National Undergraduate Fellowship Program
Plasma Physics Summer School
Howes ①
June 2009

Single Particle Motion

I. Overall Framework of Plasma Physics

Particles: Ions & Electrons

position $\underset{\sim}{x}_s$
velocity $\underset{\sim}{v}_s$ $\Rightarrow$ ρ_q, $\underset{\sim}{j}$

Lorentz Force Law

$$m_s \frac{d\underset{\sim}{v}_s}{dt} = q_s\left(\underset{\sim}{E} + \underset{\sim}{v}_s \times \underset{\sim}{B}\right)$$

Maxwell's Equations

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$$
$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$
$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$$
$$\nabla \cdot \underset{\sim}{B} = 0$$

$\underset{\sim}{B}$ & $\underset{\sim}{E}$ Fields

Single Particle Motion Description

$$\boxed{\text{What we want to study is how charged particles move in prescribed } \underset{\sim}{E} \text{ \& } \underset{\sim}{B} \text{ fields.}}$$

II. Larmor Motion: Constant, Uniform $\underset{\sim}{B}$ with $\underset{\sim}{E}=0$

A. 1. Nonrelativistic Limit $v \ll c$

2. Drop subscript "s" for species

3. Thus, for $\underset{\sim}{E}=0$,
$$m\frac{d\underset{\sim}{v}}{dt} = q\,\underset{\sim}{v}\times\underset{\sim}{B}$$

4. Take $\underset{\sim}{B} = B_0\hat{z}$ were B_0 = const

B. Solution:

1.
$$\frac{dv_x}{dt} = \frac{qB_0}{m}v_y \quad ①$$
$$\frac{dv_y}{dt} = \frac{-qB_0}{m}v_x \quad ②$$
$$\frac{dv_z}{dt} = 0 \quad \Rightarrow \quad v_z = \text{constant}$$

2. Define: $\boxed{\text{Cyclotron Frequency: } \omega_c \equiv \frac{qB_0}{m}}$

3. To Solve: a. Take $\frac{d}{dt}$(1) and substitute (2)

$$\frac{d^2 v_x}{dt^2} = -\omega_c^2 v_x$$

b. General Solution: $v_x = A e^{-i\omega_c t} + B e^{i\omega_c t}$

c. Apply Initial Conditions to Solve for A & B

i. Take $v_x = v_\perp$, $v_y = 0$ at $t=0$. $\Rightarrow A = B = \frac{v_\perp}{2}$

ii. Let $v_z = v_\parallel$ at $t=0$ also.

d. Thus,

$$v_x = v_\perp \cos \omega_c t$$
$$v_y = -v_\perp \sin \omega_c t$$
$$v_z = v_\parallel$$

e. Solve for position: $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{v} \Rightarrow$

$$x = \frac{v_\perp}{\omega_c} \sin \omega_c t + x_0$$
$$y = \frac{v_\perp}{\omega_c} \cos \omega_c t + y_0$$
$$z = v_\parallel t + z_0$$

4. Define: $\boxed{\text{Larmor Radius } r_L \equiv \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{q B_0}}$

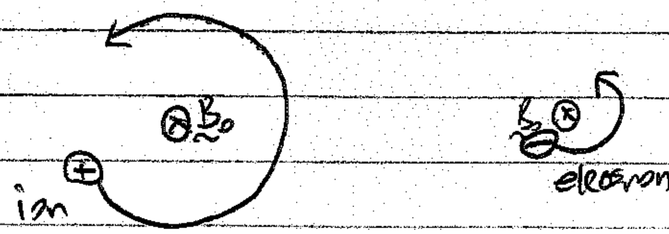
5. Summary:

a. $\underset{\sim}{x}(t) = r_L (\sin \omega_c t\, \hat{x} + \cos \omega_c t\, \hat{y}) + v_\parallel t\, \hat{z} + \underset{\sim}{x}_0$

b. $\underset{\sim}{v}(t) = v_\perp (\cos \omega_c t\, \hat{x} - \sin \omega_c t\, \hat{y}) + v_\parallel \hat{z}$

C. Properties

1. Diamagnetic:



Field due to Larmor motion opposes mean field

II.C. (Continued)

2. Constant Energy:

a. $\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \underset{\sim}{v}\cdot\left(m\frac{d\underset{\sim}{v}}{dt}\right) = \underset{\sim}{v}\cdot\left[q_s(\underset{\sim}{v}\times\underset{\sim}{B})\right] = 0$

b. Thus, $v_\perp$ = constant.

