Howes ①
Dec 2009

Lecture #3 Strong MHD Turbulence

I. Transition from Weak to Strong MHD Turbulence

A. Setup:

1. Consider a magnetofluid stirred isotropically with a velocity at the stirring scale $v_0 \ll v_A$. Mean Field $\underset{\sim}{B} = B_0 \hat{z}$.
   a. $k_{\perp 0} = k_{\parallel 0} = k_0$, $v_0 \ll v_A$.
   b. At the outer scale, the nonlinearity parameter
   $$\chi(k_\perp = k_{\perp 0}) \equiv \chi_0 = \frac{k_{\perp 0} v_0}{k_{\parallel 0} v_A} \ll 1 \Rightarrow \text{Weak Turbulence.}$$

2. In this limit, three-wave resonant interactions (involving one $k_\parallel = 0$ mode) will lead to $v_{k_\perp} \propto k_\perp^{-\frac{1}{2}}$
   b. There is no parallel cascade, so $k_\parallel = k_{\parallel 0} = $ constant.
   c. Thus
   $$\chi_{k_\perp} = \frac{k_\perp v_k}{k_\parallel v_A} \propto k_\perp^{\frac{1}{2}} \Rightarrow \chi \text{ increases with } \underline{k_\perp}!$$

B. Breakdown of Weak Turbulence Approximation $\chi \ll 1$

1. At some $k_\perp > k_{\perp 0}$, the cascade reaches $\chi \sim 1$

2. In this case, the fractional change $\delta v_k / v_k$ in a single collision is
   $$\frac{\delta v_k}{v_k} \sim \frac{k_\perp v_k}{k_\parallel v_A} \sim \chi \sim 1$$

   $\Rightarrow$ All energy at a scale $k_\perp$ cascades in a single wavepacket collision.

3. Thus, our assumption of many uncorrelated kicks leading to a random walk fails $\rightarrow$ Need a new scaling theory.

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I. B. (Continued)

4. Also, the applicability of perturbation theory ceases.

a. At $\chi \sim 1$, all terms in the perturbative expansion (three-wave, four-wave, five-wave, etc. interactions) contribute equally. ~~One can~~ No longer is the three-wave interaction term dominant.

b. The contribution of all terms leads to a resonance broadening, relaxing the strict constraints $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3$ and $\omega_1 + \omega_2 = \omega_3$.

c. The result is that the prediction of no $k_\parallel$ cascade is relaxed $\Rightarrow$ Energy may cascade in $k_\parallel$.

d. Mathematically, in GS95, this effect is included in the kinetic equation for the energy transfer by applying a frequency renormalization.

e. Heuristically, it is the hypothesis of critical balance that governs the parallel cascade in strong MHD turbulence.

C. Critical Balance:

1. GS95 proposed the hypothesis that, in strong turbulence, the parallel cascade occurs in such a manner to maintain $\chi \sim 1$ ~~for all~~ as $k_\perp$ increases.

2. ~~This idea~~ Critical Balance can be interpreted as a balance of the linear and nonlinear terms in the incompressible MHD equations.

a. $$\frac{\partial \underset{\sim}{z}^+}{\partial t} \cancel{+} \underbrace{(\underset{\sim}{v}_A \cdot \nabla)\underset{\sim}{z}^+}_{\text{linear term}} + \underbrace{(\underset{\sim}{z}^- \cdot \nabla)\underset{\sim}{z}^+}_{\text{nonlinear term}} = -\nabla p$$

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I.C. 2. (Continued)

b. Estimate linear term: $(\underset{\sim}{v_A}\cdot\nabla)\underset{\sim}{z}^+ \sim v_A k_\parallel z^+ \sim v_A k_\parallel v_k^+$

1. Take $\underset{\sim}{z}^+ = v_\perp + \frac{\delta B_\perp}{\sqrt{4\pi\rho_0}} \sim v_\perp = v_k$

c. Estimate nonlinear term: $(\underset{\sim}{z}^- \cdot\nabla)\underset{\sim}{z}^+ \sim v_k^- k_\perp v_k^+$

d. For the present, we assume a $\underline{\text{balanced}}$ turbulence, $|\underset{\sim}{z}^+|^2 \sim |\underset{\sim}{z}^-|^2$

So $v_k^+ \sim v_k^- \sim v_k$.

e. Ratio: $\frac{\text{Nonlinear term}}{\text{Linear term}} \sim \frac{v_k^- k_\perp \cancel{v_k^+}}{v_A k_\parallel \cancel{v_k^+}} \sim \frac{k_\perp v_k}{k_\parallel v_A} \sim \chi$

f. Thus, $\chi \sim 1$ signifies a balance between linear and nonlinear terms at each scale $k_\perp$.

