Chiral lattice fermions with correct vacuum polarization and chiral anomaly

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An action for chiral lattice fermions is proposed, which avoids the Nielsen-Ninomiya theorem by virtue of its nonlocality and nonbilinearity. The action is constructed by eliminating the extra fermion modes with a gauge-violating Majorana-type Wilson mass, which is then rendered invariant by an integration over gauge transformations. The free propagator is calculated, and the one-loop vacuum polarization is shown to be identical to that for Wilson fermions, even at nonzero lattice spacing. Also the chiral anomaly is shown to be the same as for Wilson fermions in the continuum limit.

INTRODUCTION

Since its introduction, lattice gauge theory[1] has been plagued by the lack of a way to include chiral fermion fields. This is disturbing because we appear to live in a chiral world, the standard model containing only left-handed fermions. The problem is summed up by the theorem of Nielsen and Ninomiya[2] which says that there is no action describing a single fermion and satisfying the conditions of Hermiticity, invariance under lattice translations, bilinearity, locality, chiral invariance, and at least global gauge invariance. Although Nielsen and Ninomiya proved their theorem rigorously, there are heuristic arguments leading to the same conclusion.[3]

According to the no-go theorem, the action must violate at least one of the conditions, though to date the only successful actions have either broken chiral symmetry or allowed more than one fermion. In this paper I will consider violating locality and bilinearity. The action will be constructed by first breaking gauge invariance, then restoring it with an integral over gauge transformations.

Let us first take a brief look at what goes wrong on the lattice. The problem begins with the action obtained by naively substituting finite differences for derivatives in the Dirac action. The naive Euclidean lattice action for a massless fermion is

$$S_N=\frac{ia^3}{2}\sum_{n,\mu}(\bar{\psi}_n\gamma_\mu U_{n,\mu}\psi_{n+\mu}-\bar{\psi}_n\gamma_\mu U^\dagger_{n-\mu,\mu}\psi_{n-\mu}) , \qquad (1)$$

where a is the lattice spacing, ψ_n and $\bar{\psi}_n$ are independent Grassmann fields defined on sites, and the γ_μ's are Euclidean γ matrices satisfying

$$\{\gamma_\mu,\gamma_\nu\}=-\delta_{\mu,\nu} . \qquad (2)$$

The indices n and μ indicate lattice sites and directions, respectively, while $n\pm\mu$ is shorthand for the site displaced one lattice vector from site n in the μ direction. $U_{n,\mu}$ is a group element defined on links by

$$U_{n,\mu}=\exp(igaA_{n,\mu}) . \qquad (3)$$

Under the gauge group these fields transform as

$$U_{n,\mu}\rightarrow V_nU_{n,\mu}V^\dagger_{n+\mu} \quad \psi_n\rightarrow V_n\psi_n, \quad \bar{\psi}_n\rightarrow\bar{\psi}_nV^\dagger_n , \qquad (4)$$

where the V_n's are group elements defined on sites. For a U(1) gauge group, $A_{n,\mu}$ is a real number, whereas for non-Abelian groups $A_{n,\mu}$ is a matrix. Throughout most of this paper explicit calculations will be done in quantum electrodynamics (QED) for simplicity, although, as in Eq. (4), daggers and transposes will be retained in general expressions.

The free fermion propagator is

$$G_N(p)=-a\sum_\mu\gamma_\mu\sin p_\mu a\left[\sum_\mu\sin^2p_\mu a\right]^{-1} , \qquad (5)$$

which has poles at $p_\mu=0$ and π/a. Since each component of p_μ can be 0 or π/a, there are 16 poles and S_N describes as many fermions. Having violated none of the conditions of the Neilsen-Ninomiya theorem, S_N gives a "doubled" fermion spectrum. However, the situation is worse than a matter of simple species multiplication. Suppose one used two-component Weyl spinors in the hope of getting 16 left-handed fermions. It turns out that the doubling mechanism produces left- and right-handed fermions instead. A careful analysis of the effective γ_5 for each of the modes shows that half the particles have one chiral charge, and the others have the opposite charge.[4] This means that one cannot choose to assign chiral fermion charges arbitrarily since the doubling produces a vectorlike theory in the end.

