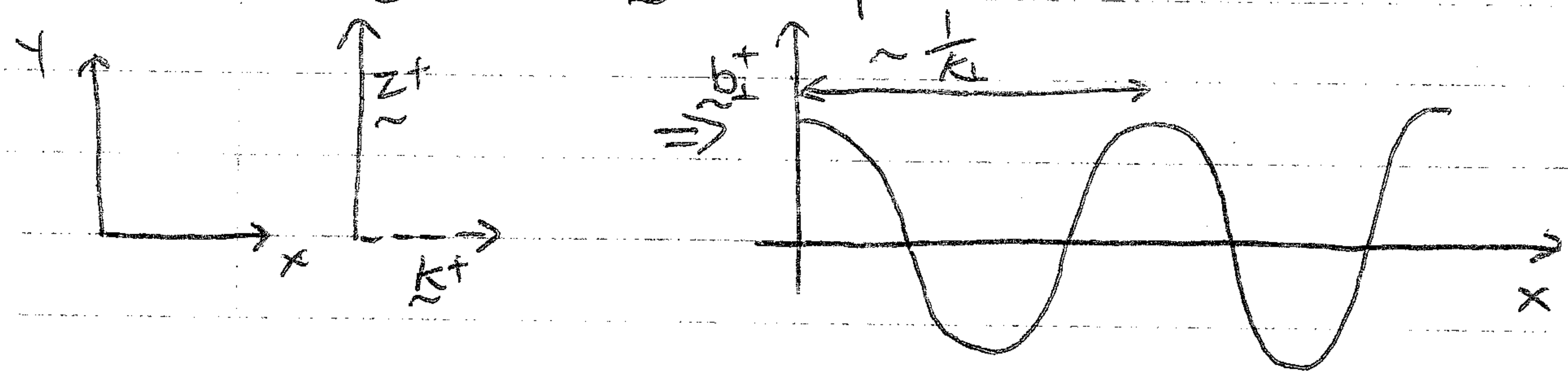
a. $\omega = k_\parallel v_A$

$\omega_{nl} \sim k_\perp v_k \theta_k$ $\Rightarrow k_\parallel v_A \sim k_\perp v_k \theta_k \sim k_\perp (\epsilon_0^{1/3} \theta_k^{-1/3} k_\perp^{-1/3}) \theta_k$

b. Thus $\boxed{k_\parallel = \frac{\epsilon_0^{1/3}}{v_A} \theta_k^{2/3} k_\perp^{2/3}}$ $k_\parallel \propto \theta_k^{2/3} k_\perp^{2/3}$

5. Structure in the $\underset{\sim}{B_0} \times \underset{\sim}{k}$ direction:

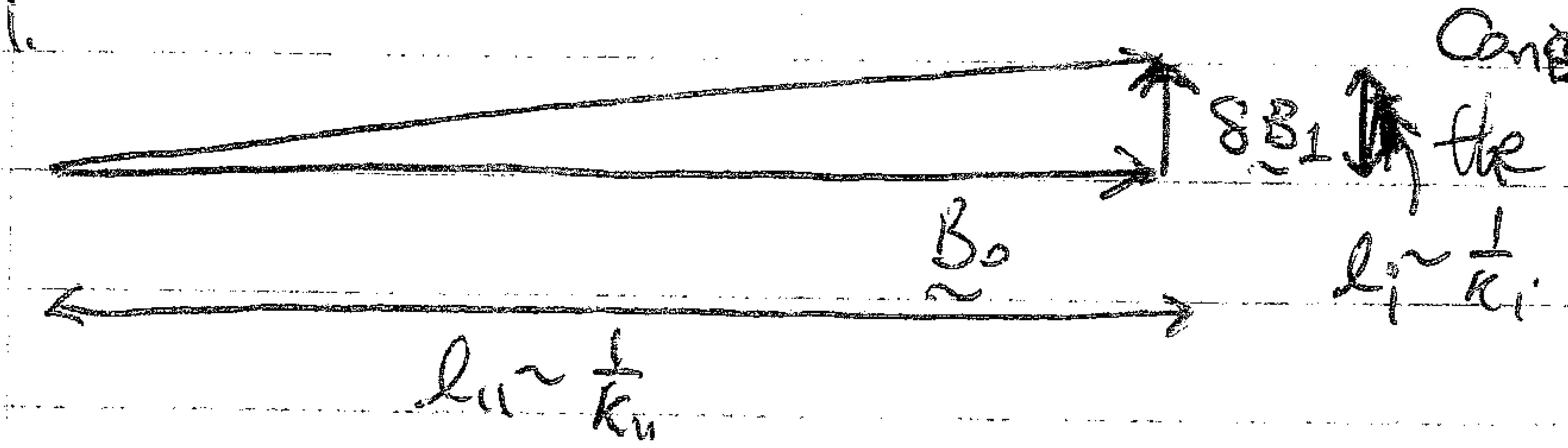
a. Consider the $\underset{\sim}{z}^+$ wave packet