3. Note that at the scale where weak turbulence first reaches $\chi \sim 1$, we find $k_\perp \gg k_\parallel$.

a. From weak turbulence scaling $\epsilon \sim \frac{k_\perp^2 v_k^4}{k_\parallel v_A} = \epsilon_0 = \frac{k_{\perp 0}^2 v_0^4}{k_{\parallel 0} v_A}$

$\rightarrow v_k = v_0 \left(\frac{k_\perp}{k_{\perp 0}}\right)^{-\frac{1}{2}}$

b. Thus $\chi \sim \frac{k_\perp v_k}{k_\parallel v_A} \sim \frac{k_\perp \left(\frac{k_\perp}{k_{\perp 0}}\right)^{-\frac{1}{2}} v_0}{k_{\parallel 0} v_A} \underset{\uparrow}{=} \left(\frac{k_\perp}{k_0}\right)^{\frac{1}{2}} \frac{v_0}{v_A} \sim 1.$

$k_{\perp 0} = k_{\parallel 0} = k_0$

c. Therefore, $\left(\frac{k_\perp}{k_0}\right)^{\frac{1}{2}} \sim \frac{v_A}{v_0}$. But, since $k_\parallel = k_{\parallel 0} = k_0$ (no parallel cascade),

$\underset{\substack{\uparrow \\ \text{at transition } \chi\sim 1}}{\frac{k_\perp}{k_\parallel}} \sim \left(\frac{v_A}{v_0}\right)^2 \gg 1.$ $\boxed{\text{Thus } k_\perp \gg k_\parallel \Rightarrow \text{Turbulence has become very anisotropic.}}$

II. Conservation Properties of Incompressible MHD:

A. Conserved Quantities:

- There are three conserved quadratic quantities:

1. Energy: $E \equiv \int d^3\mathbf{r}\, \frac{\rho_0}{2}\left(v^2 + b^2\right)$

2. Cross-Helicity: $H_c \equiv \int d^3\mathbf{r}\, \frac{1}{2}\mathbf{v}\cdot\mathbf{b}$

3. Magnetic Helicity: $H_m \equiv \int d^3\mathbf{r}\, \mathbf{A}\cdot\mathbf{B}$

where $\mathbf{b} \equiv \frac{\mathbf{B}}{\sqrt{4\pi\rho_0}}$ and $\mathbf{B} = \nabla\times\mathbf{A}$

Ref: Woltjer (1958a, b)

B. Elastic Collisions between Alfvén Wave packets

1. Consider the evolution of Elsasser energy:

a. $$\frac{\partial \mathbf{z}^+}{\partial t} = (\mathbf{v}_A\cdot\nabla)\mathbf{z}^+ - (\mathbf{z}^-\cdot\nabla)\mathbf{z}^+ - \nabla p$$

b. Dot with $\mathbf{z}^+$

$$\frac{\partial}{\partial t}\frac{|\mathbf{z}^+|^2}{2} = \underbrace{\mathbf{z}^+\cdot\left[(\mathbf{v}_A\cdot\nabla)\mathbf{z}^+\right]}_{①} - \underbrace{\mathbf{z}^+\cdot\left[(\mathbf{z}^-\cdot\nabla)\mathbf{z}^+\right]}_{②} - \underbrace{\mathbf{z}^+\cdot\nabla p}_{③}$$

c. Using $\nabla\cdot(f\mathbf{A}) = f\,\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$,

$$③ \Rightarrow \mathbf{z}^+\cdot\nabla p = \nabla\cdot(p\mathbf{z}^+) - p\,\underbrace{\cancel{\nabla\cdot\mathbf{z}^+}}_{\text{incompressible}}$$

d. ① & ② can be simplified $\mathbf{z}^+\cdot\left[(\mathbf{v}_A\cdot\nabla)\mathbf{z}^+\right] = (\mathbf{v}_A\cdot\nabla)\frac{|\mathbf{z}^+|^2}{2}$

e. Thus $$= \nabla\cdot\left(\mathbf{v}_A\frac{|\mathbf{z}^+|^2}{2}\right) - \frac{|\mathbf{z}^+|^2}{2}\underbrace{\cancel{\nabla\cdot\mathbf{v}_A}}_{\nabla\cdot\mathbf{B}=0}$$