There are two commonly used fermion actions: the Kogut-Susskind action[5] which reduces the multiplication to a factor of 8 while maintaining a discrete chiral invariance, and the Wilson action[1] which produces a single fermion, but breaks chiral symmetry. The action I will propose is very similar to the Wilson action which is obtained by adding to S_N the term

$$S_W=\frac{ra^3}{2}\sum_{n,\mu}(\bar{\psi}_nU_{n,\mu}\psi_{n+\mu}+\bar{\psi}_nU^\dagger_{n-\mu,\mu}\psi_{n-\mu}-2\bar{\psi}_n\psi_n)$$
$$=\sum_{mn}(\bar{\psi}_mW_{mn}\psi_n+\bar{\psi}_mW_{mn}\psi_n) , \qquad (6)$$

where r is an arbitrary parameter, and the second line defines the matrix W_{mn}. This gives the undesired modes momentum-dependent mass proportional to

$(1/a)\sin^2(p_\mu a/2)$ so that, in the continuum limit, the spurious modes have an infinite mass and decouple from the theory. Such a solution is satisfactory for quantum chromodynamics (QCD) which has equal numbers of left- and right-handed fields, but there is no hope for representing weak interactions or the standard model since the breaking of chiral symmetry in S_W forces there to be left- and right-handed fields at all lattice spacings, including the continuum limit.

NEW ACTION

As was pointed out by Karsten and Smit, the difficulties with lattice fermions are intimately related to the chiral anomaly.[4] In order to have a chiral anomaly, the quantum theory must be variant under the chiral transformation

$$\psi \to \exp(i\alpha\gamma_5)\psi, \quad \bar{\psi} \to \bar{\psi}\exp(i\alpha\gamma_5) , \tag{7}$$

even though the classical action is invariant. In continuum perturbation theory this comes about through the triangle graph,[6] while nonperturbatively the anomaly reflects the fact that the fermionic measure in the continuum path integral varies under chiral transformations.[7] Unfortunately, the lattice measure is manifestly chiral invariant. With no means of obtaining an anomaly in the measure, the lattice path integral will be invariant under chiral transformations unless the action itself breaks this symmetry. Wilson fermions succeed in producing the correct anomaly in this way, breaking chiral invariance in the action; although the Wilson term formally vanishes in the continuum limit, a remnant remains which gives the desired anomaly.[4,8]

It seems inescapable that the action itself must provide the variation under Eq. (7). However, Wilson fermions require *L-R* couplings which we cannot allow in a chiral gauge theory containing only left-handed fields. A term is needed which varies under Eq. (7), but only involves *L-L* (or *R-R*) interactions. These requirements force us to consider a Majorana-type Wilson term

$$S_C = \sum_{mn} \tfrac{1}{2}(\psi_m^T C W_{mn}\psi_n - \bar{\psi}_m C^\dagger W_{mn}\bar{\psi}_n^T) , \tag{8}$$

where C is the charge-conjugation matrix, satisfying $C^\dagger\gamma_\mu^T C = -\gamma_\mu$, $C^T = -C$, $C\gamma_5 = \gamma_5 C$, and W_{mn} is defined by Eq. (6).

Some facts about Eq. (8) are worthy of comment. The first is that the difference in sign between the two terms in S_C is no accident. The minus sign on the $\bar{\psi}$ term is required to make the action Hermitian when continued to Minkowski space. The other notable feature of S_C, the factor of $\frac{1}{2}$, is really a matter of convention. After all, r could be redefined. The $\frac{1}{2}$ is included to make r in Eq. (8) agree with the common definition of r used with standard Wilson fermions. Lastly, it should be pointed out that ψ is not a Majorana spinor, and is not constrained to obey the Majorana condition. The interactions in S_C will nonetheless be referred to as Majorana type.

While S_C looks promising, it is unfortunately gauge invariant, transforming as

$$\begin{aligned} \psi_m^T W_{mn}\psi_n &\to \psi_m^T V_m^T V_m W_{mn}\psi_n , \\ \bar{\psi}_m W_{mn}\bar{\psi}_n^T &\to \bar{\psi}_m W_{mn} V_n^\dagger V_n^{\dagger T}\bar{\psi}_n^T . \end{aligned} \tag{9}$$

This may be remedied by integrating the exponentiated action over all gauge transformations, making the path integral

$$Z_C = \int D\psi D\bar{\psi} \int_V \exp(-S_N - S_C) . \tag{10}$$

Equivalently, Z_C may be written as

$$\begin{aligned} Z_C &= \int D\psi D\bar{\psi}\exp(-S_{\text{eff}}) , \\ S_{\text{eff}} &= S_N - \ln\int_V \exp(-S_C) , \end{aligned} \tag{11}$$

from which it becomes apparent how the Nielsen-Ninomiya theorem is avoided; the effective action is nonlocal and nonlinear. The logarithm cannot undo the exponential since $\int_V$ sets to zero any gauge-invariant terms in $\exp(-S_C)$, and thus S_{eff} contains products of fields not possible with a local bilinear action. Using the integration rather than introducing an auxiliary field emphasizes the fact that S_{eff} may simply be written down as a sum of products of the ψ_n's and $U_{n,\mu}$'s.