Variation in $\hat{x}$ direction $\sim \frac{1}{k_\perp}$

b. What is the variation in the y direction?

i.



Consider the displacement of the magnetic field lines in y.

ii. $\frac{\delta B_\perp}{B_0} \sim \frac{l_\perp}{l_\parallel} \Rightarrow \boxed{\frac{v_k}{v_A} \sim \frac{k_\parallel}{k_\perp} \ll 1}$ Thus $\frac{k_\parallel}{k_\perp} \sim \frac{\epsilon_0^{1/3} \theta_k^{-1/3} k_\perp^{-1/3}}{v_A}$

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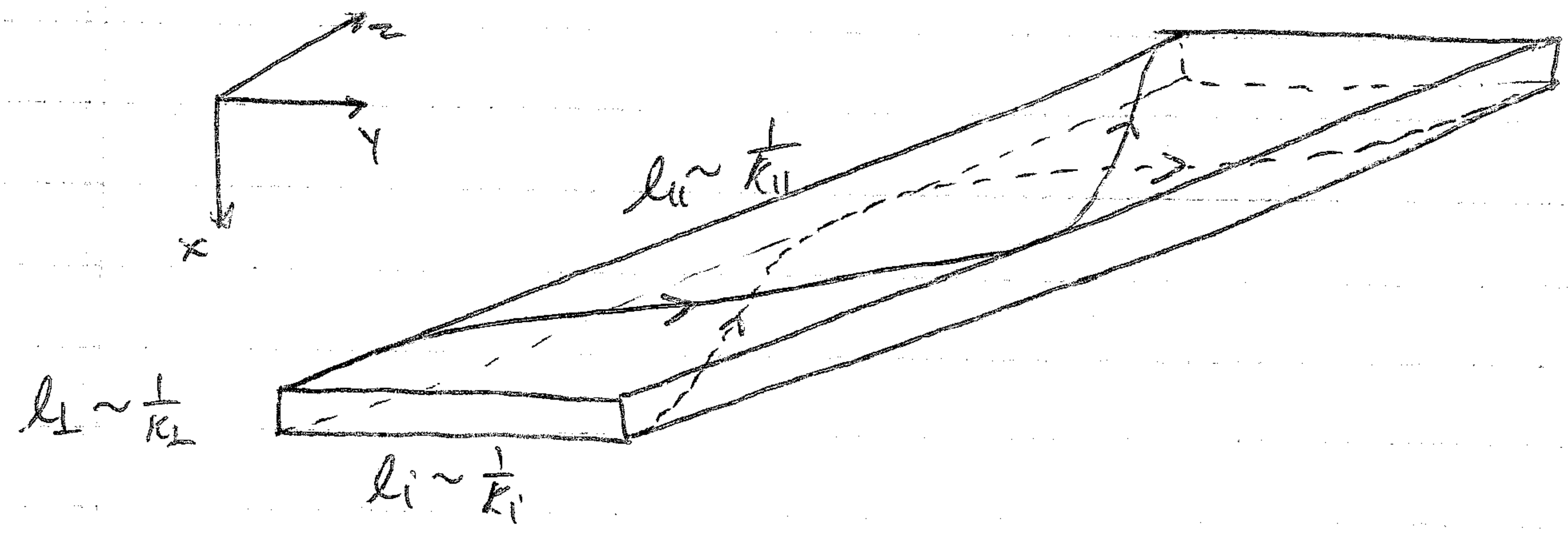
II. C. 5. b. (Continued)

$$\frac{\left(\frac{\cancel{\epsilon_0^{1/3}}\,\theta_k^{2/3}\,k_\perp^{2/3}}{\cancel{v_A}}\right)}{k_i} \sim \frac{\cancel{\epsilon_0^{1/3}}}{\cancel{v_A}}\,\theta_k^{-1/3}\,k_\perp^{-1/3} \Rightarrow \boxed{k_i \sim k_\perp \theta_k}$$

c. Thus, we find $\frac{k_\parallel}{k_i} \sim \frac{v_k}{v_A} \ll 1$ and $\frac{k_i}{k_\perp} \sim \theta_k \ll 1$, so

$$\boxed{k_\parallel \ll k_i \ll k_\perp}$$ Anisotropy in perpendicular plane as well.