III. $\underset{\sim}{E} \times \underset{\sim}{B}$ Drift: Constant, Uniform $\underset{\sim}{B}$ and $\underset{\sim}{E}$

A. Drift Motion

1. $m\frac{d\underset{\sim}{v}}{dt} = q(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})$

2. What velocity $\underset{\sim}{v}$ leads to RHS = 0? ⇒ no acceleration ⇒ drift

a. $\underset{\sim}{E} = -\underset{\sim}{v}\times\underset{\sim}{B}$

b. Cross with $\underset{\sim}{B}$: $\underset{\sim}{E}\times\underset{\sim}{B} = -(\underset{\sim}{v}\times\underset{\sim}{B})\times\underset{\sim}{B} = B_0^2(\underset{\sim}{v} - v_z\hat{z})$

c. Thus, $\underbrace{\underset{\sim}{v} - v_z\underset{\sim}{\hat{z}}}_{\text{Perpendicular to } \underset{\sim}{B_0}} = \frac{\underset{\sim}{E}\times\underset{\sim}{B}}{B_0^2}$

3. Define $\boxed{\text{"E cross B" velocity} \quad \underset{\sim}{v_E} \equiv \frac{\underset{\sim}{E}\times\underset{\sim}{B}}{B_0^2}}$

B. Motion in $\underset{\sim}{E}\times\underset{\sim}{B}$ drift Frame

1. Solve for velocity $\underset{\sim}{u}$ in $\underset{\sim}{E}\times\underset{\sim}{B}$ frame: $\underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{v_E}$

2. Substitute for $\underset{\sim}{v}$: $m\frac{d\underset{\sim}{u}}{dt} + \cancel{m\frac{d\underset{\sim}{v_E}}{dt}}^{0} = q(\underset{\sim}{E} + \underset{\sim}{v_E}\times\underset{\sim}{B} + \underset{\sim}{u}\times\underset{\sim}{B})$

a. $\underset{\sim}{v_E}\times\underset{\sim}{B} = \frac{(\underset{\sim}{E}\times\hat{z})\times\hat{z}\,\cancel{B_0^2}}{\cancel{B_0^2}} = E_z\hat{z} - \underset{\sim}{E}$

b. Thus $m\frac{d\underset{\sim}{u}}{dt} = q(E_z\underset{\sim}{\hat{z}} + \underset{\sim}{u}\times\underset{\sim}{B})$

3. Parallel Motion ($\hat{z}$): $m\frac{du_z}{dt} = qE_z \Rightarrow \boxed{u_z = \frac{qE_z}{m}t + u_{z0}}$

III. B. (Continued)

4. <u>Perpendicular Motion:</u> $\underset{\sim}{u}_\perp = \underset{\sim}{v} - v_z \hat{\underset{\sim}{z}}$

a. $m \dfrac{d\underset{\sim}{u}_\perp}{dt} = q(\underset{\sim}{u}_\perp \times \underset{\sim}{B})$ This is identical to the case with $\underset{\sim}{E} = 0$.

b. Thus, $\underset{\sim}{u}_\perp = v_\perp (\cos\omega_c t\, \hat{\underset{\sim}{x}} - \sin\omega_c t\, \hat{\underset{\sim}{y}})$

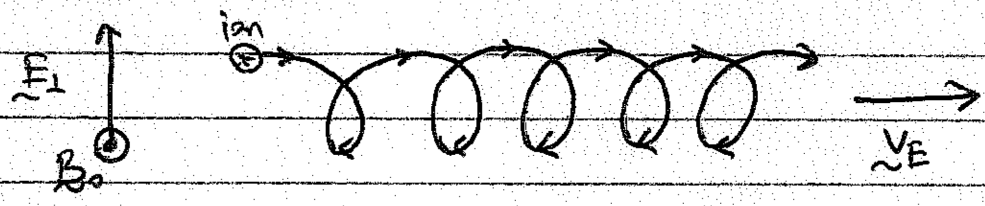
c. $\boxed{\text{In the } \underset{\sim}{E}\times\underset{\sim}{B} \text{ drift frame, you have the usual Larmor motion}}$