This construction bears a resemblance to the proposal to Aoki, Funakubo, and Kashiwa[9] (AKF) insofar as the Wilson term breaks gauge invariance. AFK used an ungauged Dirac-type Wilson mass, breaking both chiral and gauge symmetry, the latter of which was restored with a change of variables to implement the integral over gauge transformations. This approach was later shown to be equivalent[10] to the proposal of Swift and Smit,[11] in which a radially frozen Higgs field was used to give the spurious fermion modes a large mass. Unfortunately, the AFK proposal leads to a quadratically divergent vacuum polarization[12] indicating that a bare mass term must be included for A_μ in the original action. It also contains *L-R* fermion couplings.

Despite the resemblance, S_{eff} differs from the AFK action in two very important ways: it does not couple left- and right-handed fermions, and gauge fields appear in the Wilson term. The advantage of not including *L-R* couplings is clear, for it allows one to construct a theory which is chiral from the beginning, without having to resort to a limiting process or fine-tuning. In fact, the action may be written using a single left-handed Weyl fermion. The virtue of including gauge fields in the Wilson term is somewhat subtle. A minimum requirement for a lattice fermion action is that, in the continuum limit, it agrees with standard continuum perturbation theory. Examination of perturbation theory with Wilson fermions shows that the gauge-fermion interactions arising from the Wilson term are crucial. It is exactly because the AFK action omits gauge fields from the Wilson mass that the vacuum polarization is quadratically divergent. While one might argue for the necessity of using covarient derivatives as a fundamental principle, the necessity of agreement with continuum perturbation theory compels us to include gauge fields as well.

The proposed action will now be subjected to three standard tests for lattice fermions. The free propagator will be calculated to verify that it has a single pole at $p_\mu=0$, the one-loop vacuum polarization will be compared with Wilson fermions, and the chiral anomaly will be shown to be the same as for Wilson fermions in the continuum limit.

PROPAGATORS

The usual free-fermion propagator is given by

$$\langle \bar\psi_m \psi_n \rangle = \int D\psi D\bar\psi\, \bar\psi_m \psi_n \exp(-S_{\mathrm{eff}}) , \tag{12}$$

which is more conveniently evaluated as

$$\langle \bar\psi_m \psi_n \rangle = \int D\psi D\bar\psi\, \bar\psi_m \psi_n \int_V \exp(-S_N - S_C) . \tag{13}$$

Because of S_C, we must consider $\langle \psi_m^T \psi_n \rangle$ and $\langle \bar\psi_m \bar\psi_n^T \rangle$ propagators as well. All three may be calculated in momentum space by summing insertions of the S_C vertices given by

$$\begin{aligned} -\tfrac{1}{2} C W(p) &= -\tfrac{1}{4} a^{-1} r C \sum_\mu \sin^2 p_\mu a/2 , \\ \tfrac{1}{2} C^\dagger W(p) &= \tfrac{1}{4} a^{-1} r C^\dagger \sum_\mu \sin^2 p_\mu a/2 , \end{aligned} \tag{14}$$

into the naive propagator, as shown in Fig. 1. The factor of 2 on each insertion comes from the two possible orientations. The $\langle \bar\psi\psi \rangle$ propagator is

$$G_{\bar\psi\psi}(p) = G_N(p) \sum_n [-W(p) C^\dagger G_N^T(p) C W(p) G_N(p)]^n , \tag{15}$$

where the transpose refers to the spinor indices. The transpose simply reflects the order in which the spinor indices are contracted, as indicated by the directions of the arrows in Fig. 1.

Using $C^\dagger \gamma_\mu^T C = -\gamma_\mu$ and performing the sum,

$$G_{\bar\psi\psi}(p) = -a \sum_\mu \gamma_\mu \sin p_\mu a \left[\left[r \sum_\mu \sin^2 p_\mu a/2 \right]^2 + \sum_\mu \sin^2 p_\mu a \right]^{-1} . \tag{16}$$

Similarly, we find, for $\langle \psi^T \psi \rangle$ and $\langle \bar\psi \bar\psi^T \rangle$,

$$G_{\psi^T\psi}(p) = -arC \sum_\mu \sin^2 p_\mu a/2 \left[\left[r \sum_\mu \sin^2 p_\mu a/2 \right]^2 + \sum_\mu \sin^2 p_\mu a \right]^{-1} , \tag{17}$$

$$G_{\bar\psi\bar\psi^T}(p) = arC^\dagger \sum_\mu \sin^2 p_\mu a/2 \left[\left[r \sum_\mu \sin^2 p_\mu a/2 \right]^2 + \sum_\mu \sin^2 p_\mu a \right]^{-1} . \tag{18}$$

All three have the required undoubled pole structure.