6. <u>Assume All Quantities are Scale Invariant</u> (including θ_k)

a. Take $\boxed{\theta_k \propto k_\perp^{-\frac{\alpha}{3+\alpha}}}$

b. Determine scaling of v_k, k_i, and $k_\parallel$ in terms of $k_\perp$ & α:

1. $v_k \propto \theta_k^{-1/3} k_\perp^{-1/3} \propto \left[k_\perp^{\frac{\alpha}{3(3+\alpha)} - \frac{\alpha+3}{3(3+\alpha)}} \right] \propto k_\perp^{\frac{-3}{3(3+\alpha)}} \propto k_\perp^{-\frac{1}{3+\alpha}}$

$$\Rightarrow \boxed{v_k \propto k_\perp^{-\frac{1}{3+\alpha}}}$$

2. $k_i \propto k_\perp \theta_k \propto k_\perp^{\frac{3+\alpha}{3+\alpha} - \frac{\alpha}{3+\alpha}} \propto k_\perp^{\frac{3}{3+\alpha}} \Rightarrow \boxed{k_i \propto k_\perp^{\frac{3}{3+\alpha}}}$

3. $k_\parallel \propto \theta_k^{2/3} k_\perp^{2/3} \propto \left[k_\perp^{-\frac{2\alpha}{3(3+\alpha)} + \frac{2(\alpha+3)}{3(\alpha+3)}} \right] \propto k_\perp^{\frac{6}{3(3+\alpha)}} \propto k_\perp^{\frac{2}{3+\alpha}}$

$$\Rightarrow \boxed{k_\parallel \propto k_\perp^{\frac{2}{3+\alpha}}}$$

II.C.6. (Continued)

c. Thus, we have defined a one-parameter family of solutions:

1. $\theta_k \propto k_\perp^{-\frac{\alpha}{3+\alpha}}$, $v_k \propto k_\perp^{-\frac{1}{3+\alpha}}$, $k_i \propto k_\perp^{\frac{3}{3+\alpha}}$, $k_\parallel \propto k_\perp^{\frac{2}{3+\alpha}}$

2. However, α remains undetermined thus far.

3. NOTE: $\alpha = 0$ corresponds to GS95 theory:

$$\theta_k = \text{constant}, \quad v_k \propto k_\perp^{-\frac{1}{3}}, \quad \underbrace{k_i \propto k_\perp}_{\substack{\text{isotropic in} \\ \text{perpendicular plane}}}, \quad k_\parallel \propto k_\perp^{\frac{2}{3}}$$