5. <u>Full Solution:</u>

$$\underset{\sim}{v} = \underbrace{\left(\frac{qE_z}{m} + v_{z0}\right)\hat{\underset{\sim}{z}}}_{\text{Parallel Motion}} + \underbrace{v_\perp(\cos\omega_c t\, \hat{\underset{\sim}{x}} - \sin\omega_c t\, \hat{\underset{\sim}{y}})}_{\text{Larmor Motion}} + \underbrace{\left(\frac{\underset{\sim}{E}\times\underset{\sim}{B}}{B_0^2}\right)}_{\underset{\sim}{E}\times\underset{\sim}{B} \text{ drift}}$$

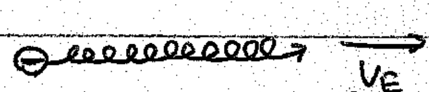
C. <u>Physical Picture:</u>

1.



a. Acceleration by $\underset{\sim}{E}$ $\Rightarrow$ r_L increases } This asymmetry leads
Deceleration by $\underset{\sim}{E}$ $\Rightarrow$ r_L decreases } to the drift

2. $\underset{\sim}{E}\times\underset{\sim}{B}$ drift is <u>independent of charge</u>. $\Rightarrow$ No net current due to $\underset{\sim}{E}\times\underset{\sim}{B}$ drift.



IV <u>Multiple Timescale Methods</u>

A. 1. A powerful approach to solving many plasma physics problems is the use of <u>multiple timescale methods</u>.

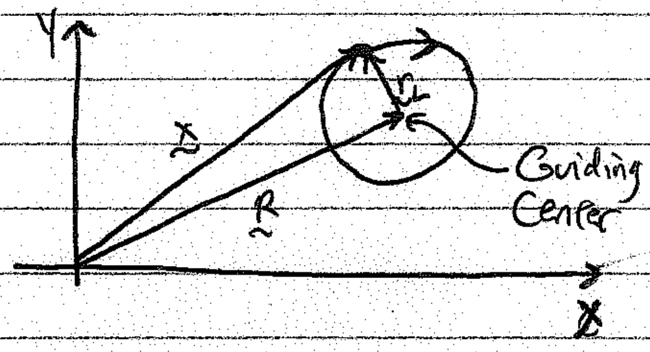
2. In many problems, different components of the motion occur on disparate timescales.

IV. A. 2. (Continued)

a. For Example, $\underset{\sim}{E} \times \underset{\sim}{B}$ drift

Decomposition of Motion:
- i. Rapid Larmor motion about field line
- ii. Slow drift across field line.

3. Define: Guiding Center



a. Position can be split into Guiding Center $\underset{\sim}{R}$ plus Larmor motion $\underset{\sim}{r_L}$

$$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{r_L}$$

4. Basic concept for multiscale methods:

a. Average over fast timescale motion:

$$\int_0^{\frac{2\pi}{\omega_c}} dt\, \underset{\sim}{r_L}(t) = 0$$

b. This leaves ~~[illegible]~~ slow timescale drift motion $\underset{\sim}{R}(t)$.

V. ∇B & Curvature Drifts: Constant, Non-uniform $\underset{\sim}{B}$ fields

A. In the fusion program, magnetic fields used to confine the plasma are neither straight nor uniform. We want to understand particle motion in $\underset{\sim}{B}$ fields of varying strength and curved $\underset{\sim}{B}$ fields.