The integral over gauge transformations is included by considering how the individual diagrams of Fig. 1 transform. Since the mass insertions come from expanding $\exp(-S_N - S_C)$ in Eq. (13), they appear under $\int_V$ and are subject to gauge transformations. From Eq. (9) we see that the C- and $C^\dagger$-type vertices transform as $V^T(p)V(p)$ and $V^\dagger(p)V^{\dagger T}(p)$, respectively. This means that any diagram containing unequal numbers of C- and $C^\dagger$-type mass insertions will transform nontrivially and will be zero under $\int_V$. This is not to say that $G_{\psi^T\psi}$ and $G_{\bar\psi\bar\psi^T}$ vanish. The reason is that a diagram may contain

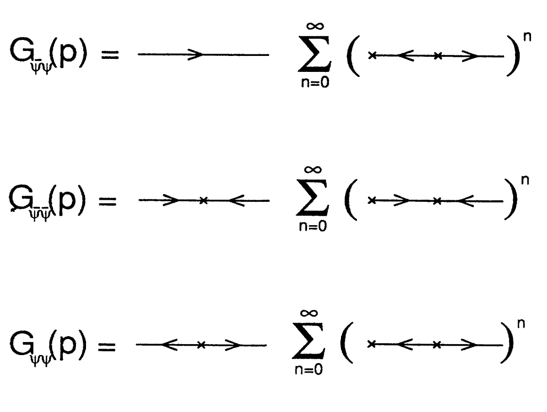

FIG. 1. The undoubled $\bar\psi\psi$, $\psi\psi$, and $\bar\psi\bar\psi$ propagators as insertions of the vertices of Eq. (13) into the doubled propagator.

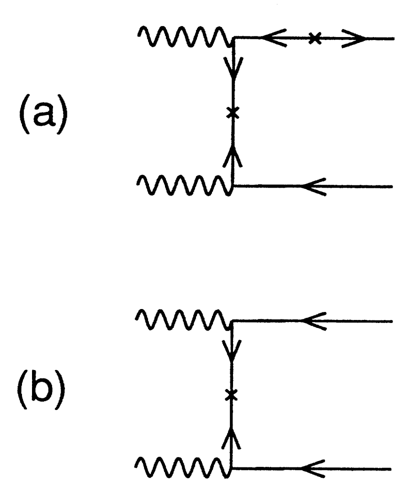

FIG. 2. (a) A diagram with a balanced number of C- and $C^\dagger$-type insertions which survives the integral over gauge transformations. (b) A diagram with one extra C-type insertion which is therefore zero under $\int_V$.

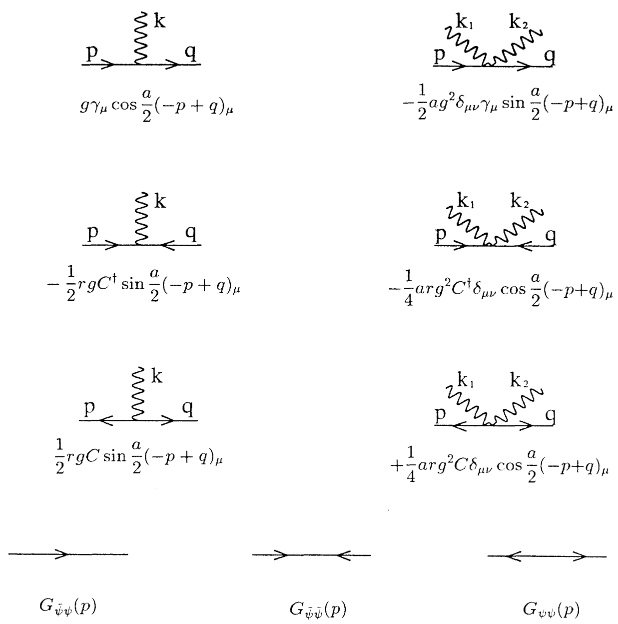

FIG. 3. Feynman rules for the proposed fermion action. All momenta are defined to flow into the vertices.