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II. B. 1. (Continued)

e. Thus $\frac{\partial}{\partial t}\left(\frac{|\underset{\sim}{z}^+|^2}{2}\right) = \nabla \cdot \left[(\underset{\sim}{v}_A - \underset{\sim}{z}^-)\frac{|\underset{\sim}{z}^+|^2}{2}\right] - \nabla \cdot (p\, \underset{\sim}{z}^+)$

f. Taking an integral over all space, we can convert the RHS using the divergence theorem: $\int_V d^3\underset{\sim}{r}\, \nabla \cdot \underset{\sim}{A} = \oint_S d\underset{\sim}{S} \cdot \underset{\sim}{A}$

$$\frac{\partial}{\partial t}\int d^3\underset{\sim}{r}\, \frac{|\underset{\sim}{z}^+|^2}{2} = \oint d\underset{\sim}{S} \cdot \left[(\underset{\sim}{v}_A - \underset{\sim}{z}^-)\frac{|z_+|^2}{2}\right] - \oint d\underset{\sim}{S} \cdot (\underset{\sim}{z}^+ p)$$

g. The integrals on the RHS $= 0$ for

1) Periodic Boundary Conditions

OR 2) Integral over all space provided $\frac{|z_+|^2}{2} \to 0$ and $p \to 0$ as $\underset{\sim}{r} \to \infty$.

h. Therefore, we find $\boxed{\frac{\partial}{\partial t}\int d^3\underset{\sim}{r}\, \frac{|\underset{\sim}{z}^+|^2}{2} = 0}$

Energy of the "+" wavepackets is not changed by nonlinear interactions with "−" wave packets.

$\Rightarrow$ $\boxed{\text{Collisions are elastic.}}$

2. Similarly $\frac{\partial}{\partial t}\int d^3\underset{\sim}{r}\, \frac{|\underset{\sim}{z}^-|^2}{2} = 0$

~~3.~~ ~~This~~ ~~properties~~ ~~are~~ is a consequence of the conservation of both energy and cross helicity.

a. $H_C = \frac{1}{8}\int d^3\underset{\sim}{r}\left(|\underset{\sim}{z}^+|^2 - |\underset{\sim}{z}^-|^2\right)$

$E = \frac{\rho_0}{4}\int d^3\underset{\sim}{r}\left(|z^+|^2 + |z^-|^2\right)$

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III. Scaling Theory for Strong MHD Turbulence

A. Setup

1. Consider turbulence stirred isotropically ($k_{\perp 0} = k_{\parallel 0} = k_0$) with velocity $v_0 = v_A$.

2. Thus, $\chi_0 \sim \frac{k_{\perp 0} v_0}{k_{\parallel 0} v_A} \sim 1 \Rightarrow$ Strong turbulence from the start

   $\Rightarrow$ No weak MHD turbulence range.

B. Estimate of Energy Cascade Rate

1. When $\chi \sim 1$, $\frac{\delta v_k}{v_k} \sim 1$ in a single collision.

2. Nonlinear transfer time $\tau_{nl} \sim \frac{1}{k_\parallel v_A} \sim \frac{1}{k_\perp v_k} \sim \frac{1}{\omega_{nl}}$

   $\Rightarrow \omega_{nl} \sim k_\perp v_k$ [Again $v_k \equiv v_\perp(k_\perp)$]

3. Energy cascade rate: $\epsilon \sim \frac{v_k^2}{\tau_{nl}} \sim v_k^2 \omega_{nl} \sim k_\perp v_k^3 = \epsilon_0$

   a. $\boxed{v_k = \epsilon_0^{\frac{1}{3}} k_\perp^{-\frac{1}{3}}}$

C. 1-D Energy Spectrum: $E_{k_\perp} \sim \frac{v_k^2}{k_\perp}$

1. Recall $E = \int_{-\infty}^{\infty} dk_\parallel \int_0^\infty 2\pi k_\perp dk_\perp E^{(3)}(\underline{k}) = \int_0^\infty dk_\perp E_{k_\perp}(k_\perp)$,

   So $E_{k_\perp}(k_\perp) = \int_{-\infty}^{\infty} dk_\parallel 2\pi k_\perp E^{(3)}(\underline{k})$

2. $E_{k_\perp} \sim \frac{v_k^2}{k_\perp} \sim \epsilon_0^{2/3} k_\perp^{-5/3}$ $\quad \boxed{E_{k_\perp} \propto k_\perp^{-\frac{5}{3}}}$ Goldreich-Sridhar Spectrum

   GS95

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III.C. (Continued)

3. Using $\epsilon_0 = k_{\perp 0} v_0^3 = k_0 v_A^3$, we can write this alternatively as $$E_{k_\perp} = \frac{v_A^2}{k_0}\left(\frac{k_\perp}{k_0}\right)^{-5/3}$$

D. Critical Balance

1. GS95 hypothesized that $\chi \sim 1$ is maintained in strong turbulence as $k_\perp$ increases.

$$k_\parallel v_A \sim k_\perp v_k \qquad (\omega \sim \omega_{nl})$$

linear ~ nonlinear

2. $k_\parallel v_A \sim k_\perp(\epsilon_0^{1/3} k_\perp^{-1/3}) = k_0^{1/3} v_A k_\perp^{2/3}$

$\Rightarrow \boxed{k_\parallel \sim k_0^{1/3} k_\perp^{2/3}}$ $\boxed{k_\parallel \propto k_\perp^{2/3}}$ Scale-dependent anisotropy

3. Scale Dependent Anisotropy $\frac{k_\perp}{k_\parallel} \sim \frac{k_\perp}{k_0^{1/3} k_\perp^{2/3}} \sim \left(\frac{k_\perp}{k_0}\right)^{1/3} \propto k_\perp^{1/3}$

$\Rightarrow$ Thus, anisotropy $\frac{k_\perp}{k_\parallel}$ increases as $k_\perp$ increases!