B. Drift due to a General Force $\underset{\sim}{F}$

1. Analogous to $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, take $m \frac{d\underset{\sim}{v}}{dt} = q \underset{\sim}{v} \times \underset{\sim}{B} + \underset{\sim}{F}$

$$\Rightarrow \boxed{\underset{\sim}{v_D} = \frac{1}{q} \frac{\underset{\sim}{F} \times \underset{\sim}{B}}{B_0^2}}$$

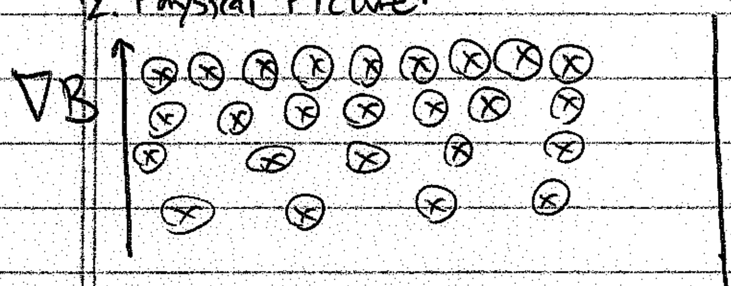
2. NOTE: The direction of this drift depends on the charge sign.

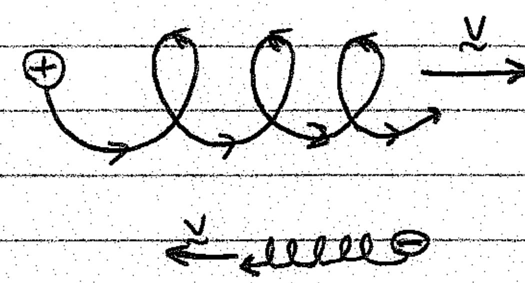
V. (Continued)

C. The ∇B ("GradB") Drift

1. Simplest Case $\nabla|\underset{\sim}{B}| \perp \underset{\sim}{B}$

2. Physical Picture:





a. Stronger $B \Rightarrow$ smaller r_L
   Weaker $B \Rightarrow$ larger r_L

3. Multiscale Approach: a. Small scale: Larmor Radius $r_L = \frac{v_\perp}{\omega_c}$
   b. Large scale: $\underset{\sim}{B}$ Scale length $L \equiv \left(\frac{\nabla B}{B}\right)^{-1}$

c. We may use a perturbative approach in the small ~~expansion parameter~~ $\epsilon \equiv \frac{r_L}{L} << 1$

d. We may derive the average force on the particle $\langle F \rangle$ (averaged over the Larmor period $T = \frac{2\pi}{\omega_c}$) due to the ∇B.

$$\langle \underset{\sim}{F} \rangle = -\frac{q v_\perp^2}{2\omega_c} \nabla B$$

4. The result is the ∇B drift, $\boxed{\underset{\sim}{V}_{\nabla B} = \frac{-v_\perp^2}{2\omega_c} \frac{\nabla B \times \underset{\sim}{B}}{B_0^2}}$

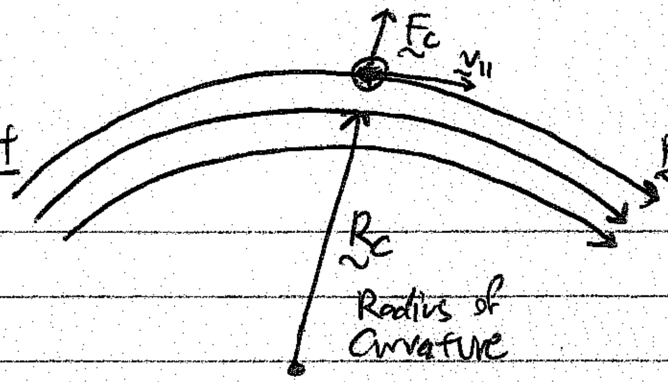
a. NOTE: Since $\omega_c = \frac{qB}{m}$, the ∇B drift depends on charge.
   $\Rightarrow$ Ions and electrons drift in opposite directions

b. Drift magnitude depends on perpendicular energy $\frac{1}{2} m v_\perp^2$.