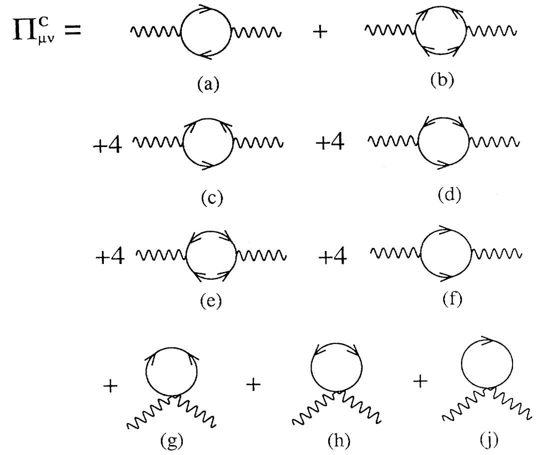

FIG. 4. One-loop vacuum polarization for the proposed action.

equal numbers of C- and $C^\dagger$-type insertions, giving a nonzero contribution to some amplitude. For example, Fig. 2(a) shows a diagram with canceling vertices. Under a gauge transformation it will be multiplied by an equal number of V's and $V^\dagger$'s, giving a singlet which survives the integral over gauge transformations. On the other hand, the diagram in Fig. 2(b) contains an extra C-type insertion, causing it to transform like V, and thus integrating to 0. The propagators $G_{\psi^T\psi}$ and $G_{\overline{\psi}\overline{\psi}^T}$ must be retained, and the Feynman rules augmented to disregard diagrams with "unbalanced arrows," such as Fig. 2(b).

VACUUM POLARIZATION

The breaking of gauge invariance in S_C is a possible source of problems, and there is good cause for fear that some irreparable damage has been done to the theory. As a test, I will compare the QED one-loop vacuum polarization $\Pi^C_{\mu\nu}$ to that of standard Wilson fermion. Rather than simply calculating $\Pi^C_{\mu\nu}$, a diagram-by-diagram comparison with Wilson fermions points to the close similarities present even in perturbation theory. Also, this way of doing the calculation makes it clear that the agreement is not a coincidence and may be generalized to higher orders.

The Majorana-type interaction vertices are given in Fig. 3. They are derived in the usual manner by expanding

$$U_{n,\mu} = \sum_m (igaA_{n,\mu})^m/m!\ . \tag{19}$$

Note that the Feynman rules do not include the gauge integration which is performed on amplitudes.

Using these vertices and the propagators in Eqs. (16)–(18), the vacuum polarization $\Pi^C_{\mu\nu}$ is given by the sum shown in Fig. 4. The symmetry factors include a factor of 2 for the two possible ways of orientating the ingoing (outgoing) arrows on the $C^\dagger$- (C-) type vertices.

For comparison, the Dirac-type interaction vertices are given in Fig. 5. These are slightly different than usual[4] because the pieces of the vertices coming from S_N and S_W have been separated, the latter being denoted by a circle on the vertex. The fermion propagator is also split into two pieces:

$$G_{W1}(p) = -a\sum_\mu \gamma_\mu \sin p_\mu a\left[\left[r\sum_\mu \sin^2 p_\mu a/2\right]^2 + \sum_\mu \sin^2 p_\mu a\right]^{-1}, \tag{20}$$

$$G_{W2}(p) = ra\sum_\mu \sin^2 p_\mu a/2\left[\left[r\sum_\mu \sin^2 p_\mu a/2\right]^2 + \sum_\mu \sin^2 p_\mu a\right]^{-1}, \tag{21}$$

with G_{W2} indicated by a circle on the line. Splitting up the Feynman rules in this manner makes it possible to compare with $\Pi^C_{\mu\nu}$ diagram by diagram.

Using these fractured Feynman rules, Fig. 6 shows $\Pi^W_{\mu\nu}$, in which those diagrams with an odd number of γ matrices have been omitted. Although the integrals are superficially quadratically divergent, a cancellation causes $\Pi^W_{\mu\nu}$ to be only logarithmically divergent.

The integrals contained in $\Pi^C_{\mu\nu}$ are easily shown to be the same as those in $\Pi^W_{\mu\nu}$. As an example, Fig. 4(f) is given by

$$\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}(-\tfrac{1}{2}C^\dagger rg\gamma_\mu \sin p_\mu a)\frac{-a\sum_\sigma \gamma_\sigma^T \sin p_\sigma a}{\left[r\sum_\sigma \sin^2 p_\sigma a/2\right]^2+\left[\sum_\sigma \sin^2 p_\sigma a\right]^2}(\tfrac{1}{2}Crg\gamma_\nu \sin p_\nu a)\frac{-a\sum_\sigma \gamma_\sigma \sin p_\sigma a}{\left[r\sum_\sigma \sin^2 p_\sigma a/2\right]^2+\left[\sum_\sigma \sin^2 p_\sigma a\right]^2}$$