7. Conservation of Cross Helicity

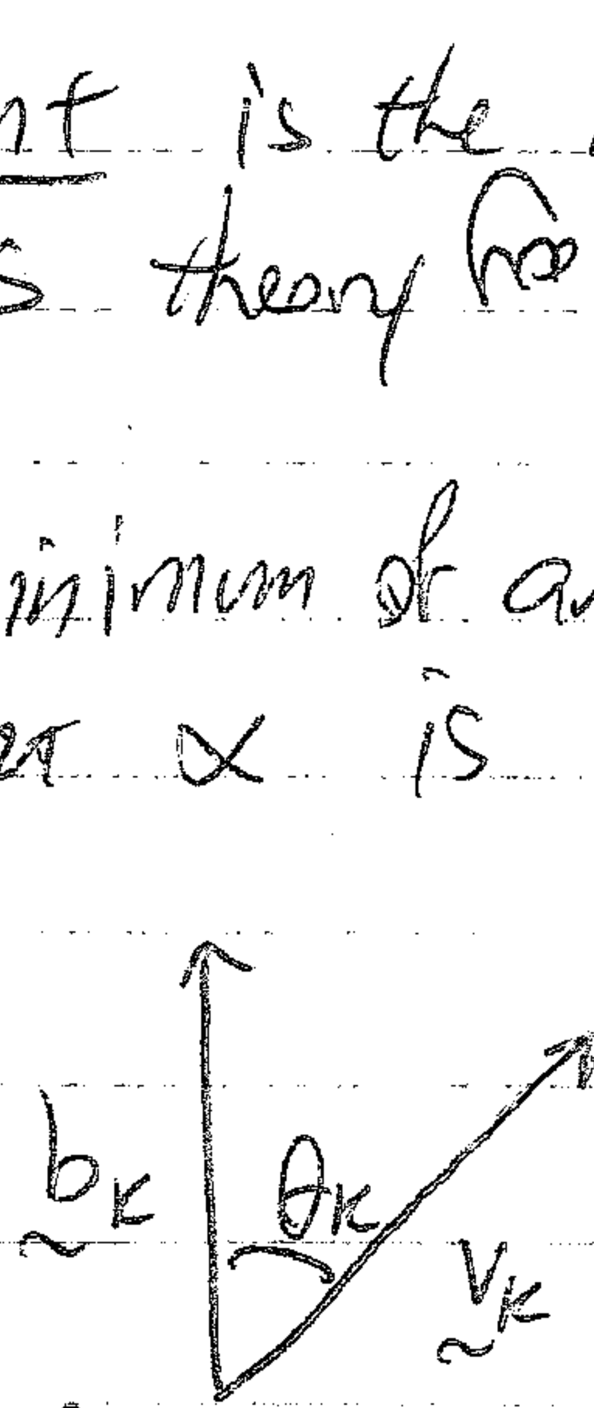
a. In incompressible MHD, $H_c \equiv \int d^3\underset{\sim}{r}\ \frac{1}{2}\, \underset{\sim}{v} \cdot \underset{\sim}{b}$
cross helicity is conserved.

b. We will choose α such that cross helicity is maximized. This is motivated by decaying MHD turbulence, in which the turbulence approaches a maximally aligned state, $\underset{\sim}{v}(\underset{\sim}{r}) = +\underset{\sim}{b}(\underset{\sim}{r})$, or $\underset{\sim}{v}(\underset{\sim}{r}) = -\underset{\sim}{b}(\underset{\sim}{r})$.

c. This concept of dynamic alignment is the new physical phenomenon distinguishing Boldyrev's theory from GS95.

d. We want maximal alignment (minimum of angular mismatch) between $\underset{\sim}{v}_k$ and $\underset{\sim}{b}_k$ as ~~[illegible]~~ α is varied.

e. In the ~~[illegible]~~ perpendicular plane,



where $\theta_k \propto k_\perp^{-\frac{\alpha}{3+\alpha}}$.

So $\alpha \to \infty$ leads to a minimum of θ_k.

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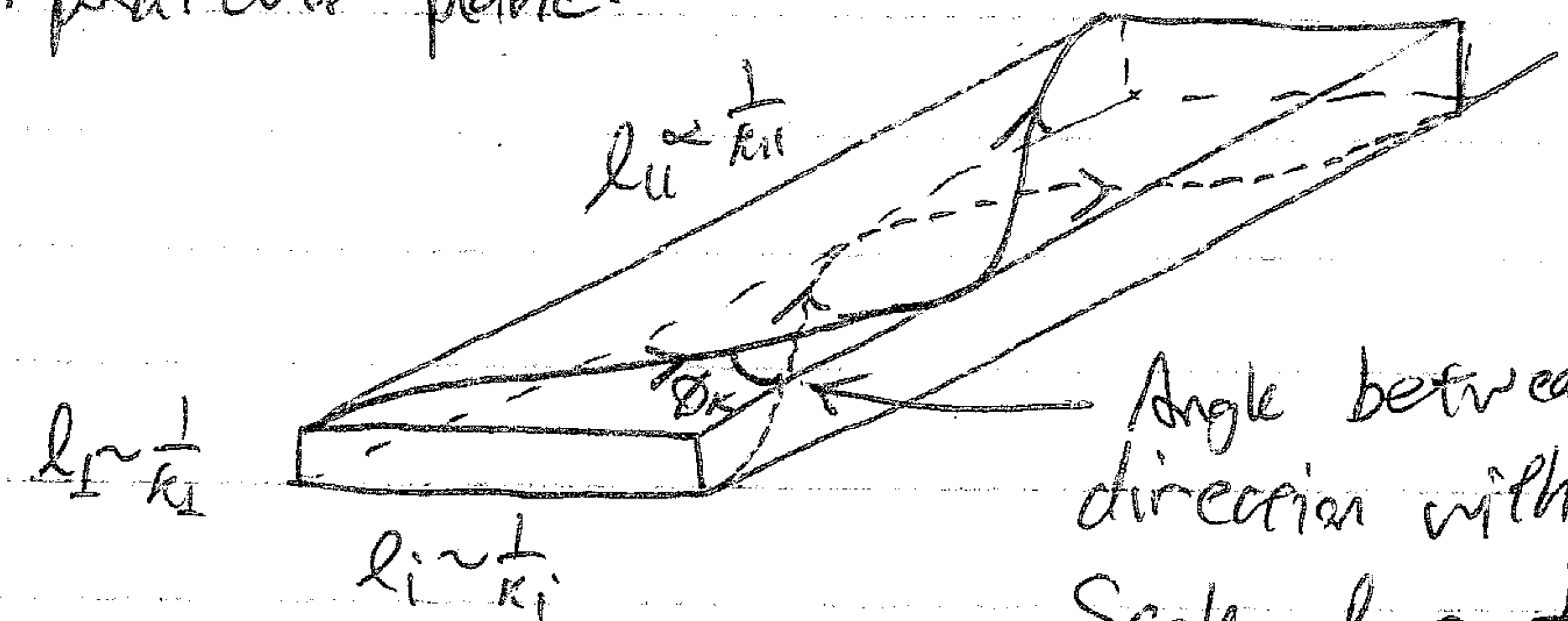
II.C.7. (Continued)