V. (Continued) Howes (7)

D. Curvature Drift

1. Physical Picture



2. Simple Estimate:

a. For a particular moving along a circular path along $\underset{\sim}{B}$, the centrifugal force felt by the particle is

$$\underset{\sim}{F_c} = \frac{m v_{\parallel}^2}{R_c}\,\underset{\sim}{\hat{r}} = \frac{m v_{\parallel}^2}{R_c^2}\,\underset{\sim}{R_c}$$

b. Treating this as the general force $\underset{\sim}{F}$, we find

Curvature Drift
$$\boxed{\underset{\sim}{V_c} \equiv \frac{m v_{\parallel}^2}{q B^2}\,\frac{\underset{\sim}{R_c} \times \underset{\sim}{B}}{R_c^2}} = \frac{v_{\parallel}^2}{\omega_c B}\,\frac{\underset{\sim}{R_c} \times \underset{\sim}{B}}{R_c^2}$$

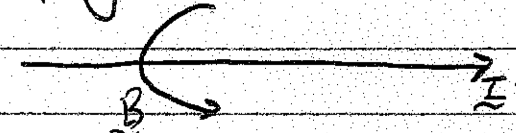
3. Properties: a. Depends on parallel energy $\frac{1}{2} m v_{\parallel}^2$

b. Again, ions & electrons drift in opposite directions

4. NOTE: When $\underset{\sim}{B}$ field lines are curved, there is typically also a gradient in $|\underset{\sim}{B}|$, so both ∇B & curvature drifts will be important.

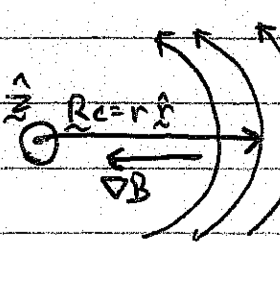
E. Example: Current Carrying Wire

Consider a wire carrying a current $\underset{\sim}{I} = I_0 \hat{z}$



1. In cylindrical coordinates (r, ϕ, z), $\underset{\sim}{B} = \frac{\mu_0 I_0}{2\pi r}\hat{\phi}$

2. End on view



From NRL Plasma Formulary p.6,

$$\nabla B = \frac{\partial B}{\partial r}\hat{r} + \cancel{\frac{1}{r}\frac{\partial B}{\partial \phi}\hat{\phi}} + \cancel{\frac{\partial B}{\partial z}\hat{z}} = -\frac{\mu_0 I_0}{2\pi r^2}\underset{\sim}{\hat{r}}$$

3. $\underline{\nabla B \text{ Drift}}$:

$$\underset{\sim}{V}_{\nabla B} = -\frac{v_\perp^2}{2\omega_c}\frac{\nabla B \times \underset{\sim}{B}}{B_0^2} = -\frac{v_\perp^2}{2\omega_c}\frac{\left(-\frac{\cancel{\mu_0 I_0}}{\cancel{2\pi} r^{\cancel{2}}}\hat{\underset{\sim}{r}}\right)\times\left(\frac{\cancel{\mu_0 I_0}}{\cancel{2\pi r}}\hat{\underset{\sim}{\phi}}\right)}{\left(\frac{\cancel{\mu_0 I_0}}{\cancel{2\pi r}}\right)^2} = +\frac{v_\perp^2}{2\omega_c r}\hat{\underset{\sim}{z}}$$

4. Curvature Drift:

$$\underset{\sim}{V}_c = \frac{v_\parallel^2}{\omega_c B}\frac{\underset{\sim}{R}_c \times \underset{\sim}{B}}{R_c^2} = \frac{v_\parallel^2}{\omega_c\left(\frac{\cancel{\mu_0 I_0}}{\cancel{2\pi r}}\right)}\frac{\left(r\,\hat{\underset{\sim}{r}}\right)\times\left(\frac{\cancel{\mu_0 I_0}}{\cancel{2\pi r}}\hat{\underset{\sim}{\phi}}\right)}{r^{\cancel{2}}} = \frac{v_\parallel^2}{\omega_c r}\hat{\underset{\sim}{z}}$$