4.a. Even for isotropic driving ($k_{\perp 0} = k_{\parallel 0} = k_0$) at large scales,

$\boxed{\text{at small scales GS95 predicts } k_\perp \gg k_\parallel.}$

b. This is very important when we consider kinetic turbulence, the continuation of MHD turbulence at scales of order or smaller than the ion Larmor radius, $k_\perp \rho_i \gtrsim 1$.

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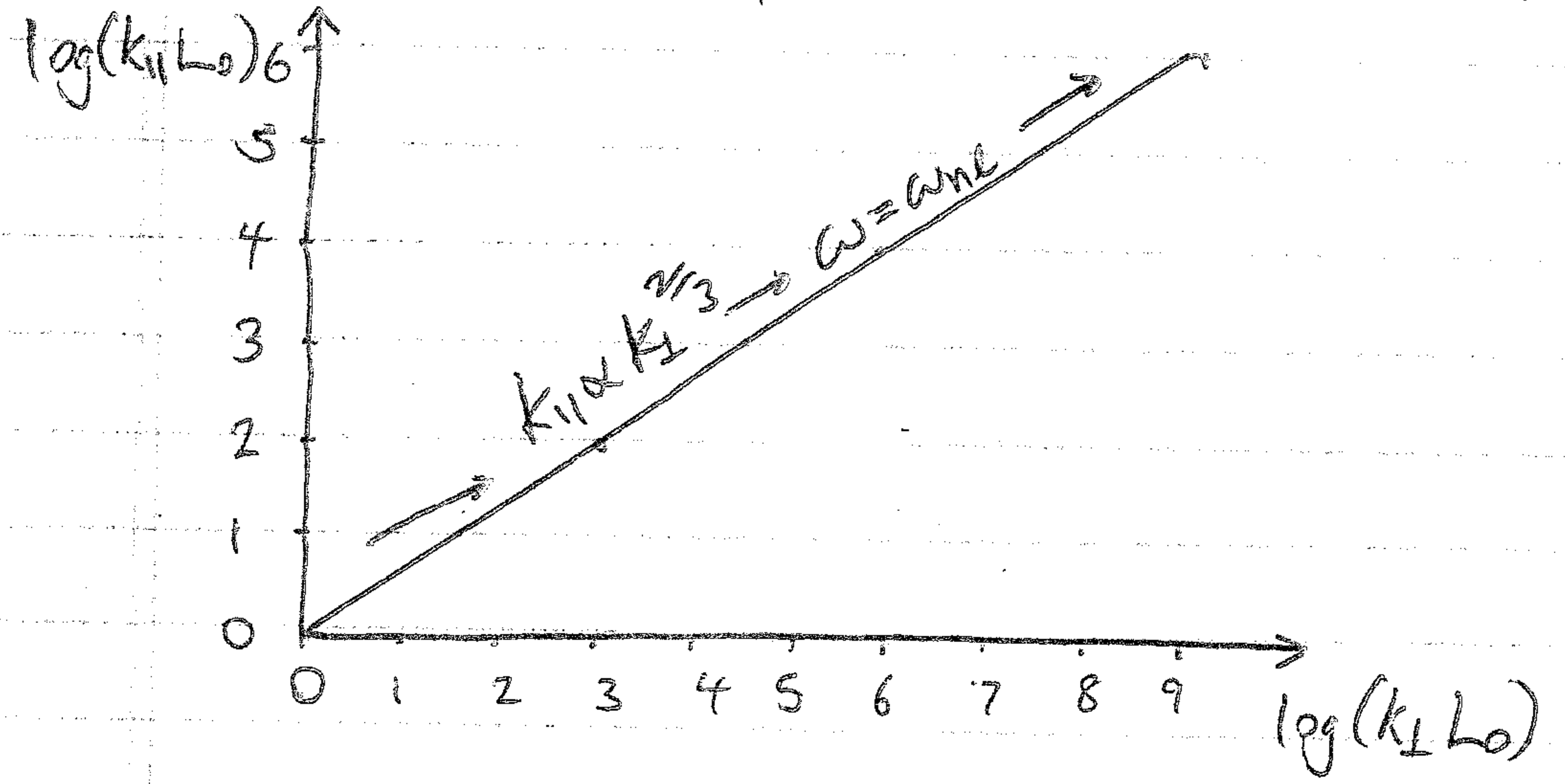
III. (Continued)

E. Transfer of Energy Through Wave vector Space

1. 