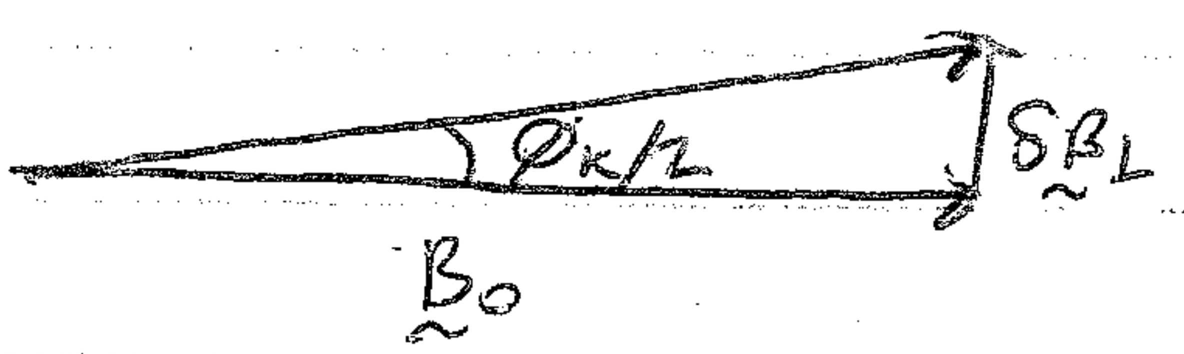
f. BUT, $\underset{\sim}{v}_k$ & $\underset{\sim}{b}_k$ are also mismatched out of the perpendicular plane:



Angle between mean magnetic field direction within perpendicular scale $l_\perp \sim \frac{1}{k_\perp}$.

$\Rightarrow$ The direction of the local magnetic field at scale $l_\perp \sim \frac{1}{k_\perp}$ cannot be defined precisely.

i. Again



$$\frac{\delta B_\perp}{B_0} \sim \frac{l_i}{l_\parallel} \sim \tan\frac{\phi_k}{2}$$

ii. Taking $\phi_k \ll 1$, $\tan\frac{\phi_k}{2} \sim \frac{\phi_k}{2}$, so $\frac{l_i}{l_\parallel} \sim \frac{k_\parallel}{k_i} \sim \frac{\phi_k}{2}$

iii. So $\phi_k \sim \frac{k_\parallel}{k_i} \sim \frac{k_\perp^{\frac{2}{3+\alpha}}}{k_\perp^{\frac{3}{3+\alpha}}} \sim k_\perp^{-\frac{1}{3+\alpha}}$

iv. The total angle is $\Theta_k = \sqrt{\theta_k^2 + \phi_k^2}$.