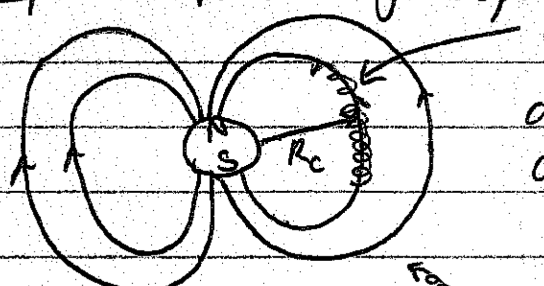
5. $\underline{\text{Net Drift}}$: $\underset{\sim}{V} = \underset{\sim}{V}_{\nabla B} + \underset{\sim}{V}_c = \frac{1}{\omega_c r}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\hat{\underset{\sim}{z}}$

a. NOTE: $\frac{1}{\omega_c r} = \frac{m}{qBr} = \frac{m\,2\pi \cancel{r}}{q\mu_0 I_0 \cancel{r}}$, so $\boxed{\underset{\sim}{V} = \frac{2\pi}{q\mu_0 I_0}\left(\frac{m v_\perp^2}{2} + m v_\parallel^2\right)\hat{\underset{\sim}{z}}}$

Velocity is independent of r!

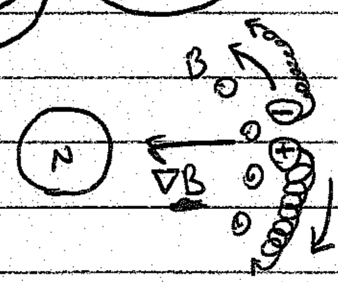
F. $\underline{\text{Example}}$: Earth's Magnetosphere

Side View:



1. Particles trapped in Earth's dipole field experience ∇B & curvature drifts

Top View:



2. These drifts produce the "ring current" in the westward direction

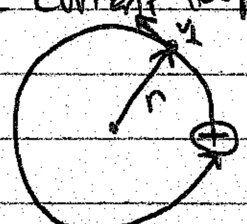
3. Strength of ring current is proportional to the energy of the particles.

$\Rightarrow$ Magnetic Storms!

VI. Magnetic Moment, The Mirror Force, and Adiabatic Invariance

A. Magnetic Moment 1. A current loop has magnetic moment $\mu = IA$

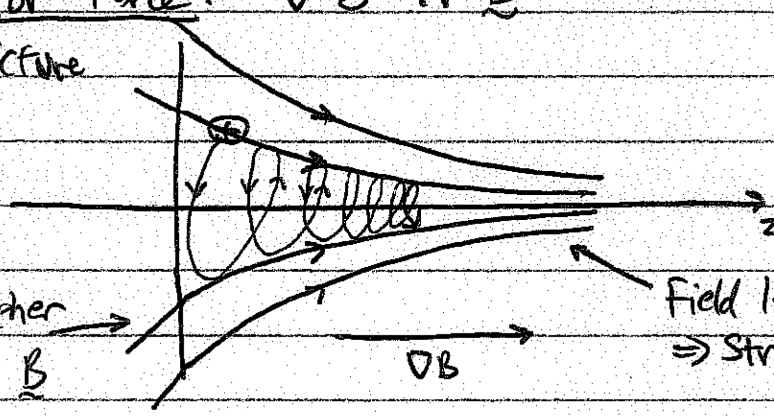
2. The current loop due to Larmor Motion gives



$$\boxed{\mu = \frac{m v_\perp^2}{2B}}$$ Magnetic Moment

B. The Mirror Force: $\nabla B \parallel \underset{\sim}{B}$

1. Physical Picture



2. Magnetic Mirror Force: $\boxed{F_z = -\mu \frac{\partial B_z}{\partial z}}$

a. Accelerates the particle along the field line in the direction of decreasing field magnitude.

b. This force can be used to confine particles (see Magnetic Mirror Machine)