$$=\tfrac{1}{4}\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}(rg\gamma_\mu \sin p_\mu a)\frac{a\sum_\sigma \gamma_\sigma \sin p_\sigma a}{\left[r\sum_\sigma \sin^2 p_\sigma a/2\right]^2+\left[\sum_\sigma \sin^2 p_\sigma a\right]^2}$$

$$\times(rg\gamma_\nu \sin p_\nu a)\frac{a\sum_\sigma \gamma_\sigma \sin p_\sigma a}{\left[r\sum_\sigma \sin^2 p_\sigma a/2\right]^2+\left[\sum_\sigma \sin^2 p_\sigma a\right]^2}\,, \qquad (22)$$

where, once again, use has been made of $C\gamma_\mu^T C^\dagger = -\gamma_\mu$, and the trace is over γ matrices. Multiplying Eq. (22) by the symmetry factor of 4, we find exactly the same integral as Fig. 4(d) for the Dirac case. Continuing in this manner, Figs. (4) and (6) give identical results.

In evaluating the diagrams in Fig. 4, one must be very careful to observe how the spinor indices are contracted. The presence of γ_σ^T in Eq. (22) reflects the ordering of spinor indices, just as in Eq. (15). Similarly, there must be a consistent convention for the contraction of indices on the interaction vertices. The spinor indices on $(\gamma_\mu)_{\alpha\beta}$ are associated with a particular orientation of the arrows on the vertex. Therefore, if the product of propagators and vertices around the loop encounters a vertex with backwards arrows, that vertex must use γ_μ^T.

Note that, if gauge fields had been omitted from S_C diagrams, (c)–(g), (i), and (j) would not appear in Fig. 4, and $\Pi^C_{\mu\nu}$ would have differed from $\Pi^W_{\mu\nu}$. It is also very interesting that the integral over gauge transformations is not required to ensure a logarithmically divergent vacuum polarization. The violation of gauge symmetry contained in S_C is surely the most benign imaginable. Inclusion of the gauge integral leaves $\Pi^C_{\mu\nu}$ unchanged. In fact, we can easily see that fermion loops will, in general, survive the integral. A fermion loop contains equal numbers of C- and $C^\dagger$-type interactions, even if some of them come from propagators. Since there are then equal numbers of interactions transforming as $V^T(p)V(p)$ and $V^\dagger(p)V^{\dagger T}(p)$, the loop will contain a singlet and remain nonzero under $\int_V$.

The diagram-by-diagram comparison reveals how similar perturbation theory for $S_{\rm eff}$ is to that for Wilson fermions. The two sets of interaction vertices are, of course, not very different. The Majorana-type vertices have C's and $C^\dagger$'s but, as was pointed out above, a loop must have equal numbers of the two vertex types. The charge-conjugation matrices then either cancel from $CC^\dagger=1$, or transpose a γ matrix on a propagator. In either case, what remains is the same integral as a Wilson fermion diagram. The normalization is taken care of by the extra factor of $\frac{1}{2}$ in Eq. (8) which cancels the symmetry factor coming from the two possible ways of inserting the Majorana-type vertices.

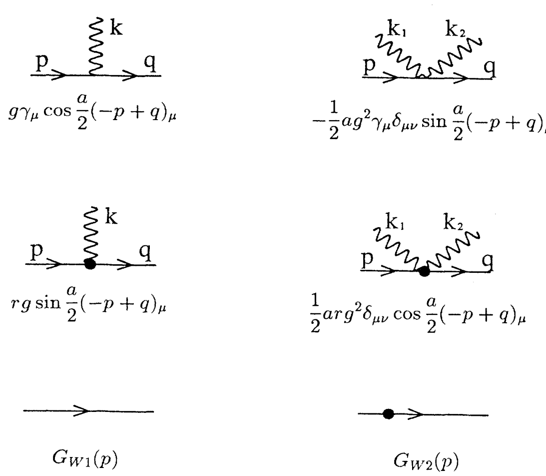

FIG. 5. Feynman rules for standard Wilson fermions, but with interactions from the naive action separated out for comparison with Fig. 3. Vertices with circles are those from S_W. All momenta are defined to flow into the vertices.