This angle is minimized, with respect to α, when $\alpha = 1$ ($\theta_k \simeq \phi_k$). $\Rightarrow$ $\boxed{\alpha = 1}$

8. Scalings: a. $\theta_k \propto k_\perp^{-\frac{1}{4}}$ c. $k_i \propto k_\perp^{\frac{3}{4}}$

b. $v_k \propto k_\perp^{-\frac{1}{4}}$ d. $k_\parallel \propto k_\perp^{\frac{1}{2}}$

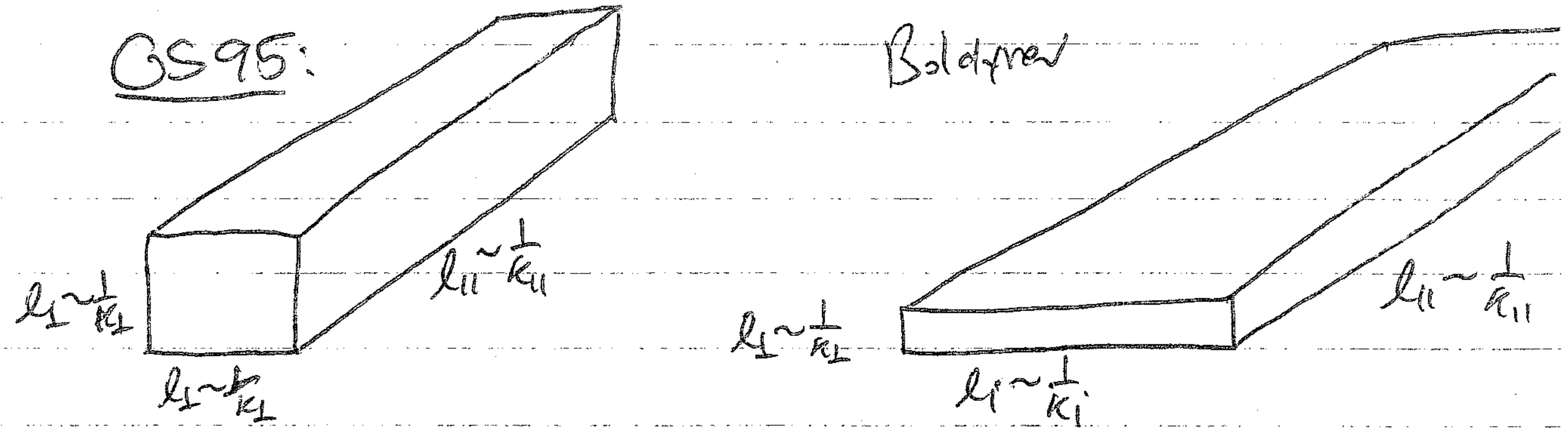
e. 1-D Energy Spectrum: $E_k \propto \frac{v_k^2}{k_\perp} \propto k_\perp^{-\frac{3}{2}}$