C. Adiabatic Invariance:

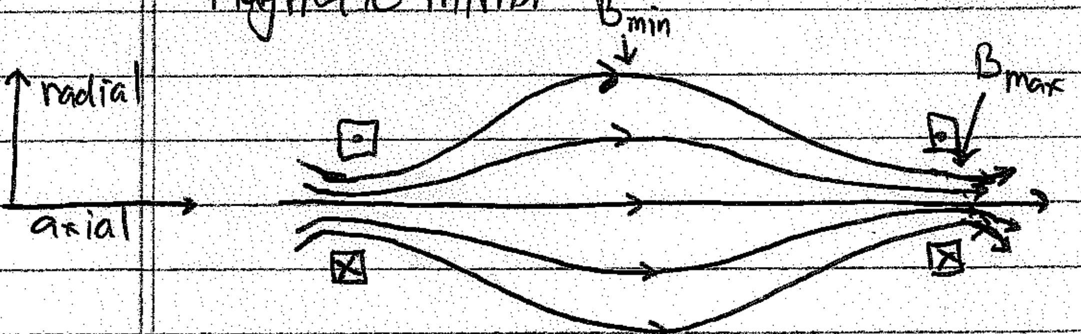
1. It can be shown that, as the charged particle moves through a spatially varying (or temporally varying) $\underset{\sim}{B}$ field, the magnetic moment is conserved, $\frac{d\mu}{dt} = 0$

2. Thus, the magnetic moment μ is a conserved quantity, or an adiabatic invariant. This invariance occurs changes of the system (spatial or temporal) occur slowly compared to the fast Larmor motion.

3. For any periodic motion, there exists a corresponding adiabatic invariant, related to the conservation of an action integral over the periodic motion from Hamiltonian mechanics.
   a. We'll see examples of a hierarchy of characteristic periodic motions in a system, leading to a hierarchy of adiabatic invariants.

D. Confinement in a Magnetic Mirror Machine

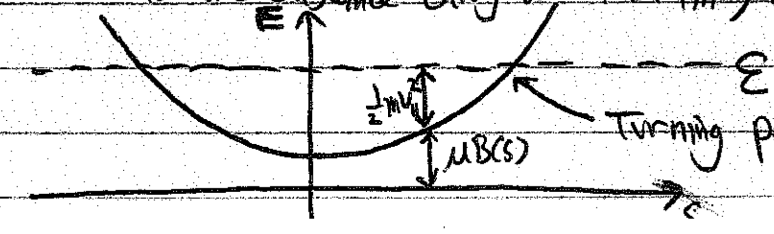
1. An early candidate for magnetic confinement fusion was the magnetic mirror:



   a. Particles are confined radially by Larmor motion about magnetic field
   b. Particles are confined axially by the mirror force.

2. a. Conservation of Energy: $\mathcal{E} = \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 = \text{constant}$.
   b. First Adiabatic Invariant: $\mu = \frac{mv_\perp^2}{2B} = \text{constant}$.
   But we can write $\frac{1}{2}mv_\perp^2 = \mu B$, so
   c. $\underset{\text{constant}}{\mathcal{E}} = \frac{1}{2}mv_\parallel^2 + \underset{\text{constant}}{\mu} B \Longleftarrow$ This relates field magnitude B to parallel velocity $v_\parallel$
   d. If s is the distance along the field line, $\mathcal{E} = \frac{1}{2}m[v_\parallel(s)]^2 + \mu B(s)$

Energy Interpretation of Mirror Force

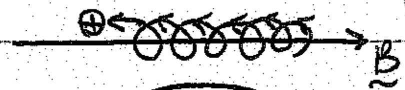


VI. D. (Continued)

3. <u>Periodic Motion and Adiabatic Invariants</u>

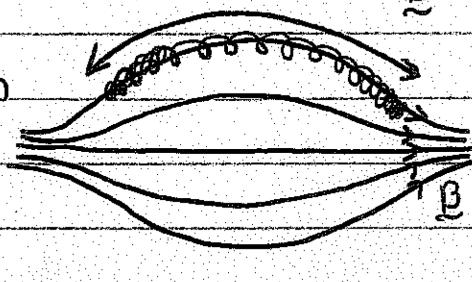
Three types of periodic motion in an axisymmetric magnetic mirror:

a. Larmor Motion



<u>1st</u>: $\mu = \frac{m v_\perp^2}{2B}$

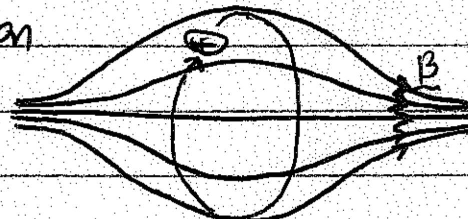
b. Parallel Bounce Motion



<u>2nd</u>: $J = m \oint v_\parallel \, ds$

c. Azimuthal Drift Motion

Due to ∇B and Curvature Drifts



<u>3rd</u>: Magnetic flux enclosed by drift orbit remains constant.