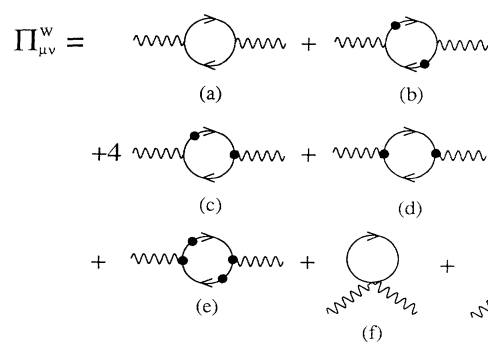

FIG. 6. One-loop vacuum polarization for standard Wilson fermions.

CHIRAL ANOMALY

An important test of any lattice fermion is the chiral anomaly, which has been investigated for several proposed fermions.[13] Instead of calculating the triangle graph, the chiral transformation properties of the path integral will be examined. Because of the similarity to Wilson fermions, the calculation is nearly identical to that found in Ref. 8. I will show that, in the continuum limit, the proposed action produces the same anomaly as does the Wilson action. This is sufficient since Wilson fermions are known to give the correct chiral anomaly.

Because S_C connects ψ with ψ, it should be written using a single matrix acting on both ψ and $\bar{\psi}$. Defining

$$Q=\frac{1}{2}\begin{bmatrix}0 & i\not{D}\\ -i\not{D}^T & 0\end{bmatrix}, \qquad R=\frac{1}{2}\begin{bmatrix}CW & 0\\ 0 & -C^\dagger\overline{W}\end{bmatrix}, \tag{23}$$

the action may be written as

$$S_N+S_C=\begin{bmatrix}\psi\\ \bar{\psi}\end{bmatrix}(Q+R)\begin{bmatrix}\psi\\ \bar{\psi}\end{bmatrix}, \tag{24}$$

where $\not{D}$, W, and $\overline{W}$ are shorthand for Eqs. (1) and (8). Because Q and R should reflect only the transformation properties of the action, $i\not{D}$ is defined to be gauge invariant while W and $\overline{W}$ transform as

$$\begin{aligned}W_{mn}&\to V_m^T V_m W_{mn}\,,\\ \overline{W}_{mn}&\to \overline{W}_{mn}V_n^\dagger V_n^{\dagger T}\,.\end{aligned} \tag{25}$$

Under an infinitesimal chiral transformation, the path integral transforms as

$$\int D\psi D\bar{\psi}\exp(-S_{\rm eff})\to \int D\psi D\bar{\psi}\int_V \exp\left[-S_N-S_C-i2\alpha\begin{bmatrix}\psi\\ \bar{\psi}\end{bmatrix}\gamma_5 R\begin{bmatrix}\psi\\ \bar{\psi}\end{bmatrix}\right]. \tag{26}$$

Performing the Grassmann integration and then using the properties of determinants, Eq. (26) becomes

$$\begin{aligned}\int_V \det(Q+R+i2a\gamma_5R)^{1/2}&=\int_V \det(Q+R)^{1/2}\det[1+i2\alpha(Q+R)^{-1}\gamma_5R]^{1/2}\\ &=\int_V \det(Q+R)^{1/2}\exp\{\mathrm{Tr}\ln[1+i2\alpha(Q+R)^{-1}\gamma_5R]^{1/2}\}\\ &\approx\int_V \det(Q+R)^{1/2}\exp\{\mathrm{Tr}[i\alpha(Q+R)^{-1}\gamma_5R]\}\,,\end{aligned} \tag{27}$$

where the last line used the fact that α is infinitesimal, and the trace is over all lattice, spinor, and gauge indices. Equation (27) shows that, under the chiral transformation, the path integral acquires an extra phase, the anomaly factor. Substituting for R and Q and expanding, we find the phase to be

$$\begin{aligned}A&=\exp\{\mathrm{Tr}[i\alpha(Q+R)^{-1}\gamma_5R]\}\\ &=\exp\{\mathrm{Tr}[i\alpha\gamma_5(Q^2+R^2+QR+RQ)^{-1}(Q+R)R]\}\\ &=\exp\left[\mathrm{Tr}\left\{i\alpha\gamma_5(G-GVG+GVGVG+\dots)\left[4QR-\begin{bmatrix}W^2 & 0\\ 0 & \overline{W}^2\end{bmatrix}\right]\right\}\right],\end{aligned} \tag{28}$$

where

$$V=\begin{bmatrix}\frac{i}{2}\sigma_{\mu\nu}[D_\mu,D_\nu] & iCW\not{D}-i\not{D}C^\dagger\overline{W}\\ i\not{D}CW-iC^\dagger\overline{W}\not{D} & \frac{i}{2}\sigma_{\mu\nu}[D_\mu,D_\nu]\end{bmatrix}, \qquad G=\begin{bmatrix}D^2-W^2 & 0\\ 0 & D^2-\overline{W}^2\end{bmatrix}^{-1}. \tag{29}$$