$\boxed{E_{k_\perp} \propto k_\perp^{-\frac{3}{2}}}$ Boldyrev Spectrum

II. (Continued)

D. Predictions of Boldyrev Theory compared to GS95

1. Turbulence is essentially 3-dimensional, with anisotropy between all axes.

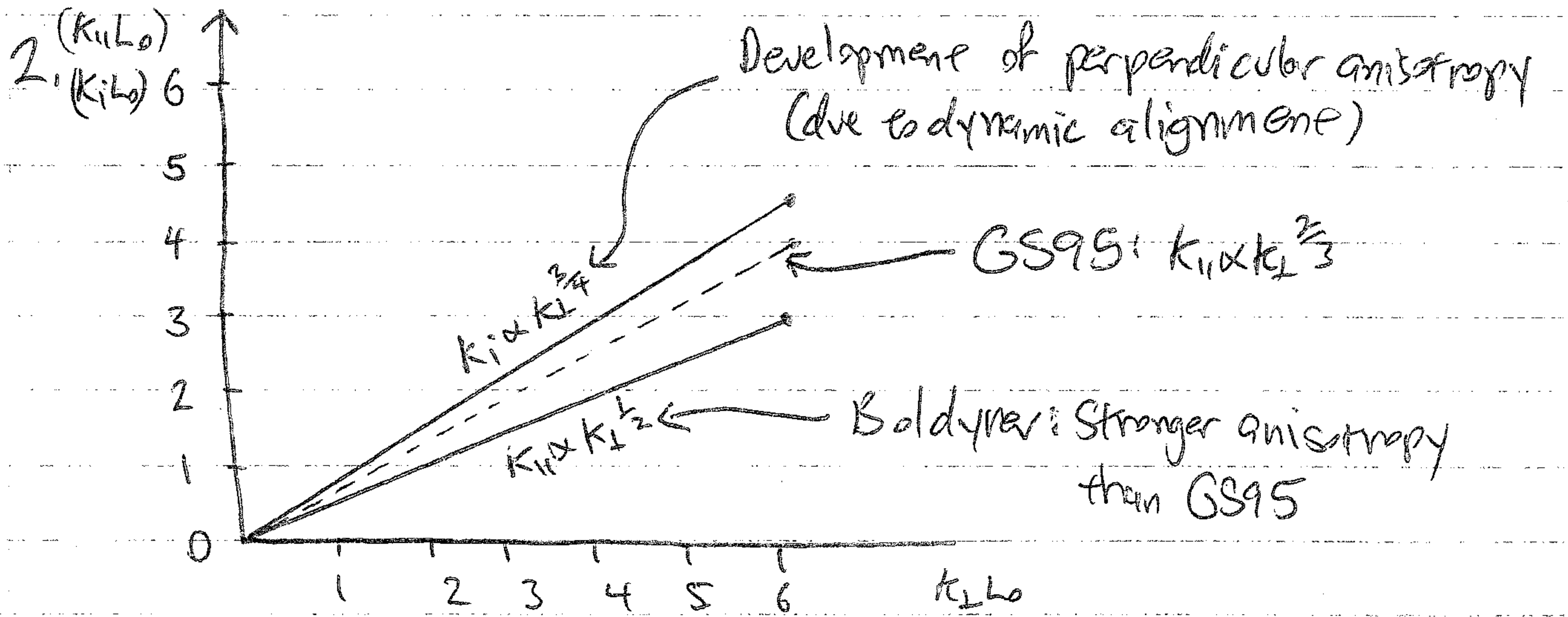


$k_\parallel \ll k_\perp$ $\qquad$ $k_\parallel \ll k_i \ll k_\perp$

a. ~~This leads to~~ Boldyrev theory leads to current sheets at small scales in MHD turbulence, consistent with simulations.

b. GS95 predicts small-scale filaments, ~~not~~ observed.

2.



a. Take isotropic driving $k_\parallel = k_\perp = k_i = k_0$ with $v_0 = v_A \Rightarrow \theta_0 = 1$

b. Scalings: $\theta_k = \theta_0 \left(\frac{k_\perp}{k_0}\right)^{-\frac{1}{4}}$, $v_k = v_A \left(\frac{k_\perp}{k_0}\right)^{-\frac{1}{4}}$, $k_\parallel = \theta_0 k_0^{\frac{1}{2}} k_\perp^{\frac{1}{2}}$, $k_i = \theta_0 k_0^{\frac{1}{4}} k_\perp^{\frac{3}{4}}$

II. (Continued)

E. Parallel Spectrum:

1. We can determine the 1-D energy spectrum in $k_\parallel$ by using

a. $E = \int_0^\infty dk_\perp \, E(k_\perp) = \int_{-\infty}^{\infty} dk_\parallel \, E(k_\parallel) = 2\int_0^\infty dk_\parallel \, E(k_\parallel)$

b. Thus $E(k_\parallel) = \frac{1}{2} E(k_\perp) / \left(\frac{dk_\parallel}{dk_\perp} \right)$

2. Boldyrev: $E_{k_\perp} \propto k_\perp^{-3/2}$, $k_\parallel \propto k_\perp^{1/2}$

a. $\frac{dk_\parallel}{dk_\perp} \propto \frac{1}{2} k_\perp^{-1/2}$

b. Thus $E(k_\parallel) \propto \frac{\cancel{\frac{1}{2}} k_\perp^{-3/2}}{\cancel{\frac{1}{2}} k_\perp^{-1/2}} \propto k_\perp^{-1} \propto k_\parallel^{-2}$

So $\boxed{E(k_\parallel) \propto k_\parallel^{-2}}$ Boldyrev 1-D parallel spectrum

3. GS95: $E_{k_\perp} \propto k_\perp^{-5/3}$ $k_\parallel \propto k_\perp^{2/3}$

a. $\frac{dk_\parallel}{dk_\perp} = \frac{2}{3} k_\perp^{-1/3}$

b. $E(k_\parallel) \propto \frac{\frac{1}{2} k_\perp^{-5/3}}{\frac{2}{3} k_\perp^{-1/3}} \propto \cancel{\frac{3}{4}}^{\sim 1} k_\perp^{-4/3} \propto k_\parallel^{-2}$

a. So $\boxed{E(k_\parallel) \propto k_\parallel^{-2}}$ GS95 1-D parallel spectrum.