Driving scale $L_0 \sim \frac{1}{k_0}$

2. Parallel distribution of energy

a. Does this theory imply all energy exists only on the line of critical balance $k_\parallel = k_0^{\frac{1}{3}} k_\perp^{\frac{2}{3}}$? No!

b. GS95 do not attempt to determine the distribution over $k_\parallel$ at each $k_\perp$, but instead propose a reasonable distribution inspired by critical balance.

c. In general, $E_{k_\perp}(k_\perp) = \int_{-\infty}^{\infty} dk_\parallel \int_0^{2\pi} d\theta \, k_\perp \, E^{(3)}(\underline{k})$

where we assume axisymmetry about $\underline{B}_0$ to find

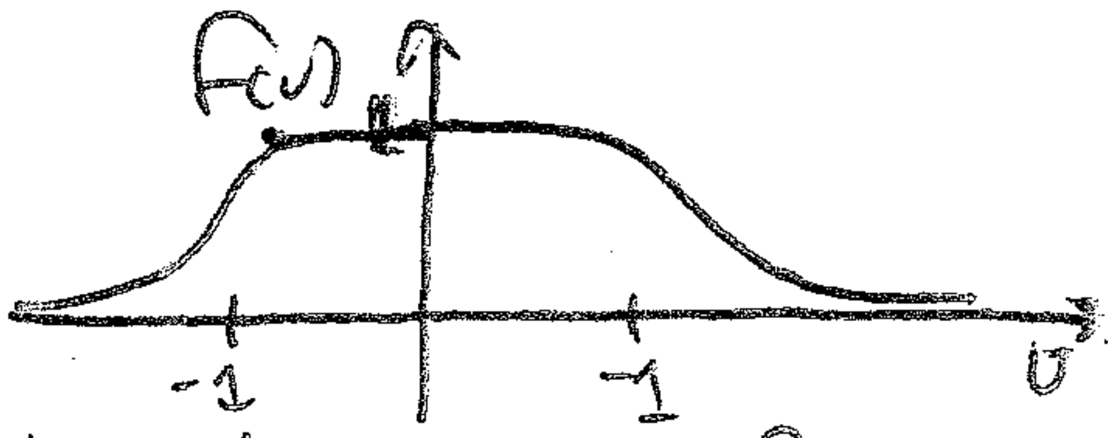
$$E^{(3)}(k_\perp \cos\theta, k_\perp \sin\theta, k_\parallel) = E^{(3)}(k_\perp, k_\parallel)$$

d. Let's assume a separable function $E^{(3)}(k_\perp, k_\parallel) = g(k_\perp), f(k_\parallel, k_\perp)$

where $g(k_\perp) = \frac{E_{k_\perp}(k_\perp)}{2\pi k_\perp} \underbrace{\left(\frac{1}{k_0^{\frac{1}{3}} k_\perp^{\frac{2}{3}}} \right)}$ → Normalization for $\int_{-\infty}^{\infty} f \, dk_\parallel$

e. Take $f(k_\parallel, k_\perp) = f(u)$ where $u = \frac{k_\parallel}{k_0^{\frac{1}{3}} k_\perp^{\frac{2}{3}}}$,

So $u = 1 \Rightarrow$ Critical balance.



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III. E. 2. (Continued)