Multiplying the matrices and tracing over the two-dimensional subspace gives nearly the same expression obtained for Wilson fermions. The only difference is that W and $\overline{W}$ transform under Eq. (25). Therefore, we may take advantage of the calculation already done for Wilson fermions.[8]

In the continuum limit only the $GVGVG$ term contributes, and terms containing $[W,D_\mu]$ vanish as well.[8] This leaves

$$A=\exp[-i\alpha\,\mathrm{Tr}\gamma_5(g\Sigma g\Sigma gW^2+\bar{g}\Sigma\bar{g}\Sigma\bar{g}\overline{W}^2)]\,, \tag{30}$$

where

$$g=(D^2-W^2)^{-1},\quad \bar{g}=(D^2-\overline{W}^2)^{-1}\,,\qquad \Sigma=\frac{i}{2}\sigma_{\mu\nu}[D_\mu,D_\nu]\,. \tag{31}$$

Terms coming from $4QR$ in Eq. (28) have been deleted since they contain an odd number of γ_μ's and are zero under the trace over spinor indices. Finally, in the continuum limit, the contributions from gauge fields in $(D^2-W^2)^{-1}$ and $(D^2-\overline{W}^2)^{-1}$ vanish.[8] This means that the factors arising from the gauge transformation in Eq. (25) do not contribute either, and we may use the ungauged, nontransforming versions of $(D^2-W^2)^{-1}$ and $(D^2-\overline{W}^2)^{-1}$. This gives the same expression as for Wilson fermions, which is then evaluated exactly as in Ref. 8, yielding the correct anomaly factor in the continuum limit.

GAUGE TRANSFORMATIONS

It is encouraging that both the vacuum polarization and the chiral anomaly did not rely on the integration over gauge transformations. To see what effect S_C does have on the gauge symmetry, we should extend the theory beyond fermions in a gauge background and consider the full path integral with gauge fields. This will shed some light on how gauge-violating interactions may be introduced.

The complete path integral is given by

$$Z=\int_V \int DU \int D\psi D\overline{\psi} \exp(-S_G-S_N-S_C) \ , \qquad (32)$$

where S_G is the gauge action and $\int_V$ has been pulled outside since the measures are gauge invariant. As it stands, Eq. (32) contains two integrals over gauge transformations, the second one comiming from $\int DU$.

Although gauge fixing is not required on the lattice, invariant Green's functions will be the same if the path integral is restricted to inequivalent configurations. Therefore, we may consider the gauge fixed

$$Z_{\rm GF}=\int_V \int' DU \exp(-S_G) \times \int D\overline{\psi} D\psi \exp(-S_N-S_C) \ , \qquad (33)$$

where the prime denotes integration over gauge-inequivalent configurations only. A trivial rearrangement of Eq. (33) gives

$$Z_{\rm GF}=\int DU \exp(-S_G) \int D\psi D\psi \exp(-S_N-S_C) \ , \qquad (34)$$

which is what one would have obtained by blindly adding S_C to the naive action.

Equation (34) has the disturbing appearance that a gauge-violating interaction has been simply added to an otherwise well-behaved theory. In fact, the gauge-violating terms coming from S_C provide a gauge-fixing function rather than destroying the symmetry outright. To see this more explicitly let

$$\det(Q+R)^{1/2}=F_I+F_V \ , \qquad (35)$$

where F_I and F_V are gauge invariant and variant, respectively. Integrating out the fermions, Eq. (34) becomes

$$Z=\int DU \exp(-S_G) F_I(1+F_V/F_I) \ , \qquad (36)$$

and we see that a gauge-fixing function[14]$(1+F_V/F_I)$ has appeared. Naive introduction of S_C provides the necessary undoubling, while its potential to break gauge symmetry is instead realized as a nonlocal gauge-fixing condition. Of course, one is not compelled to compute in this strange gauge if one is willing to do the extra integral over gauge transformations. However, by calculating in a nonlocal gauge, invariant quantities may be calculated with the single integral over all U's.

It appears that the integral over gauge transformations prevents the breakdown of gauge symmetry we would expect from naive introduction of S_C. This is made possible by the compactness of the group integration, which allows the lattice theory to be formulated without gauge fixing. This is not to say that gauge-violating interactions may be added at will, since such interactions do contribute to gauge-invariant quantities. The final test is to see that the desired theory is recovered in the continuum limit. For the proposed action this is the case, and a single chiral fermion is obtained.

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