F. Physical Difference between GS95 and Boldyrev

1. Consider Critical Balance where $\theta_k < 1$

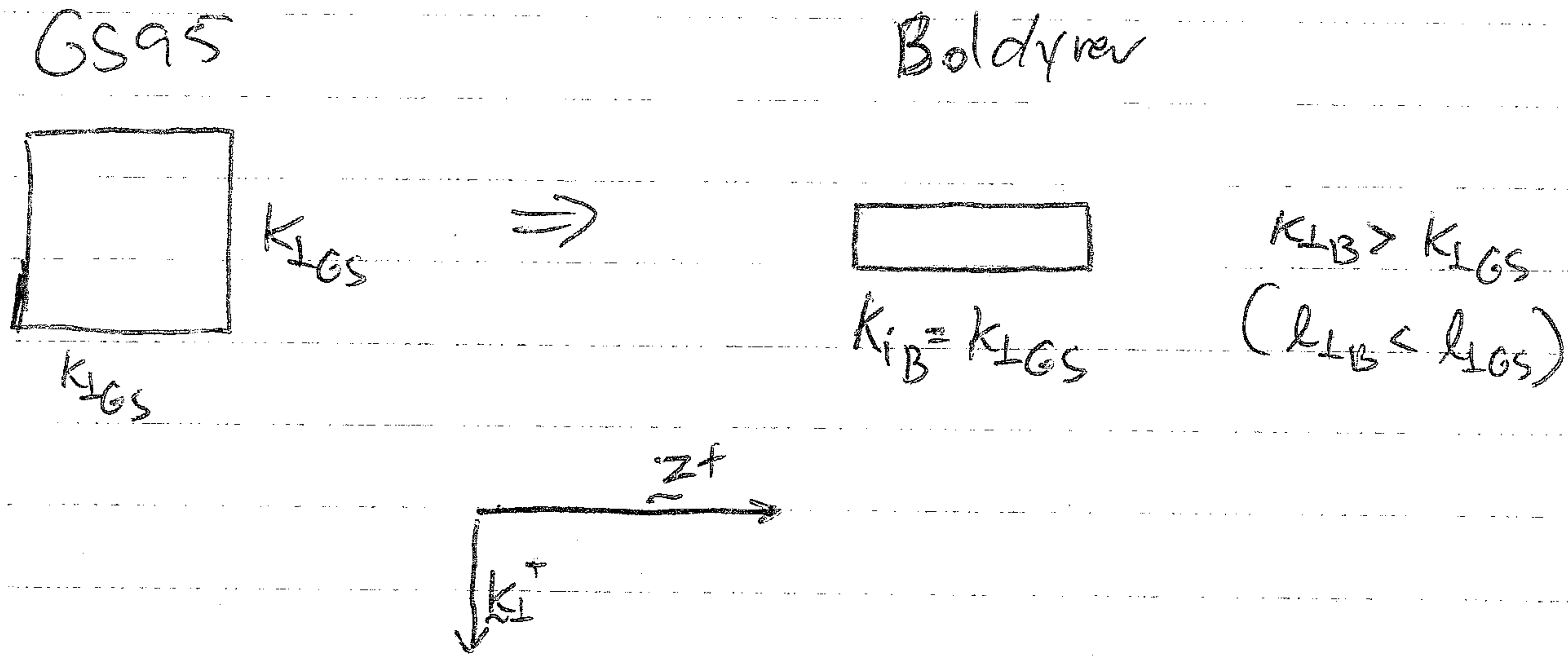
$k_\parallel v_A \sim k_\perp v_k \, \theta_k$

a. This factor weakens NL interactions requiring a higher value of $k_\perp$ to achieve critical balance (due to dynamic alignment).

b. But θ_i is a geometrical argument based on $\delta B_\perp$ & B_0, so does the change.

II.F. (Continued)

2. Thus, it is dynamic alignment that leads to a thinning of the ~~current~~ turbulent structures in the direction of $\underset{\sim}{k}_\perp$.



III. References:

1. Boldyrev, S (2006) Physical Review Letters, 96, 115002
   a. Best reference describing the theory and justifying physical arguments.
2. Boldyrev, S. (2005) ApJ Letters 626, L37–L40.
   a. Early version of the theory, with a subtly different geometry (uses II.B.3. as the basis, rather than II.B.2. as used in the 2006 paper). Not such compelling physical arguments.