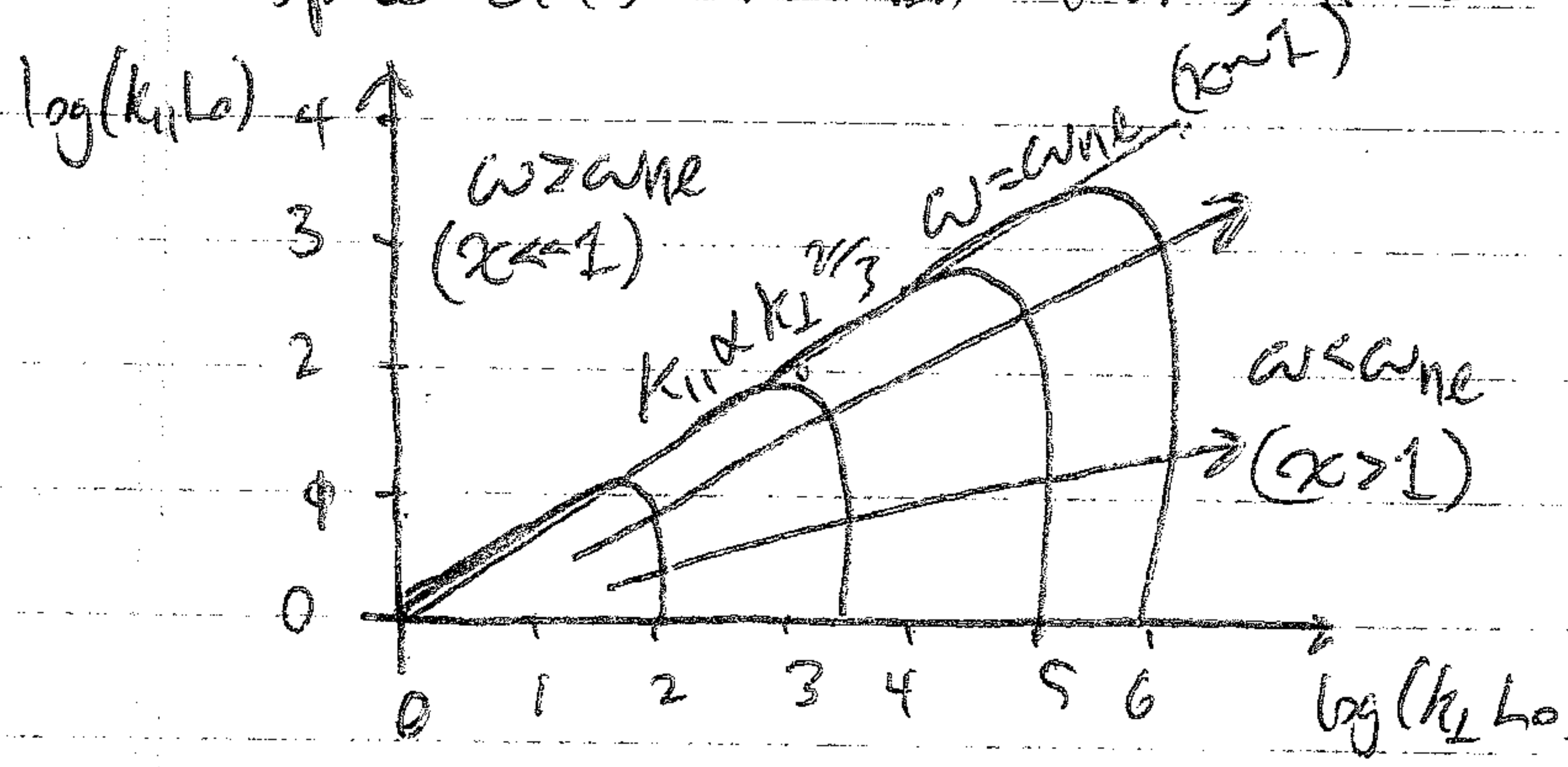
f. Assume the property $\int_{-\infty}^{\infty} dv\, f(v) = 1$ and $f(v) \to 0$ for $|v| \gg 1$, and $f(v)$ is a symmetric function of v.

g. Thus $E_{k_\perp}(k_\perp) = \int_0^{2\pi} d\theta\, k_\perp \int_{-\infty}^{\infty} dk_\parallel \left[\frac{E_{k_\perp}(k_\perp)}{2\pi k_\perp^{5/3} k_0^{1/3}} f\left(\frac{k_\parallel}{k_0^{1/3} k_\perp^{2/3}}\right)\right] = E_{k_\perp}(k_\perp)$ ✓