VII. <u>Polarization Drift:</u> Slowly varying $\underset{\sim}{E}$ and Constant $\underset{\sim}{B}$

A. <u>Multiple Timescale Analysis</u>

1. For a system in which the applied electric field $\underset{\sim}{E}$ is changing slowly in time, we can solve for the motion order by order, to find

$$\underset{\sim}{V} = \underbrace{v_\perp \left[\cos(\omega_c t + \phi)\hat{x} - \sin(\omega_c t + \phi)\hat{y} \right]}_{\text{Zeroth-order Larmor Motion}} + \underbrace{\frac{\underset{\sim}{E}(t) \times \underset{\sim}{B}}{B_0^2}}_{\substack{\text{First-order} \\ \underset{\sim}{E} \times \underset{\sim}{B} \text{ drift}}} + \underbrace{\frac{1}{\omega_c B_0} \frac{d\underset{\sim}{E}}{dt}}_{\substack{\text{Second-order} \\ \text{polarization drift.}}}$$

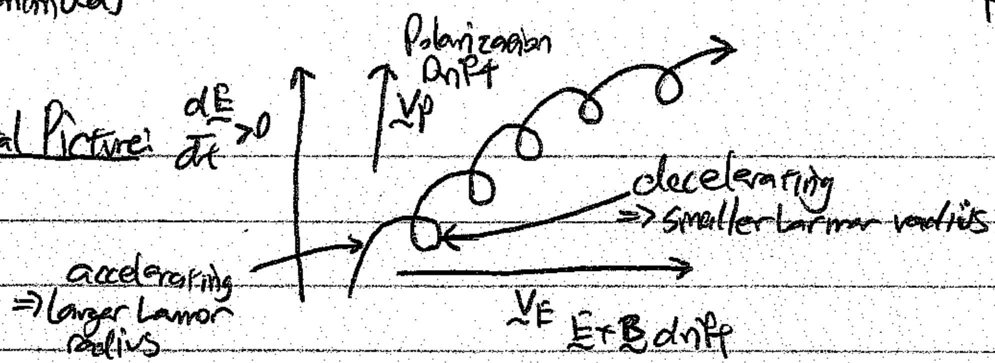
2. <u>Define</u>: Polarization Drift: $\boxed{\underset{\sim}{V}_p = \frac{1}{\omega_c B} \frac{d\underset{\sim}{E}}{dt}}$

a. NOTE: $\frac{1}{\omega_c B} = \frac{m}{q B^2}$, so polarization drift is charge dependent,

$\Rightarrow$ ions & electrons drift opposite directions

VII. A. (Continued)

3. Physical Picture: $\frac{dE}{dt} > 0$



VIII. When is the Single Particle Motion Description Useful?

A. 1. Never forget that single-particle motion is an inconsistent model of plasmas, with strict limitations.

2. The motion of charged particles in prescribed $\mathbf{E}(\mathbf{x},t)$ & $\mathbf{B}(\mathbf{x},t)$ fields is useful only when:

   a. You can guess $\mathbf{E}$ & $\mathbf{B}$ well

   b. The effect of the charge density ρ_q and current density $\mathbf{j}$ due to the plasma is small

3. Useful description for

   a. Low density plasma behavior in strong external fields

   b. For example, i. Small-scale laboratory plasmas

      ii. Laser-plasma interactions

   c. Can be helpful in building intuition about plasma behavior.

4. If the conditions above are not satisfied, single-particle motion results are unlikely to describe the plasma well

⇒ Need to use a consistent plasma model, the simplest of which is Magnetohydrodynamics (MHD).