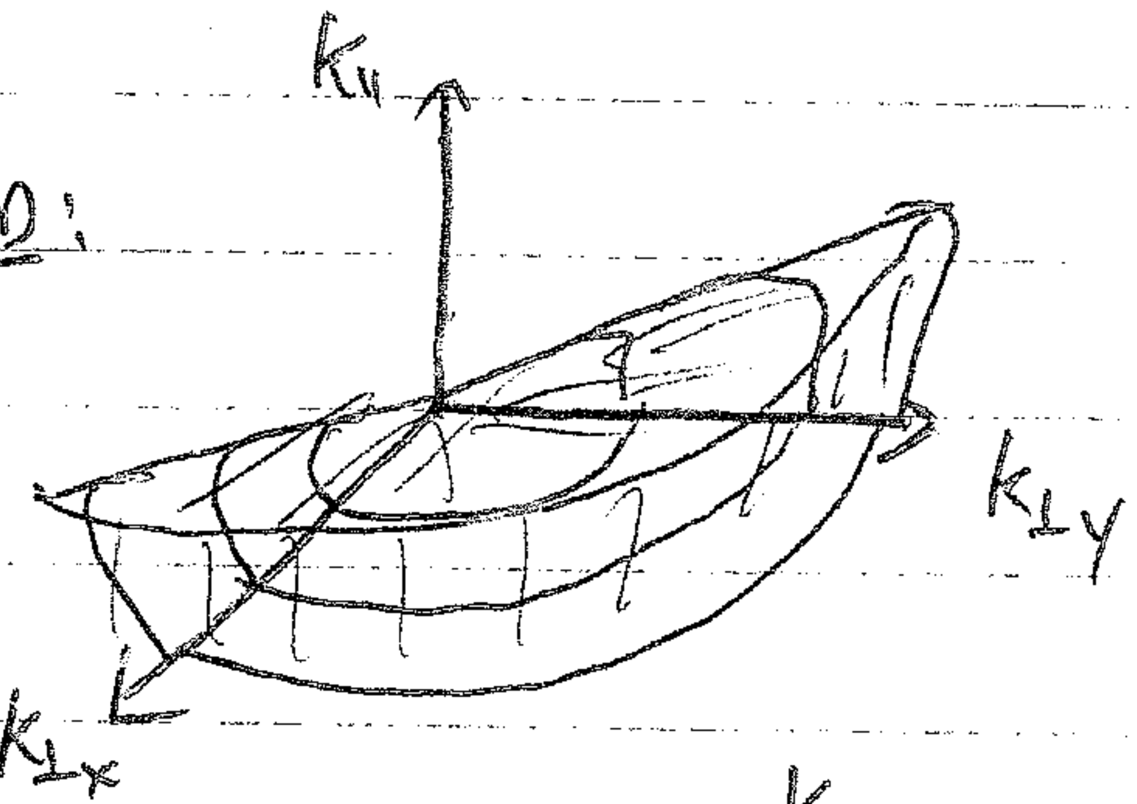
h. This means:

$$\boxed{E^{(3)}(k_\perp, k_\parallel) = \frac{V_A^2\, k_0^{1/3}}{2\pi\, k_\perp^{10/3}} f\left(\frac{k_\parallel}{k_0^{1/3} k_\perp^{2/3}}\right)}$$

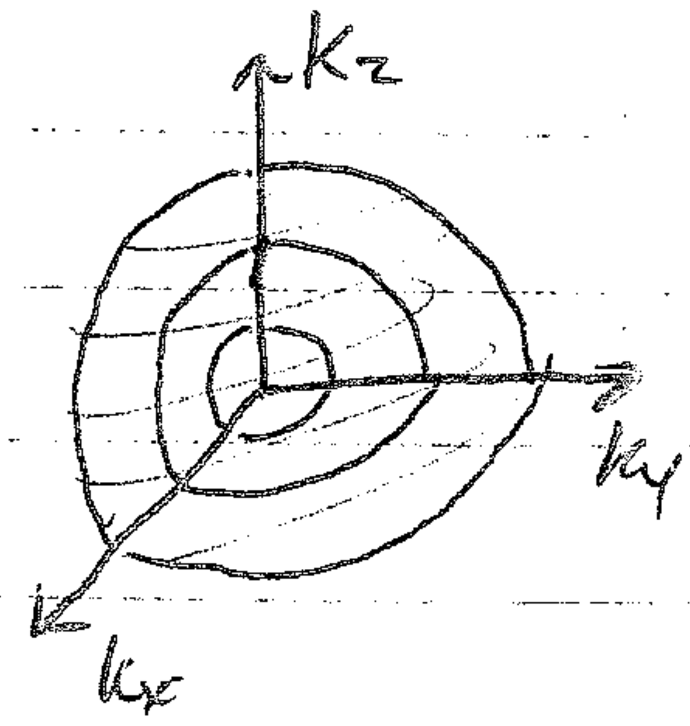
3. If we take $f(v) \simeq$ constant for $|v| \lesssim 1$, the wavevector space distribution looks like:



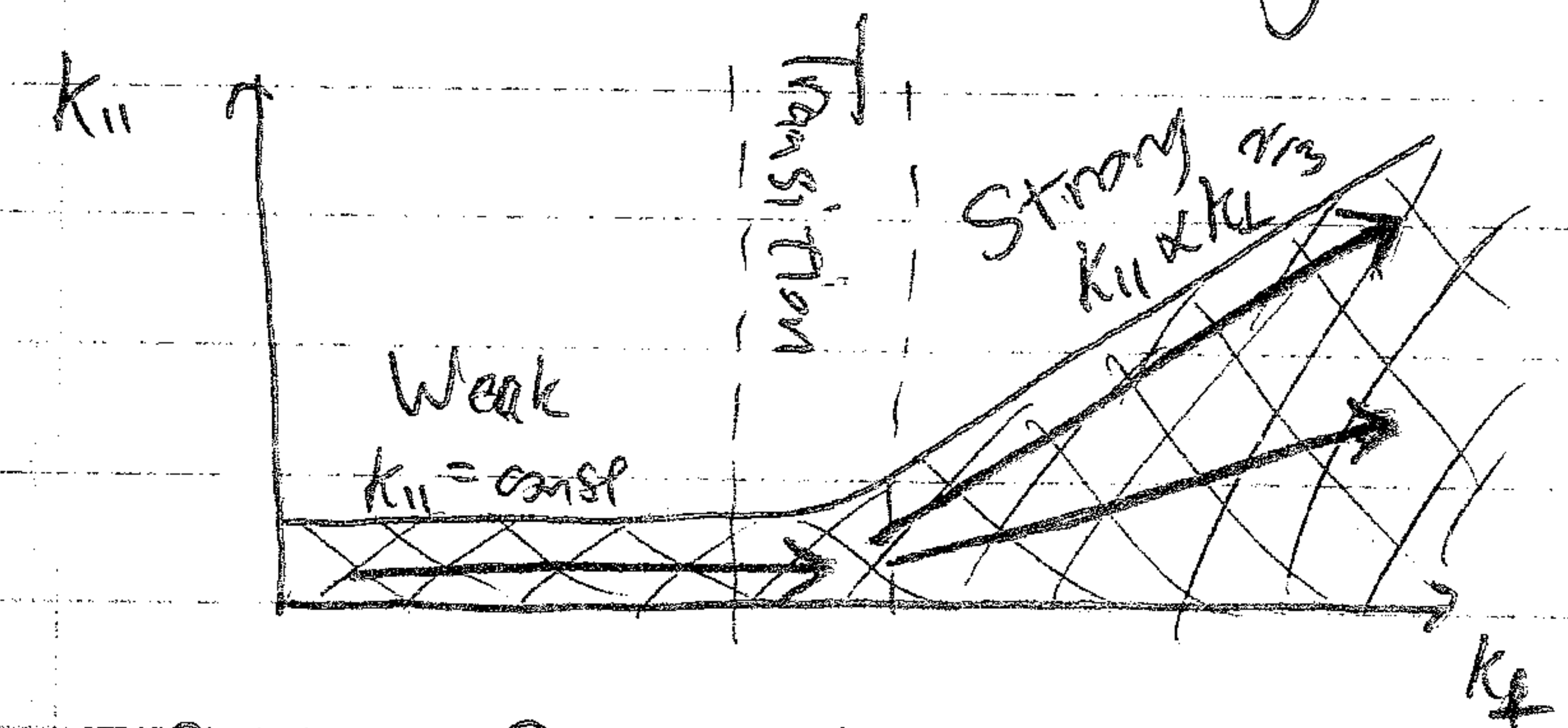
In 3-D:



Compare to Isotropic:



F. Transition from Weak to Strong Turbulence:



F. Passive Scalar Advection

1. Power spectrum of a passive scalar assumes the form of the energy spectrum of the turbulence (Lesieur, 1990).
2. Density fluctuations, if they represent entropy fluctuations at constant pressure, will act like a passive scalar $\Rightarrow E_n \propto k_\perp^{-5/3}$.

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IV. Summary:

A. Strong MHD Turbulence: General Properties

1. Collisions between oppositely directed wave packets are <u>elastic</u>.
2. Beginning from weak turbulence ($\chi \ll 1$), resonant conditions of collisions are relaxed by resonance broadening, leading to the onset of a cascade to higher $k_\parallel$ when $\chi \sim 1$.
3. <u>Critical Balance</u>: a. Balance of linear & nonlinear frequencies, $\omega \sim \omega_{nl}$
   b. Conjecture that strong turbulence maintains the condition $\chi \sim 1$.

B. Strong MHD Turbulence Scaling

1. Begin with isotropic stirring ($k_0 = k_{\perp 0} = k_{\parallel 0}$) with $v_0 = v_A$
2. All energy is transferred in a single collision, $N \sim 1$.
   $\Rightarrow \omega_{nl} \sim k_\perp v_k$
3. $v_k \propto k_\perp^{-\frac{1}{3}}$
4. 1-D Energy Spectrum: $E_{k_\perp} \propto k_\perp^{-5/3}$ Goldreich-Sridhar Spectrum
5. Scale-dependent Anisotropy:
   a. Critical Balance: $k_\parallel v_A \sim k_\perp v_k \Rightarrow k_\parallel \propto k_\perp^{2/3}$
   b. At small scales, $k_\perp \gg k_\parallel$.

V. References:

1. GS95: Goldreich, P & Sridhar, S (1995) ApJ 438, 763.
   a. First ~~modern~~ theory of anisotropic, strong MHD turbulence
   b. ~~See~~ Sec. 8 for comparison to early anisotropic theories by Montgomery, Turner, Higdon, Matthaeus, & Brown (several different papers).
2. a. Woltjer, L. 1958a, Proc. Nat. Acad. Sci. 44, 489.
   b. Woltjer, L. 1958b, Proc. Nat. Acad. Sci. 44, 833.